1 YES/NO questions

1. For any function \( f \) from \( A \neq \emptyset \) onto \( A \), \( f \) has property \( f(a) \neq a \) for certain \( a \in A \).
   Justify: \( f(x) = x \) is always "onto". n

2. All infinite sets have the same cardinality.
   Justify: \( |N| \neq |R| \) and \( N \) (natural numbers) and \( R \) (real numbers) are infinite sets. n

3. \( \{\{a,b\}\} \in 2^{\{a,b,\{a,b\}\}} \)
   Justify: \( \{\{a,b\}\} \subseteq \{a,b,\{a,b\}\} \). y

4. For any binary relation \( R \subseteq A \times A \), \( R^{-1} \) exists.
   Justify: The set \( R^{-1} = \{(b,a) : (a,b) \in R\} \) always exists. y

5. Regular language is a regular expression.
   Justify: Regular language is a language defined by a regular expression. n

6. \( L^+ = \{w_1...w_n : w_i \in L, i = 1,2,..n, n \geq 1\} \).
   Justify: definition y

7. \( L^* = L^* - \{e\} \).
   Justify: only when \( e \not\in L \). When \( e \in L \) we get that \( e \in L^+ \) and \( e \not\in L^* - \{e\} \). n

8. For any languages \( L_1, L_2 \), \( (L_1 \cap L_2) \cup L_2 = L_2 \).
   Justify: \( L_1 \cap L_2 \subseteq L_2 \) and languages are sets. y

9. \( (0^* \cap b^*) \cup 0^* \) describes a language with only one element.
   Justify: \( (\{e\} \cap \{b\}^*) \cup \{e\} = \emptyset \cup \{e\} = \{e\} \) y

10. For any \( M, L(M) \neq \emptyset \) iff the set \( F \) of its final states is non-empty.
    Justify: Let \( M \) be such that \( \Sigma = \emptyset, F \neq \emptyset, s \not\in F \), we get \( L(M) = \emptyset \). n

11. A configuration of any finite automaton \( M = (K, \Sigma, \Delta, s, F) \) is any element of \( K \times \Sigma^* \times K \).
    Justify: it is element of \( K \times \Sigma^* \) (lecture definition) n

12. If \( M = (K, \Sigma, \Delta, s, F) \) is a non-deterministic as defined in the book, then \( M \) is also non-deterministic, as defined in the lecture.
    Justify: \( \Sigma \cup \{e\} \subseteq \Sigma^* \) y

13. Let \( M \) be a finite state automaton, \( L(M) = \{\omega \in \Sigma^* : (s,\omega) \xrightarrow{s,M} (q,e)\} \).
    Justify: only when \( q \in F \) n

14. \( L(M_1) = L(M_2) \) iff \( M_1, M_2 \) are finite automata.
    Justify: one can have 2 automata that accept different languages. n

15. DFA and NDFA recognize the same class of languages.
    Justify: theorem proved in class y
2 Two definitions of a non-deterministic automaton

BOOK DEFINITION: \( M = (K, \Sigma, \Delta, s, F) \) is non-deterministic when
\[
\Delta \subseteq K \times (\Sigma \cup \{e\}) \times K
\]

OBSERVE that \( \Delta \) is always finite because \( K, \Sigma \) are finite sets.

LECTURE DEFINITION: \( M = (K, \Sigma, \Delta, s, F) \) is non-deterministic when \( \Delta \) is finite and
\[
\Delta \subseteq K \times \Sigma^* \times K
\]

OBSERVE that we have to say in this case that \( \Delta \) is finite because \( \Sigma^* \) is infinite.

SOLVING PROBLEMS you can use any of these definitions.

3 Very short questions (25pts)

For all state diagrams below do the following.

1. Determine whether it defines a finite state automaton.
2. Determine whether it is a deterministic / non-deterministic automaton.
3. Write full definition of \( M \) by listing all its components.
4. Describe the language by writing a regular expression or a property that defines it.

Q1 Solution: \( M = (K, \Sigma, s, \Delta, F) \) for \( K = \{q_0\} = F, s = q_0, \Sigma = \emptyset, \Delta = \emptyset \). \( M \) is deterministic and
\[
L(M) = \{e\} \neq \emptyset
\]

Q2 Solution: \( M = (K, \Sigma, s, \Delta, F) \) for \( \Sigma = \{a, b\}, K = \{q_0, q_1\}, s = q_0, F = \{q_0\}, \Delta = \{(q_0, a, q_1), (q_1, b, q_0)\} \). \( M \) is non deterministic; \( \Delta \) is not a function on \( K \times \Sigma \).
\[
L(M) = (ab)^*\]

Q3 Solution: \( M = (K, \Sigma, s, \Delta, F) \) for \( \Sigma = \{a, b\}, K = \{q_0, q_1, q_2, q_3\}, F = \{q_1\}, \Delta = \{(q_0, a, q_1), (q_0, b, q_1), (q_1, a, q_1), (q_1, e, q_2), (q_2, ab, q_2)\} \). It is NOT an automaton. It has no initial state.

