1 YES/NO questions

1. For any binary relation $R \subseteq A \times A$, $R^*$ exists.
   Justify: definition

2. $R^* = R \cup \{(a, b) : \text{there is a path from } a \text{ to } b\}$.
   Justify: book definition

3. $R^* = R$ for $R = \{(a, b), (b, c), (a, c)\}$.
   Justify: $(a, a) \in R^*$ (trivial path from $a$ to $a$ always exist) but $(a, a) \notin R$

4. All infinite sets have the same cardinality.
   Justify: $|N| < |2^N|$ by Cantor Theorem and $N, 2^N$ are infinite

5. Set $A$ is uncountable iff $R \subseteq A$ ($R$ is the set of real numbers).
   Justify: $R, 2^R$ are both uncountable and $R$ is not a subset of $2^R$ ($R \nsubseteq 2^R$) but $R \in 2^R$.

6. Let $A \neq \emptyset$ such that there are exactly 25 partitions of $A$. It is possible to define 20 equivalence relations on $A$.
   Justify: one can define up to 25 (as many as partitions) of equivalence classes

7. There is a relation that is equivalence and order at the same time.
   Justify: equality relation

8. Let $A = \{n \in N : n^2 + 1 \leq 15\}$. It is possible to define 8 alphabets $\Sigma \subseteq A$.
   Justify: $A$ has 4 elements, so we have $2^4 > 8$ subsets

9. There is exactly as many languages over alphabet $\Sigma = \{a\}$ as real numbers.
   Justify: $|\Sigma^*| = \aleph_0, |2^{\Sigma^*}| = |R| = C$.

10. Let $\Sigma = \{a, b\}$. There are more than 20 words of length 4 over $\Sigma$.
    Justify: There are exactly $2^4 = 16$ words of length 4 over $\Sigma$ and $16 < 20$.

11. $L^* = \{w_1...w_n : w_i \in L, i = 1, 2, .., n, n \geq 1\}$.
    Justify: $n \geq 0$.
    $L^+ = L \cup L^*$
    Justify: the problem is only with cases $e \in L$ or $e \notin L$. When $e \in L$, then $e \in L^+$, and always $e \in L^*$, hence $e \in LL^*$.
    When $e \notin L$, then $e \notin L^+$, and always $e \in L^*$, hence $e \in L \cup L^*$ and $L^+ \neq L \cup L^*$

12. $L^+ = L^* - \{e\}$.
    Justify: only when $e \notin L$. When $e \in L$ we get that $e \in L^+$ and $e \notin L^* - \{e\}$.

13. If $L = \{w \in \{0, 1\}^* : w \text{ has an unequal number of } 0\text{'s and } 1\text{'s }\}$, then $L^* = \{0, 1\}^*$.
    Justify: $1 \in L, 0 \in L$ so $\{0, 1\} \subseteq L \subseteq \Sigma^*$, hence $\{0, 1\}^* \subseteq L^* \subseteq (\Sigma^*)^* = \Sigma^* = \{0, 1\}^*$ and $L^* = \{0, 1\}^*$.
14. For any languages $L_1$, $L_2$, $(L_1 \cup L_2) \cap L_1 = L_1$.
   **Justify:** languages are sets and $(A \cup B) \cap A = A$. y

15. For any languages $L_1$, $L_2$,

   $$L_1^* = L_2^*$$

   **Justify:** consider $L_1 = \{a, e\}, L_2 = \{a\}$. Obviously, $L_1 \neq L_2$ and $L_1^* = L_2^*$. n

16. For any languages $L_1$, $L_2$, $(L_1 \cup L_2)^* = L_1^*$. 
   **Justify:** languages are sets so it is true only when $L_1 \subseteq L_2$. n

17. $((\emptyset \cap a) \cup b^*) \cap \emptyset^*$ describes a language with only one element.
   **Justify:** $\emptyset \cup b^* = b^*, b^* \cap \{e\} = \{e\}$. y

