Lecture 8: Mergesort / Quicksort

Steven Skiena

Department of Computer Science
State University of New York
Stony Brook, NY 11794–4400

http://www.cs.stonybrook.edu/~skiena

Problem of the Day

Give an efficient algorithm to determine whether two sets (of size m and n) are disjoint. Analyze the complexity of your algorithm in terms of m and n. Be sure to consider the case where m is substantially smaller than n.

Mergesort

Recursive algorithms are based on reducing large problems into small ones.

A nice recursive approach to sorting involves partitioning the elements into two groups, sorting each of the smaller problems recursively, and then interleaving the two sorted lists to totally order the elements.

`https://upload.wikimedia.org/wikipedia/commons/c/cc/Merge-sort-example-300px.gif`

Mergesort Implementation

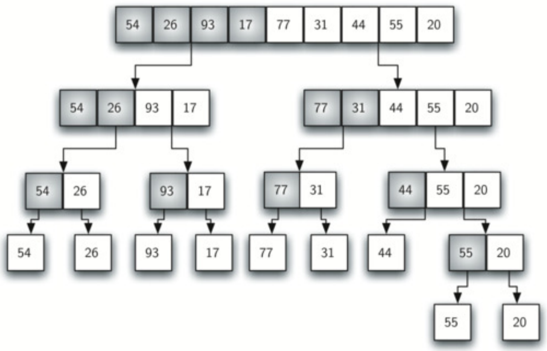
```
mergesort(item_type s[], int low, int high)
{
        int i; (* counter *)
        int middle; (* index of middle element *)

        if (low < high) {
                middle = (low+high)/2;
                mergesort(s,low,middle);
                mergesort(s,middle+1,high);

                merge(s, low, middle, high);
        }
}
```

Merging Sorted Lists

The efficiency of mergesort depends upon how efficiently we combine the two sorted halves into a single sorted list.
This smallest element can be removed, leaving two sorted lists behind, one slighly shorter than before.
Repeating this operation until both lists are empty merges two sorted lists (with a total of n elements between them) into one, using at most $n - 1$ comparisons or $O(n)$ total work
Example: $A = \{5, 7, 12, 19\}$ and $B = \{4, 6, 13, 15\}$.

Mergesort Analysis

A linear amount of work is done merging along all levels of the mergesort tree.
The height of this tree is $O(\log n)$.
Thus the worst case time is $O(n \log n)$.



Divide and Conquer

Divide and conquer is an important algorithm design technique using in mergesort, binary search the fast Fourier transform (FFT), and Strassen's matrix multiplication algorithm.
We divide the problem into two smaller subproblems, solve each recursively, and then meld the two partial solutions into one solution for the full problem.
When merging takes less time than solving the two subproblems, we get an efficient algorithm.

$$T(n) = 2 \cdot T(n/2) + \Theta(n) \rightarrow T(n) = \Theta(n \log n)$$

External Sorting

Which $O(n \log n)$ algorithm you use for sorting doesn't matter much until n is so big the data does not fit in memory. Mergesort proves to be the basis for the most efficient *external sorting* programs.

Disks are much slower than main memory, and benefit from algorithms that read and write data in long streams – not random access.

Quicksort

In practice, the fastest *internal* sorting algorithm is Quicksort, which uses *partitioning* as its main idea.

Example: pivot about 10.

Before: 17 12 6 19 23 8 5 10

After: 6 8 5 10 23 19 12 17

Partitioning places all the elements less than the pivot in the *left* part of the array, and all elements greater than the pivot in the *right* part of the array. The pivot fits in the slot between.

Note that the pivot element ends up in the correct place in the total order!

Partitioning the Elements

We can partition an array about the pivot in one linear scan, by maintaining three sections: $<$ pivot, $>$ pivot, and unexplored.

`https://upload.wikimedia.org/wikipedia/commons/9/9c/Quicksort-example.gif`

As we scan from left to right, we move the left bound to the right when the element is less than the pivot, otherwise we swap it with the *rightmost unexplored* element and move the right bound one step closer to the left.

Why Partition?

Since the partitioning step consists of at most n swaps, takes time linear in the number of keys. But what does it buy us?

1. The pivot element ends up in the position it retains in the final sorted order.

2. After a partitioning, no element flops to the other side of the pivot in the final sorted order.

Thus we can sort the elements to the left of the pivot and the right of the pivot independently, giving us a recursive sorting algorithm!

