Performance Guarantees for the TSP with a Parameterized Triangle Inequality

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July 27, 1999

Abstract

We consider the approximability of the TSP problem in graphs that satisfy a relaxed form of triangle inequality. More precisely, we assume that for some parameter $\tau \geq 1$, the distances satisfy the inequality $\text{dist}(x, y) \leq \tau \cdot (\text{dist}(x, z) + \text{dist}(z, y))$ for every triple of vertices x, y, and z. We obtain a 4τ approximation and also show that for some $\epsilon > 0$ it is NP-hard to obtain a $(1 + \epsilon\tau)$ approximation. Our upper bound improves upon the earlier known ratio of $(3\tau^2/2 + \tau/2)$ [1] for all values of $\tau > 7/3$.

Keywords: approximation algorithm, traveling salesman problem, relaxed triangle inequality.

Introduction. The approximability of the Traveling Salesman Problem (TSP) depends on the particular version of the problem. Specifically, if the distances are Euclidean, the TSP admits a polynomial time approximation scheme [2, 14]. If the distances form a metric, then the problem is MAX-SNP-hard [16] and $3/2$ is the best obtainable approximation ratio known [4]. For arbitrary (symmetric) distances, however, the problem cannot be approximated to within any factor. (This negative result follows by a reduction to the problem of finding a Hamiltonian cycle in a graph.)

It seems natural that slight variations in the distances of a problem should affect the solution to the problem only slightly, and this should hold even if a perturbed instance of metric TSP no longer obeys the Δ-inequality. This intuition suggests parameterizing TSP instances by how *strongly* they disobey the Δ-inequality. In particular, for parameter τ,

$$\text{dist}(x, z) \leq \tau \cdot (\text{dist}(x, y) + \text{dist}(y, z)).$$

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If $\tau = 1$, the distances satisfy the triangle inequality. For $\tau > 1$, the distances satisfy the *relaxed triangle inequality.*

Andreae and Bandelt [1] show a $3\tau^2/2 + \tau/2$-approximation for the parameterized triangle inequality TSP ($\Delta_\mathcal{T}$-TSP). In this paper, we provide a simple algorithm that obtains a 4τ-approximation to the $\Delta_\mathcal{T}$-TSP. This result is optimal to within a constant factor.

The Problem. An instance of the problem is a weighted graph G consisting of n vertices (representing cities). Distances between vertices satisfy the $\Delta_\mathcal{T}$-inequality and the length of an optimal TSP tour is denoted OPT. The objective of the problem is to find (in polynomial time) a tour having length at most $f(\tau)\cdot$OPT, where $f(\tau)$ is a slowly-growing function of τ. In this paper we present an algorithm having an approximation ratio of 4τ.

Intuition. Unfortunately, an approximation algorithm on a graph obeying the Δ-inequality does not automatically translate into an approximation algorithm on a graph obeying the the parameterized Δ-inequality. To understand why, notice an advantage of the triangle inequality: shortcutting vertices never increases distances, no matter how many vertices are avoided. Thus, the TSP on a *subset* of vertices is a lower bound on the TSP of all the vertices.

This advantage is lost with the parameterized Δ-inequality. Shortcutting over ℓ vertices may increase the distance substantially: the shortcutted distance may depend on the parameter τ^ℓ. Thus, the TSP on a subset of vertices may be dramatically larger than the TSP on all the vertices.

We illustrate this difficulty by examining two well-known approximation algorithms for the Δ-TSP. A 2-approximation is obtained as follows:

1. Find a minimum spanning tree (MST) for G.
2. Create a multigraph by duplicating each edge in the MST.
3. Find an Eulerian tour.
4. Shortcut over nodes to obtain a Hamiltonian cycle.

The Christofides heuristic [4] yields a 3/2 approximation:

1. Find a MST for G.
2. Identify the nodes of odd degree and find a minimum-cost perfect matching for these nodes.
3. Find an Eulerian tour in the multigraph consisting of the MST and the perfect matching.
4. Shortcut the Eulerian tour to obtain a Hamiltonian cycle.

