CSE 303 FINAL SOLUTIONS

PART 1: Yes/No Questions

Circle the correct answer. Write ONE-SENTENCE justification

No justification- **no credit**

1. Set A is uncountable iff $R \subseteq A$ (R is the set of *real* numbers).
   Justify:$R, 2^R$ are both uncountable and R is not a subset of 2^R ($R \not\subseteq 2^R$) but $R \in 2^R$.
   THIS is not the only example! **n**
2. All infinite sets have the same cardinality.
   Justify: $|N| < |2^N|$ by Cantor Theorem and $N, 2^N$ are infinite
   THIS is not the only example! **n**
3. $R^* = R \cup \{(a,b) : there\ is\ a\ path\ from\ a\ to\ b\}$.
   Justify: book definition **y**
4. $(ab \cup a^*b)^*$ is a regular language.
   Justify: this is a regular expression
   n
5. Let $\Sigma = \phi$, there is $L \neq \phi$ over Σ.
   Justify: $\emptyset^* = \{e\}$ and $L = \{e\} \subseteq \Sigma^*$
   y
6. There are uncountably many languages over $\Sigma = \{a\}$
   Justify: $|\{a\}^*| = \aleph_0$ (infinitely countable)and by definition, $L \subseteq \Sigma^*$, and $|2^{\{\aleph_0\}^*}| = \mathbf{c}$ and any set of cardinality $\mathbf{c}$ is uncountable
   Hence there uncountably many languages over $\Sigma = \{a\}$ **y**
7. Let $\mathbf{R}$ be a set of regular expressions and $\mathcal{L}$ be a function that maps $\mathbf{R}$ into set of all subsets of Σ^*. Then the following it true.
   $L \subseteq \Sigma^*$ is a regular language iff $L = \mathcal{L}$, for some $r \in \mathbf{R}$
   Justify: definition of regular language
   y
8. $L^* = \{w \in \Sigma^* : \exists_{q \in F} (s, w) \vdash_M^* (q, e)\}$
   Justify: this is part of a definition of $L(M)$, not L^*
   n
9. $(a^*b \cup \phi^*)$ is a regular expression.
   Justify: definition
   y
10. $\{a\}^*\{b\} \cup \{ab\}$ is a regular language
    Justify: 1.regular languages are closed under union, Kleene Star and concatenation , or
    2. This language is defined by a regular expression $a^*b \cup ab$ **y**
11. Let L be a language defined by $(a^*b \cup ab)$, i.e (shorthand) $L = a^*b \cup ab$. Then $L \subseteq \{a,b\}^*$.
    Justify: a, b are the only
    y
12. $\Sigma = \{a\}$, there are $\mathbf{c}$ (continuum) languages over Σ.
    Justify: $|\{a\}^*| = \aleph_0$ (infinitely countable)and by definition, $L \subseteq \Sigma^*$, and $|2^{\{\aleph_0\}^*}| = \mathbf{c}$, or for SHORT: $|2^{\{a\}^*}| = \mathbf{c}$
    y
13. $L^* = L^+ - \{e\}$
    Justify: $e \in L^*$ and $e \notin L^+ - \{e\}$
    n

14. $L^* = \{w_1 \dots w_n, w_i \in L, i = 1, \dots, n\}$.
Justify: $i = 0, 1, \dots, n\}$

n

15. For any languages $L_1, L_2, L_3 \subseteq \Sigma^*$
$L_1 \cup (L_2 \cap L_3) = (L_1 \cup L_2) \cap (L_1 \cup L_3)$
Justify: languages are sets

y

16. For any languages $L_1, L_2 \subseteq \Sigma^*$,
if $L_1 \subseteq L_2$, then $(L_1 \cup L_2)^* = L_2^*$
Justify:
languages are sets, so $(L_1 \cup L_2) = L_2$ and also $(L_1 \cup L_2)^* = L_2^*$

y

17. $((\phi^* \cap a) \cup (\phi \cup b^*)) \cap \phi^*$ represents a language $L = \emptyset$
Justify: $((\{e\} \cap \{a\}) \cup \{b\}^*) \cap \{e\} = \{b\}^* \cap \{e\} = \{e\}$

n

18. $L(M) = \{w \in \Sigma^* : (q, w) \vdash_M^* (s, e)\}$.
Justify: only when $s \in F$ and q is initial state of M

n

19. For every deterministic automaton M, $L(M) \neq \phi$
Justify: take M with $\Sigma = \phi$

n

20. If M is a nondeterministic FA, then $L(M) \neq \emptyset$
Justify: take M with $\Sigma = \phi$ or $F = \phi$

n

21. $L(M_1) = L(M_2)$ iff M_1 and M_2 are finite automata.
Justify: take as M_1 any automata such that $L(M_1) \neq \phi$ and M_2 such that $L(M_2) = \phi$
THIS is not the only example! **n**

