Untyped Lambda Calculus

Principles of Programming Languages

CSE 526

1. Syntax
2. Variables and Substitution
3. Reductions
4. Recursion
5. Nameless Representation

Compiled at 08:25 on 2020/03/10

Lambda Calculus

- A formal notation to study computability and programming.
- Can be considered as the smallest universal programming language.
  - *Universal:* Can be used to express any computation that can be performed on a Turing Machine
  - *Small:* Has only two constructs: abstraction and application.
- Brief History:
  - Introduced by Church and Kleene in 1930s.
  - Used by Church to study problems in computability.
  - Concepts have heavily influenced *functional programming*.
  - Used to study *types* and type systems in programming languages

Lambda Terms

Syntax of the λ-calculus

$$t \quad ::= \qquad \text{Terms}$$

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t	$::=$		Terms
		x	Variable

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Syntax of the λ-calculus

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\begin{array}{lcll}
t & ::= & & \text{Terms} \\
 & & x & \text{Variable} \\
 & | & \lambda x.\ t & \text{Abstraction}
\end{array}
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Textual Representation:

Use parentheses to represent trees as linear text

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  - Example: $((\lambda x.\ x)\ y)$ supplies y as the argument to the identity function.

Syntactic Conventions and Syntactic Sugar

- Parentheses can be dropped using the following conventions:
  - application is left associative
    e.g. $((f\ f)\ x)$ is same as $f\ f\ x$
  - a λ binds as much as possible to its right.
    e.g $\lambda f.\ \lambda x.\ f\ (f\ x)$ is same as $(\lambda f.(\lambda x.\ f\ (f\ x)))$
- Multiple consecutive abstractions can be combined:
  e.g. $\lambda f.\lambda x.f\ (f\ x)$ is same as $\lambda f\ x.\ f\ (f\ x)$

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Encoding Booleans in the λ-Calculus

B	λ-calculus
`true`	$\lambda x.\ \lambda y.\ x$
`false`	$\lambda x.\ \lambda y.\ y$
`&&`	$\lambda x.\ \lambda y.\ ((x\ y)\ \texttt{false})$
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$$\equiv\quad \texttt{false}$$

Encoding Natural Numbers in the λ-Calculus

N	λ-calculus
0	$\lambda s.\ \lambda z.\ z$
1	$\lambda s.\ \lambda z.\ (s\ z)$
2	$\lambda s.\ \lambda z.\ (s\ (s\ z))$
3	$\lambda s.\ \lambda z.\ (s\ (s\ (s\ z)))$
$\vdots$	
inc	$\lambda n.\ \lambda s.\ \lambda z.\ (s\ ((n\ s)\ z))$
plus	$\lambda m.\ \lambda n.\ \lambda s.\ \lambda z.\ ((m\ s)\ ((n\ s)\ z))$
times	$\lambda m.\ \lambda n.\ ((m\ (plus\ n))\ 0)$
iszero	$\lambda m.\ ((m\ (\lambda x.\ \mathtt{false}))\ \mathtt{true})$
$\vdots$	

This is known as the *Church encoding of Naturals*, or simply *Church Numerals*.

Encoding Data Structures in the λ-Calculus

pair	$\lambda f.\ \lambda s.\ \lambda c.\ ((c\ f)\ s)$
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Example: Let φ_1 and φ_2 be two arbitrary expressions.

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$$fst\ (pair\ \varphi_1\ \varphi_2)$$

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$snd\ (pair\ \varphi_1\ \varphi_2)$

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Encoding Data Structures in the λ-Calculus

pair	$\lambda f.\ \lambda s.\ \lambda c.\ ((c\ f)\ s)$
fst	$\lambda p.\ (p\ \texttt{true})$
snd	$\lambda p.\ (p\ \texttt{false})$

Example: Let φ_1 and φ_2 be two arbitrary expressions.

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\begin{array}{ll}
\multicolumn{2}{l}{pair\ \varphi_1\ \varphi_2} \\
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Evaluating Lambda Expressions: An Informal Intro.

