Language accepted by a FSA

- Let a FSA $A = (I, S, s_0, F, N)$.
  $N : S \times I \rightarrow S$ can be extended to $N : S \times I^* \rightarrow S$ to deal with strings.

  Example:
  Let a state $s_0$ and a string $abba$, $N(s_0, abba) = N(N(N(N(s_0, a), b), b), a)$.

- If $L$ is the set of strings that a FSA $A$ accepts, we say that $L$ is the language of $A$ and we write $L(A) = L$.

- Generally, we conjecture what is the language $L$ and we prove that $L(A) = L$ by proving that $L \subseteq L(A)$ and $L(A) \subseteq L$.

  To prove that $L \subseteq L(A)$, we need to prove that for all word $w \in L$, then $w \in L(A)$ i.e. $N(s_0, w) \in F$.
  To prove that $L(A) \subseteq L$, we need to prove that for all word $w \in L(A)$, then $w \in L$. So we need to prove that if $N(s_0, w) \in F$, then $w \in L$.

- Conjecture:
  Any string $w$ that ends with 00 is accepted by $A$.
  $w$ is a string of length greater than 2.
  $L = \{ w \in I^* | w = w', 00 \}$

- We prove using the structure of $w$ that if $w \in L$, i.e. $w = w', 00$ then $w$ is accepted by $A$. (We prove that $L \subseteq L(A)$).
  $w$ is a string of length $n$ that ends with 00. So $w = w', 00$. The size of $w'$ is greater than 2.
  After the first $n - 2$ symbols of $w$ have been input, $A$ is in one of its three states: $s_0$, $s_1$ and $s_2$. From any of these three states, input of the symbols 00 in will result in $A$ moving to the accept state $s_2$ ($N(s_0, 00) = s_2$ and $N(s_1, 00) = s_2$ and $N(s_2, 00) = s_2$). Hence, any string that ends in 00 is accepted by $M$.

- We prove by induction that $N(s_0, 0^n) = s_0$ ($n \geq 0$).
  $0^n$ means a sequence of 0 or more 0. $O^0$ represent the empty word denoted $\epsilon$.
  - Basis case: $n = 0$
    $N(s_0, \epsilon) = s_0$
  - Induction hypothesis: $N(s_0, 0^k) = s_0$ for an arbitrary and fixed $k \geq 0$.
  - We prove that: $N(s_0, 0^{k+1}) = s_0$.
    $N(s_0, 0^{k+1}) = N(N(s_0, 0^k), 0) = s_0$.
  - Conclusion: $N(s_0, 0^n) = s_0$ for all $n \geq 0$.

Example

- What is the language recognized by this automaton $A$?

  ![Automaton Diagram]

- 10 is not accepted
  10100 is accepted
  00 is accepted
  110010 is not accepted
  1101000 is accepted

- Ideas?
  If we are in state $s_0$ and if we read a sequence of 1, we stay in $s_0$. (We can prove this property by induction - see below).
  If we are in state $s_2$ and if we read a sequence of 0, we stay in $s_2$. (We can prove this property by induction).
  If we read 2 zeros we go in the final state $s_2$ but then if we read 1, we go out of the final state $s_2$.
Example

- Design an automaton that recognizes \( L = \{a^p b^q \mid p \geq 0 \text{ and } q > 0 \} \).

- What are the words of \( L \)?
  \( a \notin L \) \((q > 0)\)
  \( aba \notin L \)
  \( b \in L \)
  \( ab \in L \)
  \( aaaaaabbbb \in L \)

  Words that begin with 0, 1 or several \( a \)'s and terminate with \( b \)'s (at least 1).

- Automaton \( A \) - Transitions diagram:

Example

- What is the language recognized by this automaton?

- 11 is not accepted
  10101 is not accepted
  1 is not accepted
  110010 is accepted
  1101000 is accepted

- Ideas?
  If we are in state \( s0 \) and if we read a 0 we go to state \( s1 \).
  If we are in state \( s1 \) and if we read a 1 we go to state \( s0 \).

- Conjecture:
  Any string \( w \) that ends with 0 is accepted by \( A \). \( w \) is a string of length greater than 1. \( A \) recognizes even numbers.
  \( L = \{w \in \{0,1\}^* \mid |w| \text{ even} \} \)

- We can prove that \( L = L(A) \).

We prove that \( L = L(A) \).

**Sketch of the proof.**

- \( L \subseteq L(A) \)
  Let \( w \in L \). We prove that \( N(s0, w) = s1 \).
  To do that we prove:
    - \( \forall p \geq 0, N(s0, a^p) = s0 \) (by induction on \( p \)).
    - \( \forall q \geq 1, N(s0, a^q b^q) = s1 \) (by induction on \( q \)).

- \( L(A) \subseteq L \)
  Let \( w \) such that \( N(s0, w) = s1 \). We prove that \( w \in L \).
  To do that we prove:
    - \( \forall w, N(s0, w) = s0 \Rightarrow w = a^p \) (by induction on the length of \( w \)).
    - \( \forall w \) (of length greater than 1), \( N(s0, w) = s1 \Rightarrow \exists p \geq 0, \exists q > 0, w = a^p b^q \) (by induction on the length of \( w \)).
The man, the wolf, the goat and the cabbage

http://www.ecs.soton.ac.uk/ uun/CM219/HTML/ sld032.htm

Slides 32 to 41