1 YES/NO questions

Circle the correct answer (each question is worth 2pt.) Write SHORT justification.

1. For any function \( f \) from \( A \neq \emptyset \) onto \( A \), \( f \) has property
   \[
   \forall a \in A \exists b \in A (f(b) = a).
   \]
   \textbf{Justify:} definition of "onto" function.

2. Some infinite sets have the same cardinality.
   \textbf{Justify:} \(|N| = |2N|\) and \( N \) (natural numbers) and \( 2N \) (even numbers) are infinite sets.

3. \( \{\{a,b\}\} \in \{a,b,\{a,b\}\} \)
   \textbf{Justify:} \( \{\{a,b\}\} \subseteq \{a,b,\{a,b\}\} \) as \( \{a,b\} \in \{a,b,\{a,b\}\} \)

4. For any function \( R \subseteq A \times A \), \( R^{-1} \) exists.
   \textbf{Justify:} Theorem: The inverse function \( R^{-1} \) exists iff \( R \) is 1–1 and "onto".

5. A language \( L \) is regular iff \( L = \mathcal{L}(r) \) for some \( r \in \Sigma^* \).
   \textbf{Justify:} only when \( r \) is a regular expression.

6. \( L^+ = L^* - \{e\} \).
   \textbf{Justify:} only when \( e \notin L \). When \( e \in L \) we get that \( e \in L^+ \) and \( e \notin L^* - \{e\} \).

7. For any languages \( L_1, L_2, (L_1 \cup L_2) \cup L_1 = L_2 \).
   \textbf{Justify:} languages are sets, so it holds only when only when \( L_1 \subseteq L_2 \).

8. \( (\emptyset^* \cap b^*) \cup \emptyset^* \) describes a language with two elements.
   \textbf{Justify:} the set \( \{e\} \cap \{b\} \cup \{e\} = \{e\} \cup \{e\} = \{e\} \) has one element.

9. For any automata \( M, L(M) = \emptyset \) iff the set \( F \) of its final states is empty.
   \textbf{Justify:} Let \( M \) be such that \( \Sigma = \emptyset, F \neq \emptyset, s \notin F \), we get \( L(M) = \emptyset \).

10. If \( M = (K, \Sigma, \Delta, s, F) \) is a non-deterministic as defined in the book, then \( M \) is also non-deterministic, as defined in the lecture.
    \textbf{Justify:} \( \Sigma \cup \{e\} \subseteq \Sigma^* \)
11. Let $M$ be a finite state automaton, $L(M) = \bigcup_{q \in F} \{ w \in \Sigma^* : (s, w) \xrightarrow{s,M} (q, e) \}$.

**Justify:** $w \in \bigcup_{g \in F} \{ w \in \Sigma^* : (s, w) \xrightarrow{s,M} (q, e) \}$ iff there is $q \in F$ such that $(s, w) \xrightarrow{s,M} (q, e)$ iff $w \in L$.

12. For any finite automats $M_1, M_2$, $L(M_1) = L(M_2)$ iff $M_1 \equiv M_2$.

**Justify:** definition of automata equivalency.

13. DFA and NDFA recognize the same class of languages.

**Justify:** theorem proved in class.

2 Two definitions of a non-deterministic automaton

**BOOK DEFINITION:** $M = (K, \Sigma, \Delta, s, F)$ is non-deterministic when

$$\Delta \subseteq K \times (\Sigma \cup \{e\}) \times K$$

**OBSERVE** that $\Delta$ is always finite because $K, \Sigma$ are finite sets.

**LECTURE DEFINITION:** $M = (K, \Sigma, \Delta, s, F)$ is non-deterministic when $\Delta$ is finite and

$$\Delta \subseteq K \times \Sigma^* \times K$$

**OBSERVE** that we have to say in this case that $\Delta$ is finite because $\Sigma^*$ is infinite.

**SOLVING PROBLEMS** you can use any of these definitions.

3 PROBLEMS

**PROBLEM 1** Let $\Sigma = \{a, b\}$. Show that

$$(a \cup b)^* a (a \cup b)^* = \Sigma^* a \Sigma^*.$$

**Solution** Observe that

$$L(a \cup b)^* = (\{a\} \cup \{b\})^* = \{a, b\}^* = \Sigma^*.$$

Hence $(a \cup b)^* a (a \cup b)^* = \Sigma^* a \Sigma^*$.