22. A language is regular iff $L = L(M)$ and M is a deterministic automaton.
Justify: Theorem: A language is regular iff $L = L(M)$ and M is a finite automaton and
Theorem: Deterministic and nondeterministic automata are equivalent

y

23. If M is a deterministic automaton, then there is a nondeterministic N, such that $L(N) = L(M)$
Justify: a function is a special relation, or
Theorem: Deterministic and nondeterministic automata are equivalent **y**

24. Any finite language is CF
Justify: any finite language is regular and $RL \subset CFL$ or
we construct a CF grammar with set of rules $\{S \to \sigma : \quad \sigma \in \Sigma\}$

y

25. Intersection of any two regular languages is a CF language.
Justify: Regular languages are closed under intersection and $RL \subset CFL$

y

26. Union of a regular and a CF language is a CF language.
Justify: $RL \subseteq CFL$ and FCL are closed under union

y

27. L_1 is regular, L_2 is CF and $L_1, L_2 \subseteq \Sigma^*$, then $L_1 \cap L_2 \subseteq \Sigma^*$ is a CF language
Justify: theorem

y

28. If L is regular, then there is a PDA M such that $L = L(M)$.
Justify: Theorem: A language is regular iff $L = L(M)$ and M is a finite automaton and FA is a PDA operating on an empty stack
y

29. If L is regular, then there is a CF grammar G, such that $L = L(G)$.
Justify: we proved that $RL \subseteq CFL$
y

30. $L = \{a^n b^n c^n : n \geq 0\}$ is CF
Justify: is not CF, as proved by Pumping Lemma for CF languages
n

31. $L = \{a^n b^n : n \geq 0\}$ is CF.
Justify: $L = L(G)$ for G with $R = \{S \rightarrow aSb|e\}$
y

32. Let $\Sigma = \{a\}$, then for any $w \in \Sigma^*, w^R w \in \Sigma^*$
Justify: $a^R = a$, $(wa)^R = aw^R$ and by induction over length of w, $w^R = w$ for $w \in \{a\}^*$
y

33. $A \rightarrow Ax, A \in V, x \in \Sigma^*$ is a rule of a regular grammar.
Justify: this is a rule of a left-linear grammar and we defined regular grammar as a right-linear
n

34. Regular grammar has only rules $A \rightarrow xA, A \rightarrow x, x \in \Sigma^*, A \in V - \Sigma$.
Justify: not only, $A \rightarrow xB$ for $B \neq A$ is also a rule of a regular grammar
n

35. Let $G = (\{S, (,)\}, \{(,)\}, R, S)$ for $R = \{S \rightarrow SS \mid (S)\}$
$L(G)$ is regular
Justify: $L(G) = \emptyset$ and hence regular
y

36. $L = \{w \in \{a, b\}^* : w = w^R\}$ is regular.
Justify: we use Pumping Lemma; while pumping the string $a^k b a^k$ with y containing only $a's$ we get that $xy^2z \notin L$
n

37. We can always show that L is regular using Pumping Lemma
Justify: we use Pumping Lemma to prove (if possible) that L is not regular
Does not ALWAYS work!
n

38. $((p, e, \beta), (q, \gamma)) \in \Delta$ means: read nothing, move from p to q
Justify: and replace γ by β on the top of the stack
n

39. If L is not regular, then there is a CF grammar G, such that $L = L(G)$
Justify: $L = \{a^n b^n c^n : n \geq 0\}$ in not regular and not CF
n

40. There is countably many non CF languages over $\Sigma \neq \phi$

Justify: there is countably many CF languages and there is uncountably many (exactly **c**) languages over Σ so $|\Sigma^* - CF| = \mathbf{c}$, i.e. is uncountable **n**

41. Every subset of a regular language is a language.
Justify: languages are sets and a subset of a set is a set **y**

42. Any regular language is accepted by some PD automata.
Justify: $RL \equiv FA, FA \subseteq PDA$

y

43. $\Delta \subseteq (K \times \Sigma^* \times \Gamma^*) \times (K \times \Gamma^*$ is a transition relation of a pushdown automaton
Justify: Δ must be finite **n**