18. $((\emptyset \cap a) \cup b^*) \cap a^*$ is a finite regular language.
   **Justify:** $b^* \cap a^* = \{e\} = \emptyset^*$. y

19. $((\emptyset \cap a) \cup b^*) \cap \{ab\}^*$ is a finite regular language.
   **Justify:** $\{a\} \cup \{e\} \cap \{ab\}^* = \{a, e\} \cap \{ab\}^* = \{e\} = \emptyset^*$. y

20. Any regular language has a finite description.
   **Justify:** by definition $L = \mathcal{L}(r)$ and $r$ is a finite string. y

21. Any finite language is regular.
   **Justify:** $L = \{w_1\} \cup \ldots \cup \{w_1\}$ and $\{w_1\}$ has a finite description $w_i$. y

22. Every deterministic automaton is also non-deterministic.
   **Justify:** any function is a relation y

23. The set of all configurations of any non-deterministic state automata is always non-empty.
   **Justify:** $K \neq \emptyset$, because $s \in K$. If $\Sigma = \emptyset$ the set of all configuration of non-deterministic automata (book definition) is a subset of $K \times \emptyset \cup \{e\} \neq \emptyset$ as it always contains $(s, e)$. For the lecture definition, the set of all configuration is a subset of $K \times \Sigma^*$ and always $e \in \Sigma^*$ hence always $(s, e) \in K \times \Sigma^*$. y

24. Let $M$ be a finite state automaton, $L(M) = \{w \in \Sigma^* : (q, w) \stackrel{M}{\rightarrow} (s, e)\}$.
   **Justify:** $L(M) = \{w \in \Sigma^* : \exists q \in F((s, w) \stackrel{M}{\rightarrow} (q, e))\}$ n

25. For any automata $M$, $L(M) \neq \emptyset$.
   **Justify:** if $\Sigma = \emptyset$ or $F = \emptyset$, $L(M) = \emptyset$ n

26. $L(M_1) = L(M_2)$ iff $M_1$, $M_2$ are deterministic.
   **Justify:** Let $M_1$ be an automata over $\{a, b\}$ with $\Delta = \{(q_0, ab, q_0), F = \{q_0\}, s = q_0$ and let $M_2$ be an automata over $\{a, b\}$ with $\Delta = \{(q_0, ab, q_0), (q_0, e, q_1)\}, F = \{q_1\}, s = q_0$. $L(M_1) = L(M_2) = (ab)^*$ and both are non-deterministic n

27. DFA and NDFA compute the same class of languages.
   **Justify:** basic theorem y

28. Let $M_1$ be a deterministic, $M_2$ be a nondeterministic FA, $L_1 = L(M_1)$ and $L_2 = L(M_2)$ then there is a deterministic automaton $M$ such that $L(M) = (L_1 \cup (L_1 - L_2)^*)L_1$
   **Justify:** the class of finite automata is closed under $\ast, \cup, -, \cap$ y
TWO DEFINITIONS OF NON DETERMINISTIC AUTOMATA

BOOK DEFINITION: \( M = (K, \Sigma, \Delta, s, F) \) is non-deterministic when
\[
\Delta \subseteq K \times (\Sigma \cup \{ \epsilon \}) \times K
\]

OBSERVE that \( \Delta \) is always finite because \( K, \Sigma \) are finite sets.

LECTURE DEFINITION: \( M = (K, \Sigma, \Delta, s, F) \) is non-deterministic when \( \Delta \) is finite and
\[
\Delta \subseteq K \times \Sigma^* \times K
\]

OBSERVE that we have to say in this case that \( \Delta \) is finite because \( \Sigma^* \) is infinite.

SOLVING PROBLEMS you can use any of these definitions.

