The Logic of Quantified Statements

CSE 215, Foundations of Computer Science

Stony Brook University

http://www.cs.stonybrook.edu/~cse215

The Logic of Quantified Statements

All men are mortal.

Socrates is a man.

∴ Socrates is mortal.

- Propositional calculus: analysis of ordinary compound statements
- Predicate calculus: symbolic analysis of predicates and quantified statements
  - *P* is a predicate symbol

    P stands for "*is a student at SBU*"

    P(x) stands for "*x is a student at SBU*"
  - *x* is a predicate variable

Predicates and Quantified Statements

- A **predicate** is a sentence that contains a finite number of variables and becomes a **statement** when specific values are substituted for the variables.
- The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.
- Example:

  P(x) is the predicate "$x^2 > x$", *x* has as a domain the set **R** of all real numbers

  $P(2)$: $2^2 > 2$. *True.*

  $P(1/2)$: $(1/2)^2 > 1/2$. *False.*

Truth Set of a Predicate

- If $P(x)$ is a predicate and x has domain D, the truth set of $P(x)$, $\{x \in D \mid P(x)\}$, is the set of all elements of D that make $P(x)$ true when they are substituted for x.
- Example:

  $Q(n)$ is the predicate for "n is a factor of 8."

  if the domain of n is the set **Z** of all integers

  The truth set is $\{1, 2, 4, 8, -1, -2, -4, -8\}$

The Universal Quantifier: ∀

- Quantifiers are words that refer to quantities ("some" or "all") and tell for how many elements a given predicate is true.
- **Universal quantifier:** ∀ "for all"
- Example:

  ∀ human beings x, x is mortal.

  "All human beings are mortal"
- If H is the set of all human beings

  $\forall x \in H$, x is mortal

Universal statements

- A **universal statement** is a statement of the form

"$\forall x \in D, Q(x)$" where $Q(x)$ is a predicate and D is the domain of x.

  - $\forall x \in D, Q(x)$ is true if, and only if, $Q(x)$ is true for every x in D
  - $\forall x \in D, Q(x)$ is false if, and only if, $Q(x)$ is false for at least one x in D (the value for x is a **counterexample**)

- Example:

  $\forall x \in D, x^2 \geq x$ for $D = \{1, 2, 3, 4, 5\}$

  $1^2 \geq 1, \quad 2^2 \geq 2, \quad 3^2 \geq 3, \quad 4^2 \geq 4, \quad 5^2 \geq 5$

  - Hence "$\forall x \in D, x^2 \geq x$" is true.

The Existential Quantifier: ∃

- **Existential quantifier:** ∃ "there exists"
- Example:
  - "There is a student in CSE 215"

  ∃ a person *p* such that *p* is a student in CSE 215

  $\exists p \in P$ a person *p* such that *p* is a student in CSE 215

  where *P* is the set of all people

The Existential Quantifier: ∃

- An **existential statement** is a statement of the form

"$\exists x \in D$ such that $Q(x)$" where $Q(x)$ is a predicate and D the domain of x

  - $\exists x \in D$ s.t. $Q(x)$ is true if, and only if, $Q(x)$ is true for at least one x in D
  - $\exists x \in D$ s.t. $Q(x)$ is false if, and only if, $Q(x)$ is false for all x in D
- Example:
  - $\exists m \in \boldsymbol{Z}$ such that $m^2 = m$

  $1^2 = 1$ True

    - Notation: such that = s.t.

Universal Conditional Statements

- **Universal conditional statement:**

$\forall x$, *if* $P(x)$ *then* $Q(x)$

- Example:

  If a real number is greater than 2 then its square is greater than 4.

$$\forall x \in \boldsymbol{R}, \text{ if } x > 2 \text{ then } x^2 > 4$$

Equivalent Forms of Universal and Existential Statements

- $\forall x \in U$, *if P(x) then Q(x)* can be rewritten in the form

$\forall x \in D$, *Q(x)* by narrowing *U* to be the domain *D* consisting of all values of the variable *x* that make *P(x)* true.

  - Example: $\forall x$, *if x is a square then x is a rectangle*

    $\forall$ *squares x, x is a rectangle.*

- $\exists x$ *such that P(x) and Q(x)* can be rewritten in the form

"$\exists x \in D$ *such that Q(x)* where *D* consists of all values of the variable *x* that make *P(x)* true

Implicit Quantification

- $P(x) \Rightarrow Q(x)$ means that every element in the truth set of $P(x)$ is in the truth set of $Q(x)$, or, equivalently, $\forall x, P(x) \rightarrow Q(x)$
- $P(x) \Leftrightarrow Q(x)$ means that $P(x)$ and $Q(x)$ have identical truth sets, or, equivalently, $\forall x, P(x) \leftrightarrow Q(x)$.

Negations of Quantified Statements

- Negation of a Universal Statement:

The negation of a statement of the form $\forall x \in D, Q(x)$

is logically equivalent to a statement of the form

$\exists x \in D, \sim Q(x)$:

$$\sim(\forall x \in D, Q(x)) \equiv \exists x \in D, \sim Q(x)$$

- Example:
  - "All mathematicians wear glasses"
  - Its negation is: "There is at least one mathematician who does not wear glasses"
  - Its negation is NOT "No mathematicians wear glasses"

Negations of Quantified Statements

- Negation of an Existential Statement

The negation of a statement of the form $\exists x \in D, Q(x)$

is logically equivalent to a statement of the form $\forall x \in D, \sim Q(x)$:

$$\sim(\exists x \in D, Q(x)) \equiv \forall x \in D, \sim Q(x)$$

- Example:
  - “Some snowflakes are the same.”
  - Its negation is:

“No snowflakes are the same” ≡ “All snowflakes are different.”

