Lecture 16:
Introduction to Dynamic Programming

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Problem of the Day

Multisets are allowed to have repeated elements. A multiset of n items may thus have fewer than $n!$ distinct permutations. For example, $\{1, 1, 2, 2\}$ has only six different permutations: $\{1, 1, 2, 2\}$, $\{1, 2, 1, 2\}$, $\{1, 2, 2, 1\}$, $\{2, 1, 1, 2\}$, $\{2, 1, 2, 1\}$, and $\{2, 2, 1, 1\}$. Design and implement an efficient algorithm for constructing all permutations of a multiset.

Dynamic Programming

Dynamic programming is a very powerful, general tool for solving optimization problems on left-right-ordered items such as character strings.

Once understood it is relatively easy to apply, it looks like magic until you have seen enough examples.

Floyd's all-pairs shortest-path algorithm was an example of dynamic programming.

Greedy vs. Exhaustive Search

Greedy algorithms focus on making the best local choice at each decision point. In the absence of a correctness proof such greedy algorithms are very likely to fail.

Dynamic programming gives us a way to design custom algorithms which systematically search all possibilities (thus guaranteeing correctness) while storing results to avoid recomputing (thus providing efficiency).

Recurrence Relations

A recurrence relation is an equation which is defined in terms of itself. They are useful because many natural functions are easily expressed as recurrences:

Polynomials: $a_n = a_{n-1} + 1, a_1 = 1 \longrightarrow a_n = n$

Exponentials: $a_n = 2a_{n-1}, a_1 = 2 \longrightarrow a_n = 2^n$

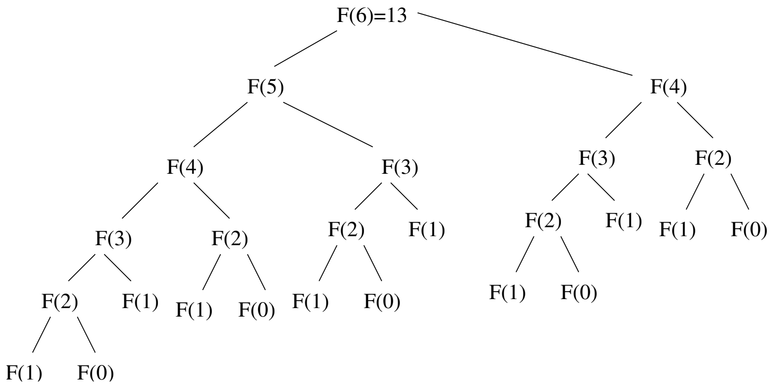
Weird: $a_n = na_{n-1}, a_1 = 1 \longrightarrow a_n = n!$

Computer programs can easily evaluate the value of a given recurrence even without the existence of a nice closed form.

Computing Fibonacci Numbers

$$F_n = F_{n-1} + F_{n-2}, F_0 = 0, F_1 = 1$$

Implementing this as a recursive procedure is easy, but slow because we keep calculating the same value over and over.



How Slow?

$$F_{n+1}/F_n \approx \phi = (1+\sqrt{5})/2 \approx 1.61803$$

Thus $F_n \approx 1.6^n$.
Since our recursion tree has 0 and 1 as leaves, computing F_n requires $\approx 1.6^n$ calls!

What about Dynamic Programming?

We can calculate F_n in linear time by storing small values:

$F_0 = 0$
$F_1 = 1$
For $i = 1$ to n

$$F_i = F_{i-1} + F_{i-2}$$

Moral: we traded space for time.

Why I Love Dynamic Programming

Dynamic programming is a technique for efficiently computing recurrences by storing partial results.

Once you understand dynamic programming, it is usually easier to reinvent certain algorithms than try to look them up!

I have found dynamic programming to be one of the most useful algorithmic techniques in practice:

- Morphing in computer graphics.
- Data compression for high density bar codes.
- Designing genes to avoid or contain specified patterns.

Avoiding Recomputation by Storing Results

The trick to dynamic programmming is to see that the naive recursive algorithm repeatedly computes the same subproblems over again, so storing the answers in a table instead of recomputing leads to an efficient algorithm.

We first hunt for a correct recursive algorithm, then we try to speed it up by using a results matrix.

Binomial Coefficients

The most important class of counting numbers are the *binomial coefficients*, where $\binom{n}{k}$ counts the number of ways to choose k things out of n possibilities.

- *Committees* – How many ways are there to form a k-member committee from n people? By definition, $\binom{n}{k}$.
- *Paths Across a Grid* – How many ways are there to travel from the upper-left corner of an $n \times m$ grid to the lower-right corner by walking only down and to the right? Every path must consist of $n + m$ steps, n downward and m to the right, so there are $\binom{n+m}{n}$ such sets/paths.

Computing Binomial Coefficients

Since $\binom{n}{k} = n!/((n-k)!k!)$, in principle you can compute them straight from factorials.

However, intermediate calculations can *easily* cause arithmetic overflow even when the final coefficient fits comfortably within an integer.

Pascal's Triangle

No doubt you played with this arrangement of numbers in high school. Each number is the sum of the two numbers directly above it:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Pascal's Recurrence

A more stable way to compute binomial coefficients is using the recurrence relation implicit in the construction of Pascal's triangle, namely, that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

It works because the nth element either appears or does not appear in one of the $\binom{n}{k}$ subsets of k elements.

Basis Case

No recurrence is complete without basis cases.

How many ways are there to choose 0 things from a set? Exactly one, the empty set.

The right term of the sum drives us up to $\binom{k}{k}$. How many ways are there to choose k things from a k-element set? Exactly one, the complete set.

Binomial Coefficients Implementation

```
long binomial_coefficient(n,m)
int n,m; (* compute n choose m *)
{
        int i,j; (* counters *)
        long bc[MAXN][MAXN]; (* table of binomial coefficients *)

        for (i=0; i<=n; i++) bc[i][0] = 1;

        for (j=0; j<=n; j++) bc[j][j] = 1;

        for (i=1; i<=n; i++)
        for (j=1; j<i; j++)
                bc[i][j] = bc[i-1][j-1] + bc[i-1][j];

        return( bc[n][m] );
}
```

Three Steps to Dynamic Programming

1. Formulate the answer as a recurrence relation or recursive algorithm.
2. Show that the number of different instances of your recurrence is bounded by a polynomial.
3. Specify an order of evaluation for the recurrence so you always have what you need.

The Gas Station Problem

Suppose we are driving from NY to Florida, and we know the positions of all gas stations g_1 to g_n we will pass on route. What is the minimum number of gas stations we will have to fill up at to make it down there?

The m_i be the mile marker where station g_i is located, and R be the driving range of the car on a full tank in miles.

Recursive Idea

Let $G[i]$ be the minimum number of fillups needed to get to gas station g_i.
If we know the best cost to get to all gas stations before i that are in driving range, we can compute $G[i]$:

$$G[i] = \min_{j<i, \text{where } ((m_j - m_i) < R)} G[j] + 1$$

The boundary case is $G[1] = 0$.

Observations

- This gives an $O(n^2)$ algorithm to minimize the number of stations.
- This problem *could* have been solved as BFS/shortest path on an unweighted directed graph.
- Many dynamic progrmaming algorithms are in fact shortest path problems on the right DAG, in disguise.
- The dynamic programming formulation can be extended (with additional state) to finding the cheapest trip if gas stations charge different prices.