1) \( R_1 \) is not a partial ordered relation, since it is not antisymmetric. Having \((1,3)\) and 
\((3,1)\) in the relation; does not imply that \(1=3\) because \(1 \neq 3\).
\( R_2 \) is a partial order because it is antisymmetric and transitive.

2) In order for a relation to be partial order, it has to be antisymmetric and transitive.
Then, for \( uRv \) where \( \text{length}(u) \leq \text{length}(v) \), we have to check the antisymmetric and transitive properties.
\( R \) is not antisymmetric and we prove this by a counter example:
Let have two strings \( s,t; s=a \) and \( t=b \). Then, \( sRt \) and \( tRs \) are in the relation since
\( \text{length}(s) \leq \text{length}(t) \) and \( \text{length}(t) \leq \text{length}(s) \) since both strings are of length 1.
However this does no imply that \( s=t \) (according to the definition of antisymmetric relation).
Then we conclude that \( R \) is not anti-symmetric, hence, not a partial order.

3) Relations on \( Z (+ -) \)
   a) For all \( m, n \) in \( Z \), \( mR_1 n \) iff every prime factor of \( m \) is a prime factor of \( n \).
   \( R_1 \) is not a partial order because it is not antisymmetric
   Proving by counter example:
   \( mR_n \) where every prime factor of \( m=2 \), \((1,2)\) is a prime factor of \( n=4 \)
   \( m=2 \) with prime factors 1 and 2
   \( n=4 \) with prime factors 1,2
   Similarly, \( nR_m \) is in the relation, because every prime factor of 4 is a prime
   factor of 2. However, \( n \neq m \) because \( 4 \neq 2 \).

   b) For all \( m, n \) in \( Z \), \( mR_2 n \) iff \( m+n \) is even.
   \( R_2 \) is not partial order because it is not antisymmetric.
   Proving by counter example:
   \( 3R_1 1 \) is in the relation because \( 3+1 \) is even; also \( 1R_3 3 \) is in the relation because
   \( 3+1=4 \) is even. However, \( 3 \neq 1 \).

4) Divides relation on: \( A=\{1,2,4,8,\ldots,2^n\} \)
   In order to prove that the divides relation on \( A \) is a total order, we have to prove that
   \( A \) is a partial order and that every pair of elements in the divides relation, \( a \) and \( b \) are
   comparable; that is to say that either \( aRb \) or \( bRa \) is an element of the relation.
   Proving partial order:
   **Transitivity:** If \( (a,b) \) is an element of the divides relation then we know that \( a \) divides
   \( b \); similarly if \( (b,c) \) is and element of the divides relation, then we know that \( b \)
divides \( c \). Transitivity holds because for elements \( (a,b), (b,c) ; (a,c) \) will be also part
of the relation and the divides property will still hold.
   **Antisymmetry:** If \( (a,b) \) is a member of the divides relation, then \( (b,a) \) would be a
member only if \( a=b \).
   **Show that \( aRb \) or \( bRa \) (a and b are comparable for all a and b):**
   Let \( a \) and \( b \), be particular but arbitrarily chosen elements of \( A \). By definition of \( A \),
there are nonnegative integer \( r \) and \( s \) such that \( a=2^r \) and \( b=2^s \) (Since \( r, s \) are the
exponents, nonnegative integers). Now either \( r < s \) or \( s < r \);
If \( r<s \), then \( b = 2^s = 2^r \cdot 2^{s-r} = a \cdot 2^{s-r} \) where \( s-r \geq 0 \). It follows, by definition of
divisibility, that \( a \) divides \( b \).
By a similar argument, if \( s < r \), then \( b \) divides \( a \). Hence \( aRb \) or \( bRa \), where \( R \) is the divides relationship.

5) Hasse diagram:

![Hasse diagram](image)

Greatest: None  
Least: 1 (it is minimal and 1 is less than equal all the elements in \( A \))  
Maximal: 20 and 15  
Minimal: 1 (this element that has not predecessors)  
Chains of length 3: 1-2-4-20; 1-5-10-20;  
Least upper bound: 60 (last common multiple of 20 and 15)  
Greatest Lower bound: 1

6) All partial order relations on \( A = \{ a, b, c \} \), where \( a \) is maximal  
\( R_1 = \{(a,a),(b,b),(c,c)\} \)  
\( R_2 = \{(a,a),(b,b),(c,c),(b,a)\} \)  
\( R_3 = \{(a,a),(b,b),(c,c),(c,a)\} \)  
\( R_4 = \{(a,a),(b,b),(c,c),(b,a),(c,a)\} \)  
\( R_5 = \{(a,a),(b,b),(c,c),(c,b),(c,a)\} \)  
\( R_6 = \{(a,a),(b,b),(c,c),(b,c),(b,a)\} \)  
\( R_7 = \{(a,a),(b,b),(c,c),(c,b),(b,a),(c,a)\} \)  
\( R_8 = \{(a,a),(b,b),(c,c),(b,c),(b,a),(c,a)\} \)  
\( R_9 = \{(a,a),(b,b),(c,c),(b,c)\} \)  
\( R_{10} = \{(a,a),(b,b),(c,c),(c,b)\} \)

7) Lexicographic order  
a) \( \preceq_l \) if is a partial order then it has to be transitive and antisymmetric.  
Recall the definition of lexicographic relation \( \preceq_l \) on \( \Sigma^* \)(see class notes)
x <L y is in the relation if x is a proper prefix of y or x = up, y = ur have a longest common prefix u and m is a predecessor of n.

Given y <L z also in the relation, we know that either y is a proper prefix of z or y = ur, z = us where p is a predecessor of r.