**PROBLEM 2** Write a regular expression $r$, such that $L = L(r)$ for $L$ over $\Sigma = \{a, b\}$ defined as

$L = \{ w \in \Sigma^* : w \text{ has no more than three a's} \}$.

**Solution**

$$r = b^* \cup b^* ab^* \cup b^* ab^* \cup b^* ab^* ab^*$$

**PROBLEM 3** Let $L$ be a language defines as follows

$L = \{ w \in \{a, b\}^* : \text{between any two a's in w there is an even number of consecutive b's} \}$. 

2
1. Describe a regular expression $r$ such that $L(r) = L$ (Meaning of $r$ is $L$).

**Solution** Remark that 0 is an even number, hence $a^* \in L$,

$$r = b^* a^* b^* \cup b^* (a(bb)^* a)^* b^* = (b^* a(bb)^* ab^*)^*$$

2. Construct a finite state automata $M$, such that $L(M) = L$.

**Solution**

**Components** of $M$ are:

- $\Sigma = \{a, b\}$
- $K = \{q_0, q_1, q_2, q_3\}$
- $s = q_0$
- $F = \{q_0, q_2, q_3\}$

$$\Delta = \{(q_0, b, q_0), (q_0, a, q_3), (q_0, a, q_1), (q_1, bb, q_1), (q_1, a, q_2), (q_2, e, q_0), (q_2, b, q_2), (q_3, b, q_3), (q_3, e, q_0)\}$$

**Some elements** of $L(M)$ as defined by the state diagram are:

- $b, a, aaaa, aabbb, bbbabaaa, abbb, abba, bbbababb, abbbababa, abbaabbaabba, ....$

**PROBLEM 4** Let

$$M = (K, \Sigma, s, \Delta, F)$$

for $K = \{q_0\}$, $s = q_0$, $\Sigma = \{a, b\}$, $F = \{q_0\}$ and

$$\Delta = \{(q_0, aba, q_0), (q_0, ab, q_0)\}$$

1. List some elements of $L(M)$.

**Solution**

- $e, ab, abab, ababa, ababaaba, ...$

2. Write a regular expression for the language accepted by $M$.

**Solution**

$$L = (ab \cup aba)^*$$

3. Use the Book Definition to define an automaton $M'$ such that $M' \equiv M$ (use the "STRETCH" technique).

$$K' = K \cup \{p_1, p_2, p_3\}, \Delta' = \Delta_{\Sigma, e} \cup \{(q_0, a, p_1), (p_1, b, p_2), (p_2, a, q_0), (q_0, b, p_3), (p_3, b, q_0)\}$$

where $\Delta_{\Sigma, e}$ denotes those elements of $\Delta$ that involve only elements of $\Sigma \cup e$.

**PROBLEM 5** (20pts)

For $M$ defined as follows

$$M = (K, \Sigma, s, \Delta, F)$$

for $K = \{q_0, q_1, q_2\}$, $s = q_0$

$\Sigma = \{a, b\}$, $F = \{q_1, q_2\}$ and

$$\Delta = \{(q_0, ab, q_1), (q_0, e, q_1), (q_0, b, q_2), (q_1, a, q_1), (q_2, bb, q_2), (q_1, e, q_2)\}$$
Write a regular expression describing \( L(M) \).

\[
aba^* (bb)^8 \cup a^*(bb)^* \cup b(bb)^*
\]

Write 5 steps of the general method of transformation the NDFA \( M \), into an equivalent deterministic \( M' \).

Reminder 1: \( E(q) = \{ p \in K : (q, e) \xrightarrow{M} (p, e) \} \) and

\[
\delta(Q, \sigma) = \bigcup \{ E(p) : \exists q \in Q, (q, \sigma, p) \in \Delta \}.
\]

Reminder 2: The above definitions apply to the Book definition of non-deterministic automata.

The proper DIAGRAM of new (book definition - use "stretch" method) \( M \) is:

\( K' = K \cup \{ p_1, p_2 \} \), \( \Delta' = \Delta_{\Sigma \cup e} \cup \{(g_0, a, p_1), (p_1, b, q_1), (q_2, b, p_2), (p_2, b, q_2)\} \)

where \( \Delta_{\Sigma \cup e} \) denes those elements of \( \Delta \) that involve only elements of \( \Sigma \cup e \).

Solution: apply definition to \( M' \) defined above.