Logic and Programs

- “All men are mortal; Socrates is a man; Hence Socrates is mortal”

\[ \forall X. \text{man}(X) \Rightarrow \text{mortal}(X) \]

\[ \text{man}(\text{socrates}) \]

- Predicate logic
  - Predicates (e.g. \text{man}, \text{mortal}) which define sets.
  - Atoms (e.g. \text{socrates}) which are data values
  - Variables (e.g. \(X\)) which range over data values
  - Rules (e.g. \( \forall X. \text{man}(X) \Rightarrow \text{mortal}(X) \)) which define relationships between predicates.
Logic Programs and Queries

\[ \forall X. \text{man}(X) \Rightarrow \text{mortal}(X) \]

\[ \text{man}(\text{socrates}) \]

Logic “Program”:

\[- \text{man}(\text{socrates}). \]
\[- \text{mortal}(X) :- \text{man}(X). \]

Queries:

?- \text{mortal}(\text{socrates}).
yes

?- \text{mortal}(X).
\text{X=socrates};
no

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Prolog

Programming in Logic

- Early development: Kowalski & van Emden (Edinburgh); Colmerauer (Marseilles) (early '70s)
- First efficient implementation: WAM of David H.D. Warren (Edinburgh) (mid '70s).
- Later developments:
  - Constraint Logic Programming: for applications in AI, planning, scheduling, etc. Jaffar & Lassez (IBM Watson)
  - Memoization: Tamaki & Sato (Tokyo); Warren et al (Stony Brook)
Prolog Systems

- SWI Prolog ([www.swi-prolog.org](http://www.swi-prolog.org))
  - Can be obtained for free and installed on Windows, Linux, Mac.
  - Has a good development environment (command completion, help, graphical debugger, etc.)
  - On compute* (Unix) servers: `~ cram/bin/swipl`

- XSB Prolog ([xsb.sourceforge.net](http://xsb.sourceforge.net))
  - Can be obtained for free and installed on Windows, Linux, Mac.
  - Supports a powerful extension (memoization) to Prolog
  - Command-line interface (e.g. no graphical debugger)
  - On compute* (Unix) servers: `~ cram/bin/xsb`

Using Prolog Systems

- Prolog programs are in files with ".pl" extension (".P" for XSB)
- Prolog systems typically support an interactive mode.
- "[filename]." to compile and load a program in filename.pl (filename.P in XSB).
- "halt." to exit the system.
Logic Programs

- Programs are a set of rules (also called clauses).
- Predicates in a logic program are analogous to procedures in imperative programs.
- One or more rules are used to define a predicate.
- Example:
  \[ \text{inc}(X,Y) \leftarrow Y \text{ is } X+1. \]
  - X and Y are variables.
  - inc is a predicate.
  - The predicate is defined using a single rule.

- "\leftarrow" separates the body of the rule from its head.
- "X" and "Y" are also "parameters" of the predicate.
  In this case, X is the input parameter, and Y is the return parameter (where the return values are stored).
- "Y is X+1" defines Y in terms of X.
- The period ("." ) marks the end of a rule.
- The predicate is called by giving values to its parameters. e.g.
  - \[ \text{inc}(6, B) \] returns with B=7.
  - \[ \text{inc}(11, B) \] returns with B=12.
Syntax of Prolog

- **Variables** are identifiers that begin with an upper case letter or underscore.
  - An underscore, by itself, represents an *anonymous variable*.
- **Predicate** names (and later, data structure symbols) are identifiers that begin with a lower case letter.
- All variables are *local* to the clause in which they occur.
- Different occurrences of the same variable in a clause denote the same data.
- Variables need not be declared, and have no type.