2 Problems

PROBLEM 1

Let \( L \) be a language defines as follows
\[
L = \{ w \in \{ a, b \}^* : \text{every } a \text{ is either immediately proceeded or followed by } b \}.
\]

1. Describe a regular expression \( r \) such that \( L(r) = L \) (Meaning of \( r \) is \( L \)).

Solution \( L = (b \cup ab \cup ba \cup bab)^* \)

2. Construct a finite state automata \( M \), such that \( L(M) = L \).

Solution

Components of \( M \) are:
\[
K = \{ s \}, \{ a, b \}, \ s, \ F = \{ s \},
\]
\[
\Delta = \{ (s, b, s), (s, ab, s), (s, b, s), (s, bab, s) \}.
\]

Some elements of \( L(M) \) are: \( b, bb, baab, abab, abbbba, bbbabbbabbbabbb \)

PROBLEM 2

Let
\[
M = (K, \Sigma, s, \Delta, F)
\]

for \( K = \{ q_0, q_1, q_2, q_3 \}, \ s = q_0 \)
\[
\Sigma = \{ a, b \}, \ F = \{ q_1, q_2, q_3 \} \text{ and }
\]
\[
\Delta = \{ (q_0, a, q_1), (q_0, b, q_3), (q_1, a, q_2), (q_1, b, q_1), (q_3, a, q_3), (q_3, b, q_2) \}.
\]

1. List some elements of \( L(M) \).

Solution \( a, b, aa, bb, aba, abba \)

2. Write a regular expression for the language accepted by \( M \). Simplify the solution.

Solution
\[
L(M) = ab^* \cup ab^*a \cup ba^* \cup ba^*b = ab^*(e \cup a) \cup ba^*(e \cup b).
\]
3. Define a deterministic $M'$ such that $M \approx M'$, i.e. $L(M) = L(M')$.

Solution We complete $M$ do a deterministic $M'$ by adding a TRAP state $q_4$ and put

$$
\Delta' = \delta = \Delta \cup \{(q_2, a, q_4), (q_2, b, q_4), (q_4, a, q_4), (q_4, b, q_4)\}
$$

Justify why $M \approx M'$.

Solution $q_4$ is a trap state, it does not influence $L(M)$.

PROBLEM 3

For $M$ defined as follows

$$
M = (K, \Sigma, s, \Delta, F)
$$

for $K = \{q_0, q_1, q_2, q_3\}$, $s = q_0$

$\Sigma = \{a, b\}$, $F = \{q_2\}$ and

$$
\Delta = \{(q_0, a, q_3), (q_0, e, q_3), (q_0, b, q_1), (q_0, e, q_1), (q_1, a, q_2), (q_2, b, q_3), (q_2, e, q_3)\}
$$

Write 2 steps of the general method of transformation the NDFA $M$ defined above into an equivalent DFA $M'$.

Step 1: Evaluate $\delta(E(q_0), a)$ and $\delta(E(q_0), b)$.

Step 2: Evaluate $\delta$ on all states that result from step 1.

Reminder: $E(q) = \{p \in K : (q, e) \xrightarrow{\delta^N} (p, e)\}$ and

$$
\delta(Q, \sigma) = \bigcup_{p \in K} \{E(p) : \exists q \in Q: (q, \sigma, p) \in \Delta\}
$$

Solution Step 1: First we need to evaluate $E(q)$, for all $q \in K$.

$E(q_0) = \{q_0, q_1, q_3\} = S$, $E(q_1) = \{q_1\}$, $E(q_2) = \{q_2, q_3\} \in F$, $E(q_3) = \{q_3\}$

$$
\delta(E(q_0), a) = \delta(\{q_0, q_1, q_3\}, a) = E(q_3) \cup E(q_2) \cup \emptyset = \{q_2, q_3\} \in F
$$

$$
\delta(E(q_0), b) = \delta(\{q_0, q_1, q_3\}, b) = E(q_1) \cup \emptyset \cup \emptyset = \{q_1\}
$$

Solution Step 2:

$$
\delta(\{q_2, q_3\}, a) = \emptyset \cup \emptyset = \emptyset
$$

$$
\delta(\{q_2, q_3\}, b) = E(q_3) \cup \emptyset = \{q_3\}
$$

$$
\delta(\{q_1\}, a) = E(q_2) = \{q_2, q_3\} \in F
$$

$$
\delta(\{q_1\}, b) = \emptyset
$$