44. Let M be a pushdown automaton
$L(M) = \{w \in \Sigma^* : (s, w, e) \models^*{}_M (f, e, e)\}$
Justify: $f \in F$ and s the initial state **n**

45. There is uncountably many regular languages
Justify: there is countably many regular expressions

n

46. Every subset of a regular language is a regular language.
Justify:$L = \{a^n b^n : n \geq 0\} \subseteq \{a^* b^*\}$ and L is not regular

n

47. A CF language is a regular language.
Justify: $L = \{a^n b^n : n \geq 0\}$ is CF and not regular

n

48. A regular language is a CF language.
Justify: Regular grammar is a special case of a context-free grammar

y

49. Every subset of a regular language is a regular language.
Justify: $L_1 = a^n b^n$ is a non-regular subset of a regular language $L_2 = a^* b^*$

n

PART 2: PROBLEMS

QUESTION 1 Let Σ be any alphabet, L_1, L_2 two languages over Σ such that $e \in L_1$ and $e \in L_2$. Show that

$$(L_1 \Sigma^{\star} L_2)^{\star} = \Sigma^{\star}$$

Solution : By definition, $L_1 \subseteq \Sigma^{\star}, L_2 \subseteq \Sigma^{\star}$ and $\Sigma^{\star} \subseteq \Sigma^{\star}$. Hence

$$(L_1 \Sigma^{\star} L_2)^{\star} \subseteq \Sigma^{\star}.$$

We have to show that also

$$\Sigma^{\star} \subseteq (L_1 \Sigma^{\star} L_2)^{\star}.$$

Let $w \in \Sigma^{\star}$ we have that also $w \in (L_1 \Sigma^{\star} L_2)^{\star}$ because $w = ewe$ and $e \in L_1$ and $e \in L_2$.

QUESTION 2

Let L be a language defines as follows

$$L = \{w \in \{a, b\}^* : every\ a\ is\ either\ immediately\ proceeded\ or\ followed\ by\ b\}.$$

Part 1 Describe a regular expression r such that $\mathcal{L}(r) = L$ (Meaning of r is L).

Solution $L = (b \cup ab \cup ba \cup aba)^*$

Part 2 Construct a *finite state automata* M, such that $L(M) = L$.

Draw a diagram and list the components of M

List some elements of L(M)

Solution

Components of M are:

$$K = \{s\}, \{a,b\}, \quad s, \quad F = \{s\},$$

$$\Delta = \{(s,b,s),(s,ab,s),(s,ba,s),(s,aba,s).\}$$

Some elements of $L(M)$ are: $b, bb, baab, abab, abbbba, bbbabbbabbbabb$

QUESTION 3 Given a **Regular grammar** $G = (V, \Sigma, R, S)$, where

$$V = \{a, b, S, A\}, \quad \Sigma = \{a, b\},$$

$$R = \{S \to aS \mid A \mid e, \quad A \to abA \mid a \mid b\}.$$

Part 1 Use the construction in the proof of **L-GTheorem:**

Language L is regular if and only if there exists a regular grammar G such that L = L(G)

to construct a finite automaton M, such that $L(G) = L(M)$.

Solution We construct a non-deterministic finite automata

$$M = (K, \Sigma, \Delta, s, F)$$

as follows:

$$K = (V - \Sigma) \cup \{f\}, \ \Sigma = \Sigma, s = S, \quad F = \{f\},$$

$$\Delta = \{(S,a,S),(S,e,A),(S,e,f),(A,ab,A),(A,a,f),(A,b,f)\}$$

Part 2 Trace a transitions of M that lead to the acceptance of the string $aaaababa$, and compare with a derivation of the same string in G.

Solution

The accepting computation is:

$$(S, aaaababa) \vdash_M (S, aaababa) \vdash_M (S, aababa) \vdash_M (S, ababa) \vdash_M (A, ababa)$$

$$\vdash_M (A, aba) \vdash_M (A, a) \vdash_M (f, e)$$

G derivation is:

$$S \Rightarrow aS \Rightarrow aaS \Rightarrow aaaS \Rightarrow aaaA \Rightarrow aaaabA \Rightarrow aaaababA \Rightarrow aaaababa$$

QUESTION 4

Define a grammar G such that $L(G) = \{a^k b^j : k < j\}$

Justify your construction

Solution 1

The grammar has the rules $S \to AB, B \to b \mid bB, A \to e \mid aAb$

Justification

the rule $A \to e \mid aAb$ produces the same amount of a's and b's, the rule $B \to bB$ adds only b's