Quicksort Pseudocode

```
Sort(A)
        Quicksort(A,1,n)


Quicksort(A, low, high)
        if (low < high)
                pivot-location = Partition(A,low,high)
                Quicksort(A,low, pivot-location - 1)
                Quicksort(A, pivot-location+1, high)
```

Partition Implementation

```
Partition(A,low,high)
        pivot = A[low]
        leftwall = low
        for $i$ = low+1 to high
                if (A[i] $<$ pivot) then
                        leftwall = leftwall+1
                        swap(A[i],A[leftwall])
        swap(A[low],A[leftwall])
```

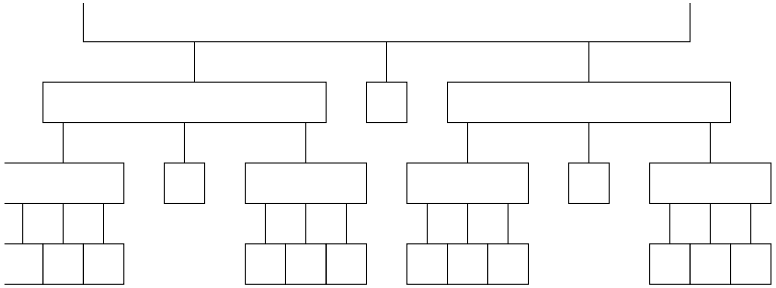
Best Case for Quicksort

Since each element ultimately ends up in the correct position, the algorithm correctly sorts. But how long does it take?

The best case for *divide-and-conquer* algorithms comes when we split the input as evenly as possible. Thus in the best case, each subproblem is of size $n/2$.

The partition step on each subproblem is linear in its size. Thus the total effort in partitioning the 2^k problems of size $n/2^k$ is $O(n)$.

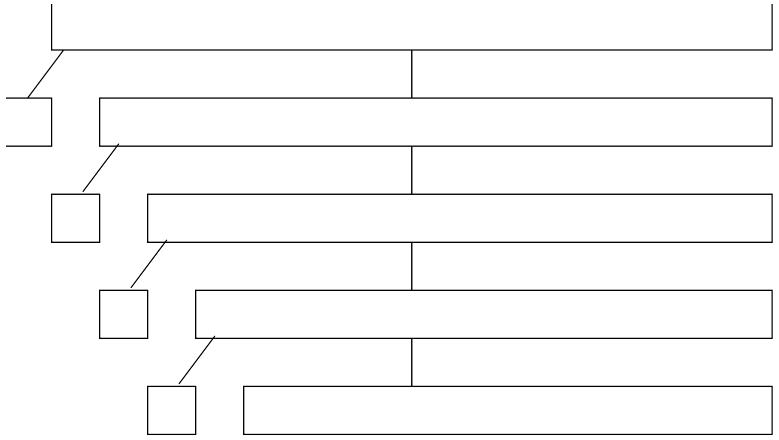
Best Case Recursion Tree



The total partitioning on each level is $O(n)$, and it take $\lg n$ levels of perfect partitions to get to single element subproblems. When we are down to single elements, the problems are sorted. Thus the total time in the best case is $O(n \lg n)$.

Worst Case for Quicksort

Suppose instead our pivot element splits the array as unequally as possible. Thus instead of $n/2$ elements in the smaller half, we get zero, meaning that the pivot element is the biggest or smallest element in the array.



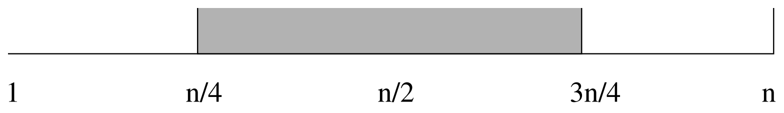
Now we have $n-1$ levels, instead of $\lg n$, for a worst case time of $\Theta(n^2)$, since the first $n/2$ levels each have $\geq n/2$ elements to partition.

To justify its name, Quicksort had better be good in the average case. Showing this requires some intricate analysis.

The divide and conquer principle applies to real life. If you break a job into pieces, make the pieces of equal size!

Intuition: The Average Case for Quicksort

Suppose we pick the pivot element at random in an array of n keys.



Half the time, the pivot element will be from the center half of the sorted array.

Whenever the pivot element is from positions $n/4$ to $3n/4$, the larger remaining subarray contains at most $3n/4$ elements.

How Many Good Partitions

If we assume that the pivot element is always in this range, what is the maximum number of partitions we need to get from n elements down to 1 element?

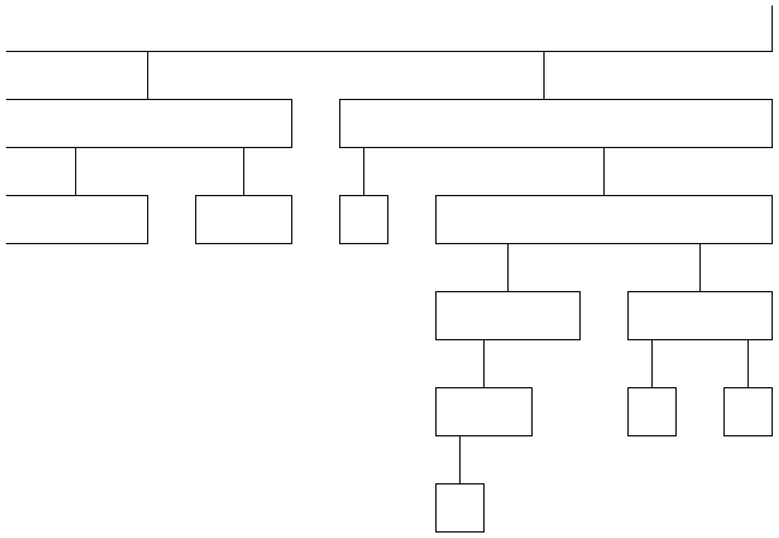
$$(3/4)^l \cdot n = 1 \longrightarrow n = (4/3)^l$$

$$\lg n = l \cdot \lg(4/3)$$

Therefore $l = \lg(4/3) \cdot \lg(n) < 2 \lg n$ good partitions suffice.