Basic reduction: $(\lambda x.\ t_1)\ t_2 \ \rightarrow\ [x \mapsto t_2]t_1$, where

$[x \mapsto t_2]t_1$ be the term obtained by replacing all "free" occurrences of x in t_1 by t_2.

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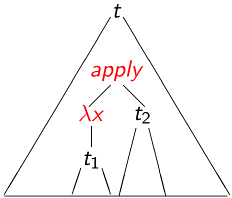

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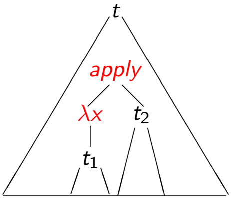

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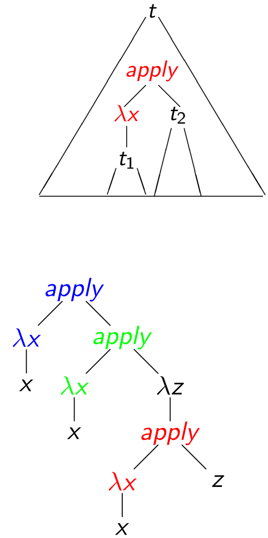
- In general, there may be many redexes in a term.
  Example: Let $id = (\lambda x.\ x)$ in term $id\ (id\ (\lambda z.\ id\ z))$



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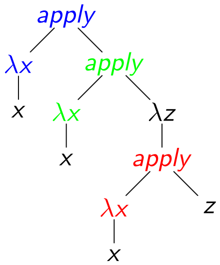

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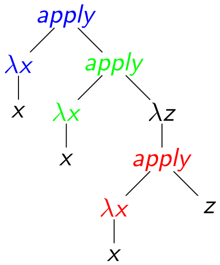

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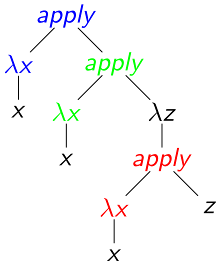

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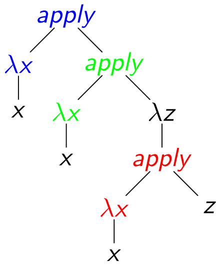

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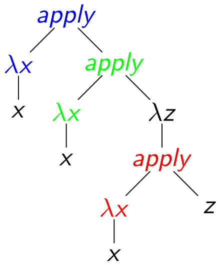

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Evaluating Lambda Expressions

- The key step in evaluating an application then is:
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- The central problem now is how we define this substitution function.

Substitutions (1^{st} attempt)

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[x \mapsto s]x & = & s & \\
[x \mapsto s]y & = & y & \text{if } y \neq x \\
[x \mapsto s](\lambda y.\ t) & = & \lambda y.\ [x \mapsto s]t & \\
[x \mapsto s](t_1\ t_2) & = & ([x \mapsto s]t_1)\ ([x \mapsto s]t_2) &
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- **Solution:** We should get $(\lambda w.\ y\ w)$ instead (by suitably renaming "local" variables).

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  - Rough meaning: free variables in a function definition are analogous to non-local variables.

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Binding Occurrence (declaration)

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  Bound Occurrence (use)

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Bound and Binding Occurrences

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  $fv(\ (\lambda x.\ x)\ (\lambda z.\ (x\ z))\ )$
  $= fv(\lambda x.\ x) \cup fv(\lambda z.\ (x\ z))$
  $= \{\ \} \cup \{x\} = \{x\}$

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- Example 3: $(\lambda y.\ x\ y)$ is *not* equivalent to $(\lambda y.\ x\ z)$ (not every occurrence of y has been replaced).
- Two terms t and t' are said to be "α-equivalent" (denoted by $t \equiv_\alpha t'$) if they are identical modulo the names of bound variables.