How Prolog Works (An Example)

```
big(bear).
big(elephant).

brown(bear).

black(cat).
small(cat).

gray(elephant).

dark(Z) :- black(Z).
dark(Z) :- brown(Z).

dangerous(X) :- dark(X), big(X).
```
Derivations

big(bear). brown(bear). dark(Z) :- black(Z).
big(elephant). black(cat). dark(Z) :- brown(Z).
small(cat). gray(elephant).

dangerous(X) :- dark(X), big(X).

dangerous(Q)
dangerous(X) :- dark(X), big(X)
dark(Q), big(Q)
dark(X) :- black(X)
black(Q), big(Q)
dark(X) :- brown(X)
brown(Q), big(Q)

black(cat) big(cat)
brown(bear) big(bear)

failure
success

How Prolog Works (the procedure)

- A query is, in general, a conjunction of goals
- To prove \( G_1, G_2, \ldots, G_n \):
  - Find a clause \( H : \neg B_1, B_2, \ldots, B_k \) such that \( G_1 \) and \( H \) match.
  - Under that substitution for variables, prove \( B_1, B_2, \ldots, B_k, G_2, \ldots, G_n \).
  - If nothing is left to prove then the proof is complete. If there are no more clauses to match, the proof attempt fails.
How Prolog Works (an example)

To prove dangerous(Q):

1. Select dangerous(X) :- dark(X), big(X) and prove dark(Q), big(Q).

2. To prove dark(Q) select the first clause of dark, i.e. dark(Z) :-
   black(Z), and prove black(Q), big(Q).

3. Now select the fact black(cat) and prove big(cat).
   
   This proof attempt fails!

4. Go back to step 2, and select the second clause of dark, i.e. dark(Z) :-
   brown(Z), and prove brown(Q), big(Q).

5. Now select brown(bear) and prove big(bear).

6. Select the fact big(bear).
   
   There is nothing left to prove, so the proof is complete

Data Representation in Prolog

- Prolog has no notion of data types
- All data is represented as terms, which can be:
  - Variables
  - Non-variable Terms
    - Atomic data (Integers, floating point numbers, constants, ...)
    - Compound Terms (Structures)
Atomic Data

- **Numeric constants**: Integers, floating point numbers (e.g. 1024, -42, 3.1415, 6.023e23 ...)
- **Atoms**:
  - Strings of characters enclosed in single quotes (e.g. ’cram’, ’Stony Brook’)
  - Identifiers: sequence of letters, digits, underscore, beginning with a letter (e.g. cram, r2d2, x_24).

Data Structures

- If \( f \) is an identifier and \( t_1, t_2, \ldots, t_n \) are terms, then \( f(t_1, t_2, \ldots, t_n) \) is a term.

In the above, \( f \) is called a *function symbol* (or *functor*) and \( t_i \) is an *argument*.
- Structures are used to group related data items together (in some ways similar to *struct* in C and objects in Java).
- Structures are used to construct trees (and, as a special case, lists).
Trees

- Example: expression trees:
  \[ \text{plus}(\text{minus}(\text{num}(3), \text{num}(1)), \text{star}(\text{num}(4), \text{num}(2))) \]

- Data structures may have variables. And the same variable may occur multiple times in a data structure.

Matching

(We'll extend this to \textit{unification} later)

- \( t_1 = t_2 \): find substitions for variables in \( t_1 \) and \( t_2 \) that make the two terms identical.

Yes, with \( X = 1 \), \( Y = 4 \).
Yes, with $X = 1$, $Y = 4$.

No! $X$ cannot be 1 and 4 at the same time.
Accessing arguments of a structure

- Matching is the common way to access a structure’s arguments.
- Let \( \text{date('Sep', 1, 2005)} \) be a structure used to represent dates, with the month, day and year as the three arguments (in that order).
- Then \( \text{date(M, D, Y)} = \text{date('Sep', 1, 2005)} \) makes \( M = \text{'Sep'}, D = 1, Y = 2005 \).
- If we want to get only the day, we can write \( \text{date(, D, )} = \text{date('Sep', 1, 2005)} \). Then we get \( D = 1 \).

