Recursion

Consider the following function definition:

$$g(n) = \mathit{if}\ n = 0\ \mathit{then} 1\ \mathit{else}\ g(n+1)$$

$g : N \to N$.

The above "equation" says that g is a function such that

- g evaluated at 0 is 1, and
- g evaluated at n is same as g evaluated at $n+1$ for $n \neq 0$.

Consider

$$f_1(n) = 1$$

and

$$f_2(n) = \mathit{if}\ n = 0\ \mathit{then} 1\ \mathit{else}\ 2$$

Both f_1 and f_2 are consistent with the definition of g!

Lifting

- Given a domain D, the lifted domain $D_\perp = \{\perp\} \cup D$, where "$\perp$" is a special symbol not in D.

  "$\perp$" will be interpreted as "undefined".

- Given a function $f : D_1 \rightarrow D_2$, the natural extension of f, denoted by $f_\perp$ is the function from $D_{1\perp}$ to $D_{2\perp}$ such that

$$
\begin{array}{rcl}
f_\perp(\perp) & = & \perp \\
f_\perp(d_1) & = & f(d_1)\ \forall d_1 \in D_1
\end{array}
$$

Interpretation of recursive functions

Given a function h defined as

$$h(n) = \textit{if } n = 0 \textit{ then} 1 \textit{ else } h(n-1)$$

consider the following sequence of functions such that

$$\begin{aligned} h_0(n) &= \perp \\ h_{i+1}(n) &= \textit{if } n = 0 \textit{ then} 1 \textit{ else } h_i(n-1) \end{aligned}$$

Observe that each element h_i of the above sequence is a nonrecursive function.

Note that $h_{i+1}(n)$ is $\perp$ only if $h_i(n)$ is $\perp$; that is, h_{i+1} is "better defined" than h_i.

To get the value of $h(n_0)$ for some given n_0, we can compute $h_i(n_0)$ for $i = 0, 1, 2, \ldots$ until we get a non-$\perp$ value.....

Functional

Given a function h defined as

$$h(n) = \mathit{if}\ n = 0\ \mathit{then} 1\ \mathit{else}\ h(n-1)$$

consider the function:

$$H\ p\ n = \mathit{if}\ n = 0\ \mathit{then} 1\ \mathit{else}\ p\ (n-1)$$

Note that H is nonrecursive

Clearly,

$$h\ n = H\ h\ n$$

In other words

$$h = Hh$$

Fixed Points

Consider the function $t(x) = x^2 - 6x$ over reals.

Let x_0 be such that $t(x_0) = x_0$.

Such x_0 is called as a *fixed point* of t.

For arbitrary real functions there may be zero, one, or more fixed points.

t has two fixed points in R: 0, and 7.

$t'(x) = x + 2$ has no fixed points in R.

$t''(x) = x$ has one fixed point: 0.

Fixed points of functionals

Consider the function

$$H\ p\ n = \mathit{if}\ n = 0\ \mathit{then} 1\ \mathit{else}\ p\ (n-1)$$

We said that a function defined as

$$h(n) = \mathit{if}\ n = 0\ \mathit{then} 1\ \mathit{else}\ h(n-1)$$

appears to be a fixed point of H.

But does H have fixed points? If so, which of them is h?

CPO

- A partial order $\preceq$ over a set A is a reflexive, antisymmetric, transitive relation over A.
- An upper bound of a subset S of A is an element $b \in A$ such that $\forall a \in S \quad a \preceq b$.
- The least upper bound of a subset S of A, denoted by $\sqcup S$ is an upper bound of A such that for all upper bounds b of S $\sqcup S \preceq b$.
- An ascending chain in a set A with partial order $\preceq$ is a sequence of elements $a_0, a_1, \ldots, a_n, \ldots$ such that $a_i \preceq a_{i+1}$ for all $i \geq 0$.
- A *complete partial order (cpo)* over A is a partial order $\preceq$ such that
  - There is an element $\perp \in A$ such that $\forall a \in A \perp \preceq a$.
  - For every ascending chain $S \subseteq A$, $\sqcup S \in A$.