Substitutions (3^{rd} attempt)

$$
\begin{array}{lcll}
[x \mapsto s]x & = & s & \\
[x \mapsto s]y & = & y & \text{if } y \neq x \\
[x \mapsto s](\lambda y.\ t) & = & \lambda y.\ [x \mapsto s]t & \text{if } x \neq y \text{ and } y \notin fv(s) \\
[x \mapsto s](t_1\ t_2) & = & ([x \mapsto s]t_1)\ ([x \mapsto s]t_2) &
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- This drawback is not serious:
- We can apply a substitution on an α-equivalent term instead.
- E.g. $[x \mapsto y](\lambda z.\ x\ z)\ = (\lambda z.\ y\ z)$

Operational Semantics: Full β-Reduction

$$\frac{t_1 \to t_1'}{t_1\ t_2 \to t_1'\ t_2} \quad \text{E-APP1}$$

$$\frac{t_2 \to t_2'}{t_1\ t_2 \to t_1\ t_2'} \quad \text{E-APP2}$$

$$\frac{t \to t'}{\lambda x.\ t \to \lambda x.\ t'} \quad \text{E-ABS}$$

$$(\lambda x.\ t_1)\ t_2 \to [x \mapsto t_2]t_1 \qquad \text{E-APPABS}$$

Operational Semantics: Call-By-Value

$$t \quad ::= \quad \ldots \qquad \text{Terms (all } \lambda\text{-terms)}$$
$$v \quad ::= \quad \lambda x.\ t \qquad \text{Values}$$

Evaluation:

$$\frac{t_1 \to t_1'}{t_1\ t_2 \to t_1'\ t_2} \quad \text{E-App1}$$

$$\frac{t_2 \to t_2'}{v_1\ t_2 \to v_1\ t_2'} \quad \text{E-App2}$$

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- In an application of the form $(t_1\ t_2)$, if t_1 is a λ-abstraction, then t_2 has to be reduced to a value before the application is done.

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- In an application of the form $(t_1\ t_2)$, if t_1 is a λ-abstraction, then t_2 has to be reduced to a value before the application is done.
- This corresponds to Call-By-Value parameter passing: evaluate the actual arguments first before passing them as parameters to a called function.

Operational Semantics: Call-By-Name

$$
\begin{array}{lll}
t & ::= & \ldots \\
v & ::= & \lambda x.\ t
\end{array}
$$

Terms (all λ-terms)
Values

Evaluation:

$$\frac{t_1 \to t_1'}{t_1\ t_2 \to t_1'\ t_2} \quad \text{E-App}$$

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- In an application of the form $(t_1\ t_2)$, if t_1 is a λ-abstraction, then t_1 has to be reduced to a value before the application is done.
- In terms of familiar languages, the actual arguments are passed *unevaluated* to the called function. They will be evaluated in the called function if needed.

Infinite and Diverging Computations in the λ-Calculus

$$\textit{omega}: \quad (\lambda x.\ x\ x)\ (\lambda x.\ x\ x)$$

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Evaluation:

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\end{aligned}
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Evaluation:

$$\begin{array}{ll} & omega \\ \equiv & (\lambda x.\ x\ x)\ \ (\lambda x.\ x\ x) \\ \rightarrow & (\lambda x.\ x\ x)\ \ (\lambda x.\ x\ x) \\ \equiv & omega \end{array}$$

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Evaluation:

$$
\begin{array}{ll}
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Recursive Functions in the λ-Calculus —(1)

- Consider the function to compute *factorial* of a natural number, written as follows:

$$fact \equiv \lambda n.\ (if\ \ (iszero\ \ n)\ 1\ (times\ n\ (fact\ (dec\ \ n))\ )\ )$$

where *dec* is the function that decrements a number by 1.

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- The solution is to "lift" factorial into a *functional*:

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- Note that F is well-defined.
- F is a very special function, as we'll see in the next...

Recursive Functions in the λ-Calculus —(2)

$$F \equiv \lambda f.\ \lambda n.\ (\mathit{if}\ (\mathit{iszero}\ n)\ 1\ (\mathit{times}\ n\ (f\ (\mathit{dec}\ n))\ )\ )$$

- Consider $\mathit{fact}_0 \equiv F\ \mathit{omega}$:

$$\begin{aligned} \mathit{fact}_0 &\equiv F\ \mathit{omega} \\ &\equiv (\lambda f.\ \lambda n.\ (\mathit{if}\ (\mathit{iszero}\ n)\ 1\ (\mathit{times}\ n\ (f\ (\mathit{dec}\ n))\ )\ ))\ \mathit{omega} \\ &\rightarrow \lambda n.\ (\mathit{if}\ (\mathit{iszero}\ n)\ 1\ (\mathit{times}\ n\ (\mathit{omega}\ (\mathit{dec}\ n))\ )\ ) \end{aligned}$$

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- *When non-strict evaluation is used,* fact_0 computes the same as *fact* for 0, but diverges elsewhere.