Lists

Prolog uses a special syntax to represent and manipulate lists.

- \([1,2,3,4]\): represents a list with 1, 2, 3 and 4, respectively.
- This can also be written as \([1 \mid [2,3,4]]\): a list with 1 as the head (its first element) and \([2,3,4]\) as its tail (the list of remaining elements).
- If \( X = 1 \) and \( Y = [2,3,4] \) then \([X\mid Y]\) is same as \([1,2,3,4]\).
- The empty list is represented by \([\ ]\).
- The symbol “\(|\)" (called cons) and is used to separate the beginning elements of a list from its tail.
  - For example: \([1,2,3,4] = [1 \mid [2,3,4]]\)
  - \( = [1 \mid [2 \mid [3,4]]]\)
  - \( = [1,2 \mid [3,4]]\)
Lists are special cases of trees. For instance, the list \([1,2,3,4]\) is represented by the following structure:

```
1
  2
  3
  4
[]
```

The function symbol \(./2\) is the list constructor. \([1,2,3,4]\) is same as \((1, (2, (3, (4, []))))\).

Programming with Lists — I

First example: \(\text{member}/2\), to find if a given element occurs in a list:

**The program:**

```
member(X, [X|_]).
member(X, [\_|Ys]) :- member(X, Ys).
```

**Example queries:**

```
member(s, [l,i,s,t])
member(X, [l,i,s,t])
member(f(X), [f(1), g(2), f(3), h(4), f(5)])
```
Programming with Lists — II

append/3: concatenate two lists to form the third list.

**The program:**

```
append([], L, L).
append([X|Xs], Ys, [X|Zs]) :- append(Xs, Ys, Zs).
```

**Example queries:**

```
append([f,i,r], [s,t], L)
append(X, Y, [s,e,c,o,n,d])
append(X, [t,h], [f,o,u,r,t,h])
```

Programming with Lists — III

Define a predicate, `len/2` that finds the length of a list (first argument).

**The program:**

```
len([], 0).
len([_|Xs], N+1) :- len(Xs, N).
```

**Example queries:**

```
len([], X)
len([l,i,s,t], 4)
len([l,i,s,t], X)
```
Arithmetic

?- 1+2 = 3.

no

- In *Predicate logic*, the basis for Prolog, the only symbols that have a meaning are the predicates themselves.
- In particular, function symbols are uninterpreted: have no special meaning and can only be used to construct data structures.
- Meaning for arithmetic expressions is given by the *built-in* predicate “is”:
  - \( X \text{ is } 1 + 2 \) succeeds, binding \( X \) to 3.
  - \( 3 \text{ is } 1 + 2 \) succeeds.
  - General form: \( R \text{ is } E \) where \( E \) is an expression to be evaluated and \( R \) is matched with the expression’s value.
  - \( Y \text{ is } X + 1 \) will give an error if \( X \) does not (yet) have a value.

The list length example revisited

Define a predicate, \texttt{length/2} that finds the length of a list (first argument).

**The program:**

\begin{verbatim}
length([], 0).
length([Xs | Xs], M) :- length(Xs, N), M is N+1.
\end{verbatim}

**Example queries:**

\begin{verbatim}
length([], X)  
length([l,i,s,t], 4)  
length([l,i,s,t], X)  
length(List, 4)
\end{verbatim}
Conditional Evaluation

Consider the computation of $n!$, i.e. the factorial of $n$.

```
factorial(N, F) :- ...
```

- $N$ is the input parameter; and $F$ is the output parameter.
- The body of the rule specifies how the output is related to the input.
- For factorial, there are two cases: $N \leq 0$ and $N > 0$.
  - $N \leq 0$: $F = 1$
  - $N > 0$: $F = N \times (N - 1)!$

```
factorial(N, F) :-
    (N > 0 -> N1 is N-1, factorial(N1, F1), F is N*F1 ; F = 1).
```