CPOs and constructed domains

Let $\preceq_A$ and $\preceq_B$ be cpos on A and B respectively. Then,

- $\preceq_{A\times B}$, defined as

$$(a, b) \preceq_{A\times B} (a', b') \text{ if and only if } a \preceq_A a' \text{ and } b \preceq_B b'$$

  is a cpo on $A \times B$.

- $\preceq_{A\uplus B}$ defined as

$$\begin{array}{rcl} (0, a) & \preceq_{A\uplus B} & (0, a') \text{ if and only if } a \preceq_A a' \\ (1, b) & \preceq_{A\uplus B} & (1, b') \text{ if and only if } b \preceq_B b' \\ \perp_{A\uplus B} & \preceq_{A\uplus B} & c \;\forall c \in A \uplus B \end{array}$$

  is a cpo on $(A \uplus B)_{\perp_{A\uplus B}}$.

- $\preceq_{A \rightarrow B}$ defined as

$$f \preceq_{A \rightarrow B} g \text{ if and only if } \forall a \in A\ f(a) \preceq_B g(a)$$

is a cpo on $A \rightarrow B$.

Continuity

- $f : A \to B$ is *monotonic* if $\forall a, a' \in A\ a \preceq_A a' \Rightarrow f(a) \preceq_B f(a')$
- $f : A \to B$ is *continuous* if for all ascending chains S in A, $f(\sqcup(S)) = \sqcup f(S)$.

  Every continuous function is monotonic.
- Let $A \hookrightarrow B$ be the set of all continuous functions from A to B. Then $\preceq_{A \hookrightarrow B}$, which is $\preceq_{A \to B}$ restricted to elements in $A \hookrightarrow B$ is a cpo on $A \hookrightarrow B$.
  - If $S = a_0 \preceq a_1 \preceq \cdots a_n \ldots$ is an ascending chain in A such that $\forall i \geq 0\ a_i \preceq d$, then $\sqcup S \preceq d$.
  - If $S_a = a_0 \preceq a_1 \preceq \cdots a_n \ldots$ and $S_b = b_0 \preceq b_1 \preceq \cdots b_n \ldots$ are ascending chains in A such that $\forall i \geq 0\ a_i \preceq b_i$, then $\sqcup S_a \preceq \sqcup S_b$.

Continuous functions and fixed points — I

- Let $f : A \to A$ be a continuous function.
- Consider the sequence $\langle x_0, x_1, x_2, \ldots, x_n, \ldots \rangle$ such that

$$
\begin{aligned}
x_0 &= \perp \\
x_{i+1} &= f(x_i)
\end{aligned}
$$

- The sequence $\langle x_0, x_1, x_2, \ldots, x_n, \ldots \rangle$ forms an ascending chain.

  $\perp = x_0 \preceq x_1$.
  Let $x_i \preceq x_{i+1}$ for all $i \leq k$. Since $x_{k-1} \preceq x_k$, we have $f(x_{k-1}) \preceq f(x_k)$ by monotonicity of f. Hence $x_k \preceq x_{k+1}$. $\square$

Continuous functions and fixed points — II

- Let $f : A \to A$ be a continuous function.
- Consider the sequence $\langle x_0, x_1, x_2, \ldots, x_n, \ldots \rangle$ such that

$$\begin{aligned} x_0 &= \perp \\ x_{i+1} &= f(x_i) \end{aligned}$$

- $\sqcup x_i$ is a fixed point of f

  $f(\sqcup x_i) = \sqcup f(x_i)$ from continuity of f. Now consider the sequence $\langle f(x_0), f(x_1), f(x_2), \ldots, f(x_n), \ldots \rangle$. This is nothing but $\langle x_1, x_2, \ldots, x_{n+1}, \ldots \rangle$. Hence $\sqcup_{i \geq 0} f(x_i) = \sqcup_{i \geq 1} x_i$.

  Now $\sqcup S = \sqcup S_\perp$, hence $\sqcup_{i \geq 1} x_i = \sqcup_{i \geq 0} x_i$.

  Hence $f(\sqcup x_i) = \sqcup x_i$. $\square$

Continuous functions and fixed points — III

- Let $f : A \to A$ be a continuous function.
- Consider the sequence $\langle x_0, x_1, x_2, \ldots, x_n, \ldots \rangle$ such that

$$\begin{array}{rcl} x_0 & = & \perp \\ x_{i+1} & = & f(x_i) \end{array}$$

- $\sqcup x_i$ is the *least* fixed point of f

  Let y be a fixed point of f. If $x_i \preceq y$ for some i, then

  $x_{i+1} = f(x_i) \preceq f(y) = y$.