Q4 \( M = (K, \Sigma, s, \Delta, F) \) for \( \Sigma = \{a, b\}, K = \{q_0, q_1, q_2, q_3\}, s = q_0, F = \emptyset, \Delta = \{(q_0, a, q_1), (q_1, b, q_2), (q_2, a, q_1), (q_0, e, q_3), (q_2, a, q_3)\} \). \( M \) is non deterministic; \( \Delta \subseteq K \times \Sigma \cup \{e\} \times K \).
\[
L(M) = \emptyset
\]

Q5 \( M = (K, \Sigma, s, \Delta, F) \) for \( \Sigma = \{a, b\}, K = \{q_0, q_1, q_2, q_3\}, s = q_0, F = \{q_1\}, \Delta = \{(q_0, ab, q_1), (q_1, e, q_0), (q_1, a, q_2), (q_1, ba, q_2), (q_2, a, q_2), (q_0, e, q_3), (q_1, a, q_3)\} \). \( M \) is non deterministic; \( \Delta \subseteq K \times \Sigma^* \times K \), \( q_2, q_3 \) are trap states.
\[
L(M) = (ab)^+
\]
4 Problems

Problem 1 Let \( L \) be a language defines as follows

\[
L = \{w \in \{a, b\}^* : \text{between any two } a\text{'s in } w \text{ there is an even number of consecutive } b\text{'s.}\}.
\]

1. Describe a regular expression \( r \) such that \( L(r) = L \).

Solution Remark that 0 is an even number, hence \( a^* \in L \),

\[
r = b^* \cup b^*ab^* \cup b^*(a(bb)^*a)^*b^* = b^*ab^* \cup b^* (a(bb)^*a)^*b^*
\]

2. Construct a finite state automata \( M \), such that \( L(M) = L \).

Solution 1 Components of \( M \) are:

\[
\Sigma = \{a, b\}, K = \{q_0, q_1, q_2, q_3\}, s = q_0, F = \{q_0, q_2, q_3\},
\]

\[
\Delta = \{(q_0, b, q_0), (q_0, a, q_3), (q_0, a, q_1), (q_1, bb, q_1), (q_1, a, q_2), (q_2, e, q_0), (q_3, b, q_3)\}
\]

Some elements of \( L(M) \) as defined by the state diagram are:

\[
a, aaa, bbb, aaaaabbb, abba, abbabbbba, abbbbbbabba, ....
\]

Solution 2 Components of \( M \) are:

\[
\Sigma = \{a, b\}, K = \{q_0, q_1, q_2\}, s = q_0, F = \{q_0, q_1, q_2\},
\]

\[
\Delta = \{(q_0, b, q_0), (q_0, e, q_1), (q_1, bb, q_1), (q_1, a, q_1), (q_1, c, q_2), (q_2, b, q_2)\}
\]

Problem 2 Let

\[
M = (K, \Sigma, s, \Delta, F)
\]

for \( K = \{q_0\}, s = q_0, \Sigma = \{a, b\}, F = \{q_0\} \) and

\[
\Delta = \{(q_0, aba, q_0), (q_0, ab, q_0)\}
\]

1. List some elements of \( L(M) \).

Solution

\[
e, ab, abab, ababa, ababaaba, ...
\]

2. Write a regular expression for the language accepted by \( M \).

Solution

\[
L = (ab \cup aba)^*
\]
Problem 3
We know that for any deterministic finite automaton \(M = (K, \Sigma, s, \delta, F)\) the following is true:
\[ e \in L(M) \iff s \in F. \]
Show that the above is not true for all non-deterministic automata.

Solution
Let \(M = (K, \Sigma, s, \Delta, F)\) for \(K = \{q_0, q_1\}, s = q_0, \Sigma = \emptyset, F = \{q_1\}\), and \(\Delta = \{(q_0, e, q_1)\}\).
\[ L(M) = \{e\} \text{ and } s \notin F. \]

Problem 4
For \(M\) defined as follows
\[ M = (K, \Sigma, s, \Delta, F) \]
for \(K = \{q_0, q_1, q_2, q_3\}\), \(s = q_0\)
\(\Sigma = \{a, b\}\), \(F = \{q_2, q_3\}\) and
\(\Delta = \{(q_0, a, q_1), (q_0, e, q_1), (q_0, b, q_2), (q_1, b, q_3), (q_1, e, q_3), (q_2, b, q_2), (q_2, e, q_3), (q_3, a, q_3)\}\)
Write a regular expression describing \(L(M)\).

Solution
\[ aa^* \cup a^* \cup aba^* \cup ba^* \cup bb^* \cup bb^* a^* \]

Write 4 steps of the general method of transformation the NDFA \(M\), into an equivalent deterministic \(M'\).

Reminder: \(E(q) = \{p \in K : (q, e) \xrightarrow{M} (p, e)\}\) and
\(\delta(Q, \sigma) = \bigcup\{E(p) : \exists q \in Q, (q, \sigma, p) \in \Delta\}\).

Solution Step 1:
\[ E(q_0) = \{q_0, q_1, q_3\}, E(q_1) = \{q_1, q_3\}, E(q_2) = \{q_2, q_3\}, E(q_3) = \{q_3\}. \]

Solution Step 2:
\[ \delta(E(q_0), a) = \delta(\{q_0, q_1, q_3\}, a) = E(q_1) \cup E(q_3) = \{q_1, q_3\} \in F, \]
\[ \delta(E(q_0), b) = \delta(\{q_0, q_1, q_3\}, b) = E(q_2) \cup E(q_3) \cup \emptyset = \{q_2, q_3\} \in F, \]

Solution Step 3:
\[ \delta(\{q_1, q_3\}, a) = \emptyset \cup E(q_3) = \{q_3\} \in F, \]
\[ \delta(\{q_1, q_3\}, b) = E(q_3) \cup \emptyset = \{q_3\} \in F, \]
\[ \delta(\{q_2, q_3\}, a) = \emptyset \cup E(q_3) = \{q_3\} \in F, \]
\[ \delta(\{q_2, q_3\}, b) = \{q_2\} \cup \emptyset = \{q_2\} \]

Solution Step 4:
\[ \delta(\{q_3\}, a) = E(q_3) = \{q_3\} \in F, \]
\[ \delta(\{q_3\}, b) = \emptyset, \]
\[ \delta(\emptyset, a) = \emptyset, \delta(\emptyset, b) = \emptyset. \]

End of the construction.