Recursive Functions in the λ-Calculus —(3)

$$F \equiv \lambda f.\ \lambda n.\ (if\ (iszero\ n)\ 1\ (times\ n\ (f\ (dec\ n))\ )\ )$$

- Now consider $fact_1 \equiv F\ fact_0$:

$$\begin{array}{lll} fact_1 & \equiv & F\ fact_0 \\ & \equiv & (\lambda f.\ \lambda n.\ (if\ (iszero\ n)\ 1\ (times\ n\ (f\ (dec\ n))\ )\ ))\ fact_0 \\ & \rightarrow & \lambda n.\ (if\ (iszero\ n)\ 1\ (times\ n\ (fact_0\ (dec\ n))\ )\ ) \end{array}$$

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- *When non-strict evaluation is used,* $fact_1$ computes the same as $fact$ for 0 and 1, but diverges elsewhere.

Recursive Functions in the λ-Calculus —(4)

- Consider the sequence of functions $fact_0, fact_1, fact_2, \ldots$ such that $fact_0 = omega$, and $fact_{n+1} = (F\ fact_n)$.

Recursive Functions in the λ-Calculus —(4)

- Consider the sequence of functions $fact_0, fact_1, fact_2, \ldots$ such that $fact_0 = omega$, and $fact_{n+1} = (F\ fact_n)$.
- None of these functions is same as *fact*, but as we construct more and more members of this sequence, we get functions that approximate *fact* closer and closer.

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- *fact* is indeed the limit of this sequence of functions!

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- None of these functions is same as *fact*, but as we construct more and more members of this sequence, we get functions that approximate *fact* closer and closer.
- *fact* is indeed the limit of this sequence of functions!
- If only we had a way, in the λ-calculus, to generate such a sequence...

The Y-Combinator

$$Y = \lambda f.\ (\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x))$$

- Consider $(Y\ F)$:

$$\begin{aligned}(Y\ F) \quad &\equiv \quad (\lambda f.\ (\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ F \\ \to \quad &(\lambda x.\ F\ (x\ x))\ (\lambda x.\ F\ (x\ x)) \\ \to \quad &F\ (\ (\lambda x.\ F\ (x\ x))\ (\lambda x.\ F\ (x\ x))\ ) \\ \cong \quad &F\ (Y\ F)\end{aligned}$$

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- Recall $F \equiv \lambda f.\ \lambda n.\ (if\ (iszero\ n)\ 1\ (times\ n\ (f\ (dec\ n))\ )\ )$.

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- Recall $F \equiv \lambda f.\ \lambda n.\ (\mathit{if}\ (\mathit{iszero}\ n)\ 1\ (\mathit{times}\ n\ (f\ (\mathit{dec}\ n))\ )\ )$.
- Putting it all together:
  $$\begin{array}{ll} (Y\ F) & \cong\ F\ (Y\ F) \\ \equiv & (\lambda f.\ \lambda n.\ (\mathit{if}\ (\mathit{iszero}\ n)\ 1\ (\mathit{times}\ n\ (f\ (\mathit{dec}\ n))\ )\ ))\ (Y\ F) \\ \to & \lambda n.\ (\mathit{if}\ (\mathit{iszero}\ n)\ 1\ (\mathit{times}\ n\ ((Y\ F)\ (\mathit{dec}\ n))\ )\ ) \end{array}$$

The Y-Combinator

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- Putting it all together:
  $$\begin{array}{rcl} (Y\ F) & \cong & F\ (Y\ F) \\ & \equiv & (\lambda f.\ \lambda n.\ (\mathit{if}\ (\mathit{iszero}\ n)\ 1\ (\mathit{times}\ n\ (f\ (\mathit{dec}\ n))\ )\ ))\ (Y\ F) \\ & \rightarrow & \lambda n.\ (\mathit{if}\ (\mathit{iszero}\ n)\ 1\ (\mathit{times}\ n\ ((Y\ F)\ (\mathit{dec}\ n))\ )\ ) \end{array}$$
- $(Y\ F)$ looks like the mythical function *fact*.