More Prolog Syntax

- Assignments with arithmetic expressions is done using the keyword “is”.
- If-then-else is written as ( `cond` -> `then-part` ; `else-part` )
- If more than one action needs to be performed in a rule, they are written one after another, separated by a comma.
- Arithmetic expressions are not directly used as arguments when calling a predicate; they are first evaluated, and then passed to the called predicate.
Arithmetic Operators

- Integer/Float operators: +, -, *, /
- Integer operators: mod, // (div)
- Int ↔ Float operators: floor, ceiling
- Comparison operators: <, >, =<, >=, =:, =\n
Sequences, revisited

append/3: concatenate two lists to form the third list (sometimes called conc/3).

The program:
append([], L, L).
append([X|Xs], Ys, [X|Zs]) :- append(Xs, Ys, Zs).

Example queries:
- append([f,i,r], [s,t], L)
- append(X, Y, [s,e,c,o,n,d])
- append(X, [t,h], [f,o,u,r,t,h])
Mystery Program

\[ m(X, X). \]
\[ m(X1, X5) :- a(X1, X2), m(X2, X3), b(X3, X4), m(X4, X5). \]
\[ a([0|Y], Y). \]
\[ b([1|Y], Y). \]

?- \( m([0,1,0,0,1,1], L) \).
\[
L=[0,1,0,0,1,1]
L=[0,0,1,1]
L=[]
\]

?- \( m([0,0,1,1,1,0], L) \).
\[
L=[0,1,0,0,1,1]
L=[1,0]
\]

Definite Clause Grammars

\[ m \rightarrow []. \]
\[ m \rightarrow a, m, b, m. \]
\[ a \rightarrow [0]. \]
\[ b \rightarrow [1]. \]

?- \( m([0,1,0,0,1,1], L) \).
\[
L=[0,1,0,0,1,1], \ldots
\]

?- \( \text{phrase}(m, [0,1,0,0,1,1]) \equiv m([0,1,0,0,1,1], []) \)
\[
\text{yes}
\]

?- \( \text{phrase}(m, L) \).
\[
L=[]
L=[0,1]
L=[0,1,0,1]
\]
Definite Clause Grammars (Magic?)

\[ r([\]) \rightarrow []. \]
\[ r([X|Xs]) \rightarrow r(Xs), [X]. \]

\[- \text{phrase}(r([1,2,3,4]), L). \]
\[ L=[4,3,2,1] \]
\[- \text{phrase}(r(Q), [1,2,3,4]). \]
\[ Q=[4,3,2,1] \]

Definite Clause Grammars (Trick exposed!)

\[ r([\]) \rightarrow []. \]
\[ r([X|Xs]) \rightarrow r(Xs), [X]. \]

Translated to:
\[ r([], X, X). \]
\[ r([X|Xs], Z1, Z3) \leftarrow r(Xs, Z1, Z2), Z2 = [X|Z3]. \]

Equivalent to:
\[ r([], X, X). \]
\[ r([X|Xs], Z1, Z3) \leftarrow r(Xs, Z1, [X|Z3]). \]

\[- \text{phrase}(r([1,2,3,4]), L). \]
\[ \equiv r([1,2,3,4], L, []) \]
\[ L=[4,3,2,1] \]

- A way to reverse a list in \textit{polynomial time!}
Unification

- Operation done to “match” the goal atom with the head of a clause in the program.
- Forms the basis for the matching operation we used for Prolog evaluation.
  - \( f(a, Y) \) and \( f(X, b) \) unify when \( X=a \) and \( Y=b \).
  - \( f(a, X) \) and \( f(X, b) \) do not unify.
  - \( X \) and \( f(X) \) do not unify
    (but they “match” in Prolog!)