Then, x <L z will be in the relation because if x is a proper prefix of y, then x is a proper prefix of z. In addition, if x had a common long prefix with y and y had a common substring with z, then x and z have a common prefix and p is a predecessor of s. This proves transitivity.

x <L y is antisymmetric.

By contradiction, x <L y is not antisymmetric then y <L x is a member and x ≠ y.

However, this will never be the case because y <L x will never be in the set since x has to be a proper substring of y and y’s length will have to be at least one more than x’s length. Therefore we know that y <L x is not a member;

b) Prove sketch

Prove partial order for Σ* based on Σ (Basically, for the two cases of x <L y; x will be prefix of y as in <L define for Σ; and if x = wm, y = vw then w is the longest prefix in common and m is a predecessor of v.

Show that for this relation either x <L y or y <L x (i.e. they are comparable) for the cases that x is a proper prefix of y and when x and y have a long prefix in common.

8) Lexicographic order: aaa, aaab, aab, ab, abb, abba, abbb, ba, bba, bba, bbb
   Standard order: aa, ba, aaa, aab, abb, bba, bbb, aaab, abba, abbb

9) Let f(x, y) be the Ackerman function defined as:
   f(x, y) = if x = 0 then y + 1
           else if y = 0 then f(x - 1, 1)
           else f(x - 1, f(x, y - 1))

   Probing by well-founded induciton

   Let (x, y) be an element of the well founded set N x N.

   Now, let’s prove that f(x, y) is defined for all (x, y) ∈ N x N. We do it in two steps:

   a) Base Case:

   prove that f(x, y) is true for all minimal elements (x, y) ∈ N x N

   the minimal element is (0, 0)

   The Ackerman function is defined for (0, 0) returning 0 + 1 = 1 for the condition x = 0

   Therefore, it holds for the base case,

   b) Let us choose an arbitrary element, (m, n) of N x N and assume that the Ackerman function is given for all predecessor of (m, n):

   f(m - 1, n) ------(1)
   f(m, n - 1) ------(2)
   f(m - 1, n - 1) ------(3)

   Then we prove that it works for f(m, n)

   f(m, n) =
cases:
    if m = 0 then n+1 ; since we chose a value for m,n in NxN, we know that that n is define, therefore n+1 is also defined.
else if n=0 the f(m-1, 1) ; based on our assumption (3), we know that f(m-1, n) is defined for any arbitrary value of n; in this case n=1
else f(m-1, f(m,n-1)) ; based on our assumption (2), we know that f(m,n-1) will be defined and return a natural value s; then, f(m—1,s) is also define as in (1) for any n=s.

10) To prove that the relation < on NxN is well founded, we need to prove that there are not infinite descending chains or sequences.
For (a,b) < (c,d) if an only if max {a,b} < max {c,d}
By contradiction: (a,b)<(c,d) is not well-founded; then it must have a infinite descending chain. Since (a,b) < (c,d) is defined in terms of max{a,b} < max {c,b} then, it should be the case that max{a,b} < max {c,b} has an infinite descending chain also. However, since the relation is applied to NxN, we can have for instance a chain max{(0,0)}< max{0,1}<max(1,2)< …. ; but there is not predecessor of (0,0).
Therefore max{a,b} < max {c,d} is well founded. Since we obtained the latter based on our assumption, we conclude by contradiction the (a,b) < (c,d) does not have an infinite descending chain and therefore is well-founded.

11)
   a) Let S_n denote the sum 1.3 + 2.4+3.5+...+n(n+2) . To prove that S_n = n(n+1)(2n+7)/6
   Base case: n=1, S_1 = 1.3 = 3
       (1)(2)(2+7)/6 = 3
   Induction Step: Assume that S_n is true for all n >= 1 .
   Writing the sum for S_{n+1} we have:
   S_{n+1} = 1.3+2.4+3.5+...+n(n+2)+(n+1)(n+3)
   S_{n+1} = S_n + (n+1)(n+3).
   Replacing S_n
   S_{n+1} = n(n+1)(2n+7)/6 + (n+1)(n+3).
   S_{n+1} = (n+1)( 2n^2 + 7n + 6n + 18)/6
   S_{n+1} = (n+1)( 2n^2 +13n +18)/6
   S_{n+1} = (n+1)(n+2)(2(n+1) + 7)/6
   Which is the same formula as the inductive step S_{n+1}, proving this way S_n

   b) Base case: n=5
   \( n^2=25 \); 2^5 = 32; since 25 < 32 then base case holds
   Induction Step: Assume \( n^2 < 2^n \) for n > 4
   Then for n+1
   \( (n+1)^2 < 2^{(n+1)} \)
   Consider \( (n+1)^2 = n^2+ 2n + 1 \)
Using our assumption, \((n+1)^2 < 2^n + 2n + 1 < 2^n + 2^n < 2^{n+1}\). We can write this because for \(n > 4\), we have \(2n+1 < 2^n\). Therefore we have proved the claim that \(n^2 < 2^n\) for \(n > 4\).

c) Let \(S_n = 1 + 2 + 4 + 8 + \ldots + 2^{n-1}\). To prove that \(S_n = 2^n - 1\)

**Base Case:** \(n=1\). We just have one term in the sum and that is 1. From the formula we get \(2^1 - 1\), which is 1. Therefore, base case holds.

**Induction Step:** Assume that \(S_n = 2^n - 1\). Now, \(S_{n+1} = 1 + 2 + 4 + 8 + \ldots + 2^{n-1} + 2^n\). Using \(S_n\), we have \(S_{n+1} = S_n + 2^n = 2^n - 1 + 2^n = 2^{n+1} - 1\), thus we prove that \(S_n = 2^n - 1\).