The Z-Combinator

- $(Y\ F) \cong F\ (Y\ F)$
- With *call-by-name* evaluation strategy, the next steps in reduction will first substitute the formal parameter of F with $(Y\ F)$.
- With *call-by-value* strategy, $F\ (Y\ F)$ will first reduce $(Y\ F)$, which result in:

$$\begin{array}{ll} \to^* & F\ (F\ (Y\ F)) \\ \to^* & F\ (F\ (F\ (Y\ F))) \\ \to^* & \ldots \end{array}$$

- For *call-by-value* strategy, we should use the Z combinator instead:

$$Z = \lambda f.\ (\lambda x.\ f\ (\lambda y.\ x\ x\ y))\ (\lambda x.\ f\ (\lambda y.\ x\ x\ y))$$

Recursive Functions in the λ-Calculus —(5)

$$
\begin{array}{rcl}
Y & = & \lambda f.\ (\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)) \\
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- Note that the definitions of Y, F and *fact* are all non-recursive.

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- Note that the definitions of Y, F and $fact$ are all non-recursive.
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- Say, we have a mythical recursive definition $f = \lambda x.\ e$ where e uses f.
- We simply rewrite the definition as $f = (Y\ (\lambda f.\ \lambda x.\ e))$.

Nameless Representation of Terms

- Consider variables in a λ-term as named "holes" to be filled in.
- Instead of using symbolic names for variables, one can name the holes w.r.t. the λ that binds them.

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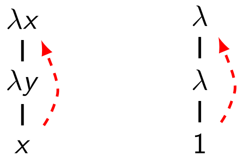

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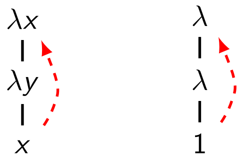

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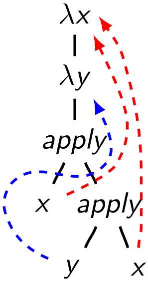

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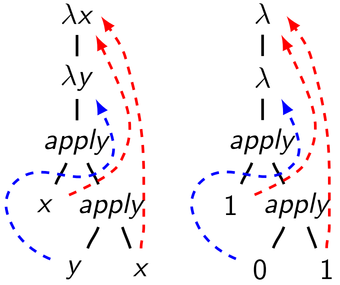

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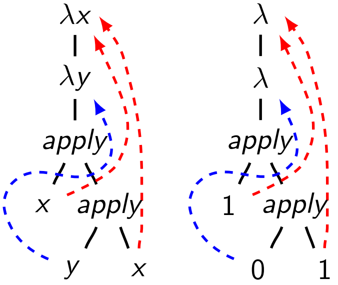

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n-Terms

De Bruijn terms are defined by a family of sets (each set being a set of terms) $\{\mathcal{T}_0, \mathcal{T}_1, \ldots\}$ such that $\mathcal{T}_n$ represents λ-terms with n or fewer free variables

Formally, $\mathcal{T}$ is the smallest family of sets $\{\mathcal{T}_0, \mathcal{T}_1, \ldots\}$ such that

- $k \in \mathcal{T}_n$ whenever $0 \leq k < n$
- if $t_1 \in \mathcal{T}_n$ then $\lambda.\ t_1 \ \in \mathcal{T}_{n-1}$
- if $t_1, t_2 \in \mathcal{T}_n$ then $(t_1\ t_2) \in \mathcal{T}_n$

α-equivalent closed λ-terms will have the same de Bruijn representation.