Substitutions

A substitution is a mapping between variables and values (terms).
- Denoted by \( \{ X_1 \mapsto t_1, X_2 \mapsto t_2, \ldots, X_n \mapsto t_n \} \) such that
  - \( X_i \neq t_i \), and
  - \( X_i \) and \( X_j \) are distinct variables when \( i \neq j \).
- Empty substitution is denoted by \( \epsilon \).
- A substition is said to be a renaming if it is of the form
  \( \{ X_1 \mapsto Y_1, \ldots, X_n \mapsto Y_n \} \) and \( Y_1, \ldots, Y_n \) is a permutation of \( X_1, \ldots, X_n \).
- Example: \( \{ X \mapsto Y, Y \mapsto X \} \) is a renaming substitution.
Substitutions and Terms

- Application of a substitution: 
  - \( X\theta = t \) if \( X \mapsto t \in \theta \).
  - \( X\theta = X \) if \( X \mapsto t \notin \theta \) for any term \( t \).
- Application of a substitution \( \{ X_1 \mapsto t_1, \ldots, X_n \mapsto t_n \} \) to a term \( s \): 
  - is a term obtained by \textit{simultaneously} replacing every occurrence of \( X_i \) in \( s \) by \( t_i \).
  - Denoted by \( s\theta \) and \( s\theta \) is said to be an \textit{instance} of \( s \).
- Example:
  \[
  p(f(X, Z), f(Y, a)) \{ X \mapsto g(Y), Y \mapsto Z, Z \mapsto a \} \\
  = p(f(g(Y), a), f(Z, a))
  \]

Composition of Substitutions

- Composition of substitutions \( \theta = \{ X_1 \mapsto s_1, \ldots, X_m \mapsto s_m \} \) and \( \sigma = \{ Y_1 \mapsto t_1, \ldots, Y_n \mapsto t_n \} \):
  - First form the set \( \{ X_1 \mapsto s_1\sigma, \ldots, X_m \mapsto s_m\sigma, Y_1 \mapsto t_1, \ldots, Y_n \mapsto t_n \} \).
  - Remove from the set \( X_i \mapsto s_i\sigma \) if \( s_i\sigma = X_i \).
  - Remove from the set \( Y_j \mapsto t_j \) if \( Y_j \) is identical to some variable \( X_i \).
- Example: Let \( \theta = \sigma = \{ X \mapsto g(Y), Y \mapsto Z, Z \mapsto a \} \). Then \( \theta\sigma = \)
  \[
  \{ X \mapsto g(Y), Y \mapsto Z, Z \mapsto a \} \{ X \mapsto g(Y), Y \mapsto Z, Z \mapsto a \} \\
  = \{ X \mapsto g(Z), Y \mapsto a, Z \mapsto a \}
  \]
- More examples: Let \( \theta = \{ X \mapsto f(Y) \} \) and \( \sigma = \{ Y \mapsto a \} \)
  - \( \theta\sigma = \{ X \mapsto f(a), Y \mapsto a \} \)
  - \( \theta\sigma = \{ X \mapsto f(Y), Y \mapsto a \} \)
- Composition is not \textit{commutative} but is \textit{associative}: \( \theta(\sigma\gamma) = (\theta\sigma)\gamma \)
- Also, \( E(\theta\sigma) = (E\theta)\sigma \)
Idempotence