More formally, let's look at the derivations

$$S \Rightarrow AB \Rightarrow .... \Rightarrow a^n b^n B \Rightarrow ... \Rightarrow a^n b^n b^k B \Rightarrow a^n b^n b^k$$

$$S \Rightarrow AB \Rightarrow .... \Rightarrow a^n b^n B \Rightarrow a^n b^n b$$

we get $a^n b^{n+k} \in L(G)$ and $n < n+k$, and $a^n b^{n+1} \in L(G)$ and $n < n+1$

Solution 2

The grammar has the rules $\quad S \to aSB \mid Sb \mid b$

QUESTION 5

Prove that the Class of context-free languages is NOT closed under intersection

Solution :

$L_1 = \{a^n b^n c^m, n, m \geq 0\}$ is CF, $L_1 = \{a^m b^n c^n, n, m \geq 0\}$ is CF, but $L_1 \cap L_2 = \{a^n b^n c^n, n \geq 0\}$ is not CF

QUESTION 6 Construct a **pushdown** automaton M such that

$$L(M) = \{w \in \{a, b\}^* : \ w = w^R\}$$

Draw a DIAGRAM

Write the components

Solution 1 We define M as follows: $M = (K, \Sigma, \Gamma, \Delta, s, F)$

M **components** are

$$K = \{s, f\}, \Sigma = \{a, b\}, \Gamma = \{a, b\}, F = \{f\}$$

$$\Delta = \{((s, a, e), (s, a)),\ ((s, b, e), (s, b)),\ ((s, e, e), (f, e)),\ ((s, a, e), (f, e)),$$
$$((s, b, e), (f, e)),\ ((f, a, a), (f, e)),\ ((f, b, b), (f, e))\}$$

Trace a transitions of M that lead to the acceptance of the string $ababa$.

Solution

$$S \quad ababa \quad e$$

$$S \quad baba \quad a$$

$$S \quad aba \quad ba$$

$$f \quad ba \quad ba$$

$$\begin{array}{lll} f & a & a \\ f & e & e \end{array}$$

QUESTION 7

Construct a PDA M, such that

$$L(M) = \{b^n a^{2n} : n \geq 0\}.$$

Draw a DIAGRAM

Write the components

Solution $M = (K, \Sigma, \Gamma, \Delta, s, F)$ for

$$K = \{s, f\}, \Sigma = \{a, b\}, \Gamma = \{a\}, s, F = \{f\},$$

$$\Delta = \{((s, b, e), (s, aa)), ((s, e, e), (f, e)), ((f, a, a), (f, e))\}$$

Explain the construction. Write motivation.

Solution M operates as follows: Δ pushes aa on the top of the stock while M is reading b, switches to f (final state) non-deterministically; and pops a while reading a (all in final state). M puts on the stock two a's for each b, and then remove all a's from the stock comparing them with a's in the word while in the final state.

Trace a transitions of M that leads to the acceptance of the string $bbaaaa$.

Solution The accepting computation is:

$$(s, bbaaaa, e) \vdash_M (s, baaaa, aa) \vdash_M (s, aaaa, aaaa) \vdash_M (f, aaaa, aaaa)$$
$$\vdash_M (f, aaa, aaa) \vdash_M (f, aa, aa) \vdash_M (f, a, a) \vdash_M (f, e, e)$$

Solution 2 $M = (K, \Sigma, \Gamma, \Delta, s, F)$ for

$$K = \{s, f\}, \Sigma = \{a, b\}, \Gamma = \{b\}, s, F = \{f\},$$

$$\Delta = \{((s, b, e), (s, b)), ((s, e, e), (f, e)), ((f, aa, b), (f, e))\}$$

QUESTION 8 (Extra Credit)

Use closure under union for CF languages to show that $L = \{a^n b^n : \ n \neq m\}$ is CF language

Solution 1 We know that $L_1 = \{a^m b^n : m > n\}$ and $L_2 = \{a^m b^n : m < n\}$ are context-free languages (we constructed proper grammars for both of them). $L = L_1 \cup L_2$, hence L is context free as the class of context free languages is closed under union.

Solution 2 Observe that $L_1 = \{a^m b^n : m > n\} = \{a\}^+\{a^n b^n : n \in N\}$ We proved that $\{a^n b^n : n \in N\}$, is context free, $\{a\}^+$ is regular and hence CF and the class of context free languages is closed under concatenation, hence L_1 is also context free.

Similarly, $L_2 = \{a^m b^n : m < n\} = \{a^n b^n : n \in N\}\{b\}^+$, so L_2 is context free. $L = L_1 \cup L_2$, hence L is context free as the class of context free languages is closed under union.