  $x_0 = \perp \preceq y$.

  Hence $\forall i \geq 0\ x_i \preceq y$, and so $\sqcup x_i \preceq y$. $\Box$

Fixed points of functionals

Consider the function

$$H\ p\ n = \mathit{if}\ n = 0\ \mathit{then} 1\ \mathit{else}\ p\ (n-1)$$

We said that a function defined as

$$h(n) = \mathit{if}\ n = 0\ \mathit{then} 1\ \mathit{else}\ h(n-1)$$

appears to be a fixed point of H.

We define $h = \mathit{fix}(H)$ where $\mathit{fix}(\cdot)$ is the least fixed point operator.

We know that h is well-defined if H is continuous.

Continuity of functionals — I

- Composition of two continuous functions is continuous.
- The identity function and constant functions are continuous.
- Let A and B be primitive domains. Let $f_\perp$ be a natural extension of $f : A \rightarrow B$ such that $f_\perp$ is a total function from $A_\perp$ to $B_\perp$. Then $f_\perp$ is continuous.

Continuity of functionals — II

- Let $A = A_1 \times A_2$ be a cartesian product of primitive domains, B be a primitive domain and let $f : A_1 \times A_2 \to B$ be a function. Extend f to $f_\perp$ as follows:

$$
\begin{array}{rcl}
f_\perp(\perp_{A_1}, \_) & = & \perp_B \\
f_\perp(\_, \perp_{A_2}) & = & \perp_B \\
f_\perp(a_1, a_2) & = & f(a_1, a_2)
\end{array}
$$

  Then $f_\perp$ is continuous.

  This naturally generalizes to higher-dimension product domains.

  $f_\perp$ is said to be *strict*.

Conditionals and continuity

- Let A_1, A_2 be primitive domains, and let B be the boolean domain. Let $\mathit{cond} : B \times A_1 \times A_1 \rightarrow A_2$ be a function defined as:

$$
\begin{aligned}
\mathit{cond}(\mathit{true}, a_1, a_2) &= a_1 \\
\mathit{cond}(\mathit{false}, a_1, a_2) &= a_2
\end{aligned}
$$

  Let $\mathit{cond}_\perp : B_{\perp_B} \times A_{1\perp_{A_1}} \times A_{1\perp_{A_1}} \rightarrow A_{2\perp_{A_2}}$ be an extension of cond defined as:

$$
\begin{aligned}
\mathit{cond}_\perp(\perp_B, a_1, a_2) &= \perp_{A_2} \\
\mathit{cond}_\perp(\mathit{true}, a_1, a_2) &= a_1 \\
\mathit{cond}_\perp(\mathit{false}, a_1, a_2) &= a_2
\end{aligned}
$$

  Then $\mathit{cond}_\perp$ is continuous.

Continuity of functionals

Any functional H defined by the composition of

- identity function
- constant functions
- naturally extended functions on primitive domains and their cartesian products
- specially extended function $cond_{\perp}$ as defined above
- function parameters

is continuous.

Thus the least fixed point of functionals is well-defined.

Denotational semantics: while statement

$$\mathcal{S}[\![\textbf{while } b \textbf{ do } s]\!]\sigma = \begin{cases} \sigma & if\ \mathcal{B}[\![b]\!]\sigma = \mathit{false} \\ \mathcal{S}[\![\textbf{while } b \textbf{ do } s]\!](\mathcal{S}[\![s]\!]\sigma) & \text{otherwise} \end{cases}$$

will now be defined as:

$$\mathcal{S}[\![\textbf{while } b \textbf{ do } s]\!]\sigma = \mathit{fix}(\psi)\sigma$$

where

$$\psi\ \varphi\ \sigma = \begin{cases} \sigma & if\ \mathcal{B}[\![b]\!]\sigma = \mathit{false} \\ (\varphi \circ (\mathcal{S}[\![s]\!]))\ \sigma & \text{otherwise} \end{cases}$$