- A substitution \( \theta \) is **idempotent** iff \( \theta \theta = \theta \).
- Examples:
  - \( \{ X \mapsto g(Y), Y \mapsto Z, Z \mapsto a \} \) is not idempotent since
    \[
    \{ X \mapsto g(Y), Y \mapsto Z, Z \mapsto a \} \{ X \mapsto g(Y), Y \mapsto Z, Z \mapsto a \}
    = \{ X \mapsto g(Z), Y \mapsto a, Z \mapsto a \}
    \]
  - \( \{ X \mapsto g(Z), Y \mapsto a, Z \mapsto a \} \) is not idempotent either since
    \[
    \{ X \mapsto g(Z), Y \mapsto a, Z \mapsto a \} \{ X \mapsto g(Z), Y \mapsto a, Z \mapsto a \}
    = \{ X \mapsto g(a), Y \mapsto a, Z \mapsto a \}
    \]
  - \( \{ X \mapsto g(a), Y \mapsto a, Z \mapsto a \} \) is idempotent
- For a substitution \( \theta = \{ X_1 \mapsto t_1, \ldots, X_n \mapsto t_n \} \),
  - \( \text{Dom}(\theta) = \{ X_1, X_2, \ldots X_n \} \)
  - \( \text{Range}(\theta) = \) set of all variables in \( t_1, \ldots t_n \)
- A substitution \( \theta \) is idempotent iff \( \text{Dom}(\theta) \cap \text{Range}(\theta) = \emptyset \)

Unifiers

- A substitution \( \theta \) is a **unifier** of two terms \( s \) and \( t \) if \( s\theta \) is identical to \( t\theta \).
- \( \theta \) is a unifier of a set of equations \( \{ s_1 \equiv t_1, \ldots, s_n \equiv t_n \} \), if for all \( i \), \( s_i\theta = t_i\theta \).
- A substitution \( \theta \) is more general than \( \sigma \) (written as \( \theta \succeq \sigma \)) if there is a substitution \( \omega \) such that \( \sigma = \theta \omega \)
- A substitution \( \theta \) is a **most general unifier** (mgu) of two terms (or a set of equations) if for every unifier \( \sigma \) of the two terms (or equations) \( \theta \succeq \sigma \)
- Example: Consider two terms \( f(g(X), Y, a, b) \) and \( f(Z, W, X, b) \).
  - \( \theta_1 = \{ X \mapsto a, Y \mapsto b, Z \mapsto g(a), W \mapsto b \} \) is a unifier
  - \( \theta_2 = \{ X \mapsto a, Y \mapsto W, Z \mapsto g(a) \} \) is also a unifier
  - \( \theta_2 \) is a most general unifier
Equations and Unifiers

- A set of equations $\mathcal{E}$ is in **solved form** if it is of the form
  \[ \{X_1 \equiv t_1, \ldots, X_n \equiv t_n\} \]
  iff
  - all $X_i$'s are distinct, and
  - no $X_i$ appears in any $t_j$.

- Given a set of equations in solved form $\mathcal{E} = \{X_1 \equiv t_1, \ldots, X_n \equiv t_n\}$
  the substitution $\{X_1/t_1, \ldots X_n/t_n\}$ is an idempotent mgu of $\mathcal{E}$.

- Two sets of equations $\mathcal{E}_1$ and $\mathcal{E}_2$ are said to be **equivalent** iff they have the same set of unifiers.

- To find the mgu of two terms $s$ and $t$, find a set of equations in solved form that is equivalent to $\{s \equiv t\}$.
  If there is no equivalent solved form, there is no mgu.

A Simple Unification Algorithm (via Examples)

- Example 1: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

  \[ \{f(X, g(Y)) \equiv f(g(Z), Z)\} \Rightarrow \{X \equiv g(Z), g(Y) \equiv Z\} \]

  \[ \Rightarrow \{X \equiv g(Z), Z \equiv g(Y)\} \]

- Example 2: Find the mgu of $f(X, g(X), b)$ and $f(a, g(Z), Z)$

  \[ \{f(X, g(X), b) \equiv f(a, g(Z), Z)\} \Rightarrow \{X \equiv a, g(X) \equiv g(Z), b \equiv Z\} \]

  \[ \Rightarrow \{X \equiv a, g(a) \equiv g(Z), b \equiv Z\} \]

  \[ \Rightarrow \{X \equiv a, a \equiv Z, b \equiv Z\} \]

  \[ \Rightarrow \{X \equiv a, Z \equiv a, b \equiv Z\} \]