The performance guarantees follow because the MST is a lower bound on OPT and the minimum cost perfect matching is a lower bound on OPT/2.

$\Delta_{\mathcal{T}}$-TSP. Now consider what happens to these algorithms for the $\Delta_{\mathcal{T}}$-TSP. The MST remains a lower bound on OPT, however the minimum cost perfect matching on the odd-degree vertices is no longer a lower bound on OPT/2.

The $\Delta_{\mathcal{T}}$-TSP algorithm of [1] finds an MST and doubles edges to create an Eulerian tour. Then, it obtains a Hamiltonian cycle by shortcutting. In the algorithm, at most 2 consecutive nodes are removed from any part of the tour, and thus the approximation ratio is $O(\tau^2)$. It is never necessary to shortcut *more* than 2 nodes, because the cube of any connected graph is Hamiltonian [19]. (The cube of G, denoted G^3, contains edge (u, v) if there is a path from u to v in G of at most 3 edges.) This approach cannot obtain an approximation better than $O(\tau^2)$, because there are instances when the MST is a factor of $O(\tau^2)$ smaller than OPT.

$O(\tau)$-Approximation Algorithm. In order to obtain a better approximation ratio, we begin with a tighter lower bound. A graph S is 2 *node connected* if the deletion of any node from S leaves it connected. Any Hamiltonian cycle is 2 node connected. Thus, the *minimum weight* 2-*node-connected subgraph* is a lower bound on OPT. Notice that it is NP-hard to find the minimum weight 2-node-connected subgraph. However, a 2-approximation can be computed in polynomial time [10, 18].

Let S^2 denote the square of graph S. (Thus, (u, v) is an edge in S^2 if there is a path in S between u and v using at most 2 edges.) Our algorithm is described below.

TSP-APPROX (G)

1. Find an (approximately) minimum cost 2-node-connected subgraph of G.
   Call this graph S.
2. Find a Hamiltonian cycle in S^2.

In 1970, Fleischner proved that the square of a 2-node-connected graph is Hamiltonian [7, 8], and in his Ph.D. thesis Lau provided a constructive proof [11, 12] that finds a Hamiltonian cycle in polynomial time. We note that Fleischner's result is applied to the TSP in a simpler way by Parker and Rardin [17], who show a 2-approximation for the bottleneck-TSP.

Theorem 1 *Algorithm* TSP-APPROX *is a* 4τ *approximation algorithm.*

Proof. The graph S has weight at most $2 \cdot$ OPT. The goal is to find a Hamiltonian cycle in S^2 that has low weight. (Notice that it is not sufficient for the analysis to bound the total weight of

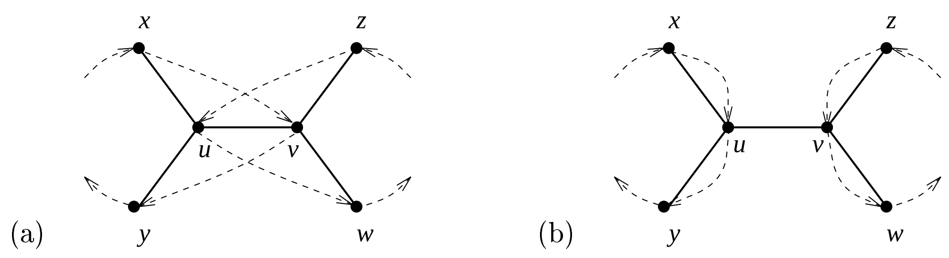

Figure 1: In (a), edge $\{u, v\}$ is crossed 4 times by cycle C. In (b), cycle C is modified so that $\{u, v\}$ is never crossed.

the edges in S^2, because the weight could be a factor of $O(n\tau)$ larger than the weight of the edges in S.) Thus, we need a more careful analysis.

Consider any Hamiltonian cycle C in S^2, and orient this cycle. Let (u, v) be a (directed) edge in C. We say that $(u, w) \in C$ *crosses* (undirected) edges $\{u, v\}$ and $\{v, w\}$ in S if $\{u, w\} \notin S$ and otherwise crosses edge $\{u, w\} \in S$.

