Q1. Let $S$ be a set, and $R$ be a binary relation over $S$ (i.e., $\subseteq S \times S$). Consider the following three definitions, all attempting to define the transitive closure of $R$:

$R^+_I$: The inductive transitive closure of $R$, denoted by $R^+_I$, is the smallest set such that:
1. if $(s_1, s_2) \in R$ then $(s_1, s_2) \in R^+_I$.
2. if $(s_1, s_2) \in R$ and $(s_2, s_3) \in R^+_I$, then $(s_1, s_3) \in R^+_I$.

$R^+_C$: The constructive transitive closure of $R$, denoted by $R^+_C$, is the union of all $T_i$, $i \geq 0$ (i.e. $R^+_C = \bigcup_{i \geq 0} T_i$), where:

$$
T_0 = \emptyset
$$

$$
T_{i+1} = \left\{ \begin{array}{l} R \\ \cup \{(s_1, s_3) | \exists (s_1, s_2) \in R, (s_2, s_3) \in T_i \} \end{array} \right.
$$

$R^+_D$: The recursive doubling transitive closure of $R$, denoted by $R^+_D$, is the union of all $V_i$, $i \geq 0$ (i.e. $R^+_D = \bigcup_{i \geq 0} V_i$) where:

$$
V_0 = \emptyset
$$

$$
V_{i+1} = \left\{ \begin{array}{l} R \\ \cup \{(s_1, s_3) | \exists (s_1, s_2) \in V_i, (s_2, s_3) \in V_i \} \end{array} \right.
$$

Q1. (a) Show that $R^+_I = R^+_C$.
(b) Show that $R^+_C = R^+_D$. 

Q2. Consider the language $B_N$ of Boolean expressions from the text whose syntax and single-step operational semantics are given below.

<table>
<thead>
<tr>
<th>Terms and Values:</th>
<th>Evaluation Rules:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t ::= \begin{align*} \text{true} \ \text{false} \ \text{nand}(t, t) \end{align*}$</td>
<td>$\begin{align*} \text{nand}(\text{false}, t_2) &amp; \rightarrow \text{true} &amp; \text{E-False} \ \text{nand}(\text{true}, \text{false}) &amp; \rightarrow \text{true} &amp; \text{E-TrueFalse} \ \text{nand}(\text{true}, \text{true}) &amp; \rightarrow \text{false} &amp; \text{E-TrueTrue} \end{align*}$</td>
</tr>
<tr>
<td>$v ::= \begin{align*} \text{true} \ \text{false} \end{align*}$</td>
<td>$\begin{align*} \frac{t_2 \rightarrow t'_2}{\text{nand}(\text{true}, t_2) \rightarrow \text{nand}(\text{true}, t'_2)} &amp; \text{E-True} \ \frac{t_1 \rightarrow t'_1}{\text{nand}(t_1, t_2) \rightarrow \text{nand}(t'_1, t_2)} &amp; \text{E-Nand} \end{align*}$</td>
</tr>
</tbody>
</table>

Q2. (a) Does determinacy hold for $B_N$? That is, for all $t, t', t''$, if $t \rightarrow t'$ and $t \rightarrow t''$ then $t' = t''$? Justify.

(b) Does uniqueness of normal forms hold in $B_N$? That is, for all $t, v, v'$, if $t \rightarrow^* v$ and $t \rightarrow^* v'$ then $v = v'$? Justify.

(c) Does termination hold in $B_N$? Justify.

For all three parts above, your justification should be as follows. If the property holds, give a formal proof. If the property does not hold give a completely-specified counter example, and explain how it is a counter example.