How Many Bad Partitions?

How often when we pick an arbitrary element as pivot will it generate a decent partition?
Since any number ranked between $n/4$ and $3n/4$ would make a decent pivot, we get one half the time on average.
If we need $2 \lg n$ levels of decent partitions to finish the job, and half of random partitions are decent, then on average the recursion tree to quicksort the array has $\approx 4 \lg n$ levels.



Since $O(n)$ work is done partitioning on each level, the average time is $O(n \lg n)$.

Average-Case Analysis of Quicksort

To do a precise average-case analysis of quicksort, we formulate a recurrence given the exact expected time $T(n)$:

$$T(n) = \sum_{p=1}^{n} \frac{1}{n}(T(p-1) + T(n-p)) + n - 1$$

Each possible pivot p is selected with equal probability. The number of comparisons needed to do the partition is $n-1$.

We will need one useful fact about the Harmonic numbers H_n, namely

$$H_n = \sum_{i=1}^{n} 1/i \approx \ln n$$

It is important to understand (1) where the recurrence relation comes from and (2) how the log comes out from the summation. The rest is just messy algebra.

$$T(n) = \sum_{p=1}^{n} \frac{1}{n}(T(p-1) + T(n-p)) + n - 1$$

$$T(n) = \frac{2}{n} \sum_{p=1}^{n} T(p-1) + n - 1$$

$$nT(n) = 2 \sum_{p=1}^{n} T(p-1) + n(n-1) \quad \text{multiply by n}$$

$$(n-1)T(n-1) = 2 \sum_{p=1}^{n-1} T(p-1) + (n-1)(n-2) \quad \text{apply to n-1}$$

$$nT(n) - (n-1)T(n-1) = 2T(n-1) + 2(n-1)$$

rearranging the terms give us:

$$\frac{T(n)}{n+1} = \frac{T(n-1)}{n} + \frac{2(n-1)}{n(n+1)}$$

substituting $a_n = A(n)/(n+1)$ gives

$$a_n = a_{n-1} + \frac{2(n-1)}{n(n+1)} = \sum_{i=1}^{n} \frac{2(i-1)}{i(i+1)}$$

$$a_n \approx 2 \sum_{i=1}^{n} \frac{1}{(i+1)} \approx 2 \ln n$$

We are really interested in $A(n)$, so

$$A(n) = (n+1)a_n \approx 2(n+1)\ln n \approx 1.38 n \lg n$$

Randomized Quicksort

Suppose you are writing a sorting program, to run on data given to you by your worst enemy. Quicksort is good on average, but bad on certain worst-case instances.

If you used Quicksort, what kind of data would your enemy give you to run it on? Exactly the worst-case instance, to make you look bad.

But suppose you picked the pivot element at *random*.

Now your enemy cannot design a worst-case instance to give to you, because no matter which data they give you, you would have the same probability of picking a good pivot!

Randomized Guarantees

Randomization is a very important and useful idea. By either picking a random pivot or scrambling the permutation before sorting it, we can say:

> "With high probability, randomized quicksort runs in $\Theta(n \lg n)$ time."

Where before, all we could say is:

> "If you give me random input data, quicksort runs in expected $\Theta(n \lg n)$ time."

Importance of Randomization

Since the time bound how does not depend upon your input distribution, this means that unless we are *extremely* unlucky (as opposed to ill prepared or unpopular) we will certainly get good performance.

Randomization is a general tool to improve algorithms with bad worst-case but good average-case complexity.

The worst-case is still there, but we almost certainly won't see it.

Pick a Better Pivot

Having the worst case occur when they are sorted or almost sorted is *very bad*, since that is likely to be the case in certain applications.
To eliminate this problem, pick a better pivot:

1. Use the middle element of the subarray as pivot.
2. Use a *random* element of the array as the pivot.
3. Perhaps best of all, take the median of three elements (first, last, middle) as the pivot. Why should we use median instead of the mean?

Whichever of these three rules we use, the worst case remains $O(n^2)$.

Is Quicksort *really* faster than Heapsort?

Since Heapsort is $\Theta(n \lg n)$ and selection sort is $\Theta(n^2)$, there is no debate about which will be better for decent-sized files.

When Quicksort is implemented well, it is typically 2-3 times faster than mergesort or heapsort.

The primary reason is that the operations in the innermost loop are simpler.

Since the difference between the two programs will be limited to a multiplicative constant factor, the details of how you program each algorithm will make a big difference.