Notice that no Hamiltonian cycle crosses any edge more than 4 times. This is because each node u in S is incident to exactly two edges in C, one edge entering u and one edge leaving u (see Figure 1(a)). This observation immediately yields an 8τ-approximation.

We obtain a 4τ-approximation by finding a new Hamiltonian cycle C' that crosses every edge in S at most twice. We obtain C' by adjusting overlapping edges. There are 2 cases. Figure 1(a) depicts the first case, where $\{u, v\}$ is crossed 4 times. The solid lines in the figure represent edges in S. The dotted lines represent edges in C. Cycle C can be modified as shown in Figure 1(b), so that $\{u, v\}$ is never crossed.

Case 2 is shown in Figure 2(a), where $\{u, v\}$ is crossed 3 times. Cycle C can be modified as in Figure 2(b) so the $\{u, v\}$ is crossed only once. After fixing these 2 cases, no edge is crossed more

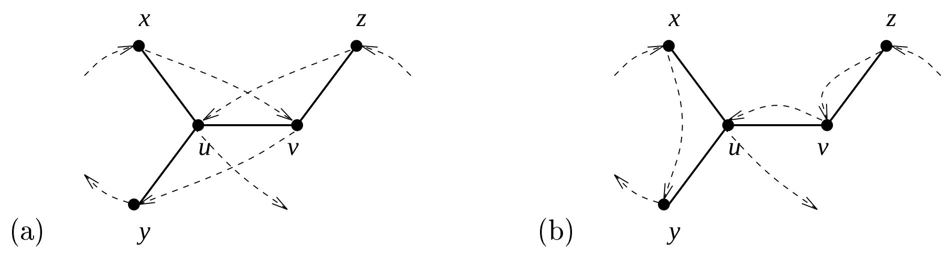

Figure 2: In (a), edge $\{u, v\}$ is crossed 3 times by cycle C. In (b), cycle C is modified so that $\{u, v\}$ is crossed once.

than twice. It is easily seen that identifying the two cases and transforming as described above can be accomplished in polynomial time. □

Theorem 2 *There exists some fixed constant $\epsilon > 0$ such that it is NP-hard to approximate $\Delta_{\mathcal{T}}$-TSP to within a factor better than $(1 + \epsilon\tau)$.*

Proof. We use a reduction from TSP with distances one and two which is MAX-SNP-hard [16]. The proof in [16] establishes that there exists an infinite class of graphs $\mathcal{G}$ with the following properties. Let $G = (V, E)$ be a graph in $\mathcal{G}$. There exists a fixed constant $\delta > 0$ such that either G is Hamiltonian, or every Hamiltonian cycle in the complete graph on V requires at least ϵn edges from $\bar{E} = V \times V - E$. Further, given a graph from $\mathcal{G}$ it is NP-Complete to distinguish which of the above two cases applies. MAX-SNP-hardness of one-two TSP follows from this. We use the same class of graphs to obtain our result. Given a graph $G = (V, E)$ from $\mathcal{G}$ we create an instance of $\Delta_{\mathcal{T}}$-TSP in the obvious way by making the graph complete with edges in E having cost 1, and the rest having cost 2τ. If G is Hamiltonian the cost of an optimal TSP tour in G is n, otherwise it is at least $(1 - \delta)n + 2\tau\delta n$. Since it is NP-Complete to distinguish between these two cases the theorem follows. □

Open Problems. Our upper and lower bounds are within a constant factor of each other; however, the gap is large. Can we obtain a $k\tau$ approximation for some k smaller than 4? One approach might be to combine the two steps in our algorithm in some clever way. It is natural to consider the same relaxation of triangle inequality for the Asymmetric TSP problem. Although an $O(\log n)$ approximation is known with the triangle inequality, no approximation factor of the form $O(\tau^{\ell} \log n)$ is known for the relaxed version for any constant ℓ. The relaxed triangle inequality is useful in practice (see [6, 15] for applications to pattern matching of images) and finding robust algorithms whose performance degrades only mildly with τ is of interest.

Acknowledgments.
We warmly thank Santosh Vempala for many helpful discussions.

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