Naming Context

- When a λ-term has free variables, we need information on their relative positions.
- E.g. given $\{v \mapsto 2, w \mapsto 1, x \mapsto 0\}$:

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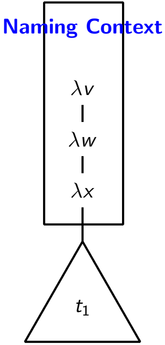

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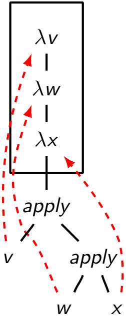

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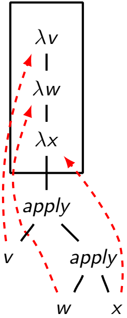

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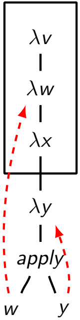

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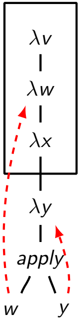

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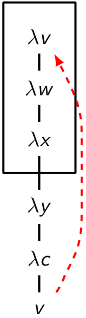

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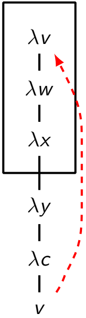

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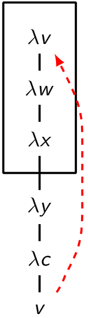

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- Naming contexts are often written as a sequence, where $x_n, x_{n-1}, \ldots, x_1, x_0$, represents a context where each x_i has de Bruijn index i.

Substitution

- Term $(\lambda y.\ \lambda z.\ (x\ y)\ (w\ z))$ under naming context v, w, x has the following de Bruijn representation:

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- Hence, when carrying out substitution, we need to renumber the indices of free variables in the replacement term, and retain the indices of bound variables.

  This will be done using the shifting operation, defined next.

Shifting

For substitution, we need to

- renumber the indices of free variables (say, by d), and
- retain the indices of bound variables (say, those numbered below c).

This is done using the *shifting* operation, defined as follows:

$$
\begin{aligned}
\uparrow_c^d (k) &= \begin{cases} k & \text{if } k < c \\ k + d & \text{if } k \geq c \end{cases} \\
\uparrow_c^d (\lambda.\ t_1) &= \lambda.\ \uparrow_{c+1}^d (t_1) \\
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- $\uparrow^2 (\lambda.\ 0\ 1\ (\lambda.\ 0\ 1\ 2)) = \lambda. 0\ 3\ (\lambda.\ 0\ 1\ 4)$

Substitution using Shifting

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[j \mapsto s]k & = & \begin{cases} s & \text{if } k = j \\ k & \text{otherwise} \end{cases} \\
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Examples:

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- $[0 \mapsto 1](\lambda.\ (0\ 2)) = \lambda.\ (0\ 2)$

Evaluation

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$$(\lambda x.\ t_1)\ t_2 \to [x \mapsto t_2]t_1 \qquad \text{E-AppAbs}$$

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  - The result of the application is $w\ (\lambda y.\ w\ y)\ v$.

Evaluation

In the calculus with symbolic term representation:

$$(\lambda x.\ t_1)\ t_2 \rightarrow [x \mapsto t_2]t_1 \qquad \text{E-AppAbs}$$

In the calculus with de Bruijn representation:

$$(\lambda.\ t_1)\ t_2 \rightarrow \uparrow^{-1} ([0 \mapsto \uparrow^{1} (t_2)]t_1) \qquad \text{E-AppAbs}$$

- The outer λ is removed after application, so the indices have to shift *down* by 1.
- Indices in argument (t_2) should *not* be changed in the end, so we shifting them *up* by 1 first.
  - Consider $(\lambda x.\ w\ x\ v)\ (\lambda y.\ (w\ y))$, whose de Bruijn representation is $(\lambda.\ 1\ 0\ 2)\ (\lambda.\ 1\ 0)$ (assuming naming context v, w).
  - The result of the application is $w\ (\lambda y.\ w\ y)\ v$.
  - $\uparrow^{1} (\lambda.\ 1\ 0) = \lambda.\ 2\ 0$

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  - $\uparrow^{-1} (1\ (\lambda.\ 2\ 0)\ 2) = 0\ (\lambda.\ 1\ 0)\ 1$