  \[ \Rightarrow \{X \equiv a, Z \equiv a, b \equiv a\} \]

  \[ \Rightarrow \text{fail} \]
A Simple Unification Algorithm

Given a set of equations \( E \):

```
repeat
  select \( s \cdot t \in E \);
  case \( s \cdot t \) of
    1. \( f(s_1, \ldots, s_n) \cdot f(t_1, \ldots, t_n) \):
      replace the equation by \( s_i \cdot t_i \) for all \( i \)
    2. \( f(s_1, \ldots, s_n) \cdot g(t_1, \ldots, t_m) \), \( f \neq g \) or \( n \neq m \):
      halt with failure
    3. \( X \cdot X \): remove the equation
    4. \( t \cdot X \): where \( t \) is not a variable
      replace equation by \( X \cdot t \)
    5. \( X \cdot t \): where \( X \neq t \) and \( X \) occurs more than once in \( E \):
      if \( X \) is a proper subterm of \( t \)
        then halt with failure (5a)
        else replace all other \( X \) in \( E \) by \( t \) (5b)
  until no action is possible for any equation in \( E \)
return \( E \)
```

Example 1: Find the mgu of \( f(X, g(Y)) \) and \( f(g(Z), Z) \)

\[
\{f(X, g(Y)) \cdot f(g(Z), Z)\} \quad \Rightarrow \quad \{X \cdot g(Z), g(Y) \cdot Z\} \quad \text{case 1}
\]
\[
\Rightarrow \quad \{X \cdot g(Z), Z \cdot g(Y)\} \quad \text{case 4}
\]
\[
\Rightarrow \quad \{X \cdot g(g(Y)), Z \cdot g(Y)\} \quad \text{case 5b}
\]

Example 3: Find the mgu of \( f(X, g(X)) \) and \( f(Z, Z) \)

\[
\{f(X, g(X)) \cdot f(Z, Z)\} \quad \Rightarrow \quad \{X \cdot Z, g(X) \cdot Z\} \quad \text{case 1}
\]
\[
\Rightarrow \quad \{X \cdot Z, g(Z) \cdot Z\} \quad \text{case 5b}
\]
\[
\Rightarrow \quad \{X \cdot Z, Z \cdot g(Z)\} \quad \text{case 4}
\]
\[
\Rightarrow \quad \text{fail} \quad \text{case 5a}
\]
Complexity of the unification algorithm

Consider
\[ E = \{ g(X_1, \ldots, X_n) \equiv g(f(X_0, X_0), f(X_1, X_1), \ldots, f(X_{n-1}, X_{n-1}) \} \].

- By applying case 1 of the algorithm, we get
  \[ \{ X_1 = f(X_0, X_0), X_2 = f(X_1, X_1), \ldots, X_n = f(X_{n-1}, X_{n-1}) \} \]

- If terms are kept as trees, the final value for \( X_n \) is a tree of size \( O(2^n) \).
- Recall that for case 5 we need to first check if a variable appears in a term, and this could now take \( O(2^n) \) time.
- There are linear-time unification algorithms that share structures (terms as DAGs).
- \( X = t \) is the most common case for unification in Prolog. The fastest algorithms are linear in \( t \).
- Prolog cuts corners by omitting case 5a (the occur check), thereby doing \( X = t \) in constant time.

Most General Unifiers

- Note that mgu stands for a most general unifier.
- There may be more than one mgu. E.g. \( f(X) \equiv f(Y) \) has two mgus:
  - \( \{ X \mapsto Y \} \)
  - \( \{ Y \mapsto X \} \)
- If \( \theta \) is an mgu of \( s \) and \( t \), and \( \omega \) is a renaming, then \( \theta\omega \) is an mgu of \( s \) and \( t \).
- If \( \theta \) and \( \sigma \) are mgus of \( s \) and \( t \), then there is a renaming \( \omega \) such that \( \theta = \sigma\omega \).