Negations of Quantified Statements

- More Examples:
  - ~(∀ primes *p*, *p* is odd) ≡ ∃ a prime *p* such that *p* is **not** odd
  - ~(∃ a triangle *T* such that the sum of the angles of *T* equals 200°) ≡ ∀ triangles *T*, the sum of the angles of *T* **does not** equal 200°
  - ~(∀ politicians *x*, *x* is **not** honest) ≡ ∃ a politician *x* such that *x* is honest **(by double negation)**
  - ~(∀ computer programs *p*, *p* is finite) ≡ ∃ a computer program *p* that is not finite
  - ~(∃ a computer hacker *c*, *c* is over 40) ≡ ∀ computer hacker *c*, *c* is 40 or under
  - ~(∃ an integer *n* between 1 and 37 such that 1,357 is divisible by *n*) ≡ ∀ integers *n* between 1 and 37, 1,357 is not divisible by *n*

Negations of Universal Conditional Statements

- $\sim(\forall x, P(x) \rightarrow Q(x)) \equiv \exists x$ such that $P(x) \wedge \sim Q(x)$
- Proof:

$\sim(\forall x, P(x) \rightarrow Q(x)) \equiv \exists x$ such that $\sim(P(x) \rightarrow Q(x))$

$\sim(P(x) \rightarrow Q(x)) \equiv \sim(\sim P(x) \vee Q(x)) \equiv \sim\sim P(x) \wedge \sim Q(x)) \equiv P(x) \wedge \sim Q(x)$

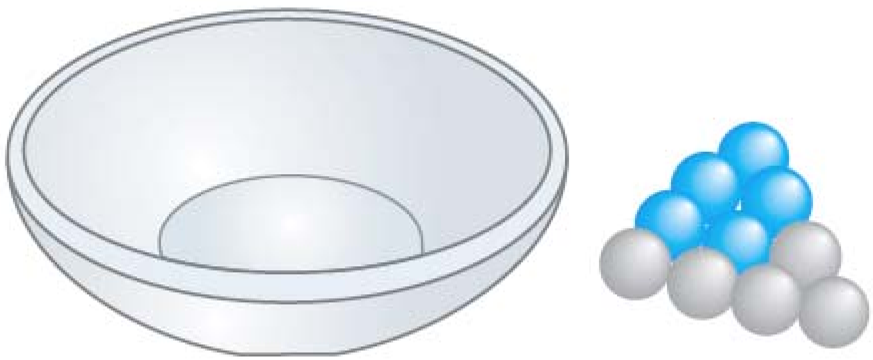
- Examples:
  - $\sim(\forall$ people p, if p is blond then p has blue eyes) $\equiv$
    $\exists$ a person p such that p is blond and p does not have blue eyes
- $\sim$(If a computer program has more than 100,000 lines, then it contains a bug) $\equiv$ There is at least one computer program that has more than 100,000 lines and does not contain a bug

The Relation among ∀, ∃, ∧, and ∨

- $D = \{x_1, x_2, \ldots, x_n\}$ and $\forall x \in D, Q(x) \quad \equiv$
  $Q(x_1) \land Q(x_2) \land \cdots \land Q(x_n)$

- $D = \{x_1, x_2, \ldots, x_n\}$ and $\exists x \in D$ such that $Q(x) \quad \equiv$
  $Q(x_1) \lor Q(x_2) \lor \cdots \lor Q(x_n)$

Vacuous Truth of Universal Statements



All the balls in the bowl are blue True

$\forall x$ in D, if $P(x)$ then $Q(x)$ is *vacuously true* or *true by default* if, and only if, $P(x)$ is false for every x in D

Variants of Universal Conditional Statements

- Universal conditional statement: $\forall x \in D$, if $P(x)$ then $Q(x)$
- **Contrapositive:** $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$

  $\forall x \in D$, if $P(x)$ then $Q(x) \equiv \forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$

  Proof: for any x in D by the logical equivalence between statement and its contrapositive
- **Converse:** $\forall x \in D$, if $Q(x)$ then $P(x)$.
- **Inverse:** $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$.
- Example:

  $\forall x \in R$, if $x > 2$ then $x^2 > 4$

  Contrapositive: $\forall x \in R$, if $x^2 \leq 4$ then $x \leq 2$

  Converse: $\forall x \in R$, if $x^2 > 4$ then $x > 2$

  Inverse: $\forall x \in R$, if $x \leq 2$ then $x^2 \leq 4$

Necessary and Sufficient Conditions

- Necessary condition:

"∀x, r (x) is a **necessary condition** for s(x)" means

"∀x, if ~r (x) then ~s(x)" ≡ "∀x, if s(x) then r(x)" (*)

(*)(by contrapositive and double negation)

- Sufficient condition:

"∀x, r (x) is a **sufficient condition** for s(x)" means

"∀x, if r (x) then s(x)"

Necessary and Sufficient Conditions

- Examples:
  - Squareness is a **sufficient condition** for rectangularity;

  Formal statement: ∀x, if x is a square, then x is a rectangle
  - Being at least 35 years old is a **necessary condition** for being President of the United States

  ∀ people x, if x is younger than 35, then x cannot be President of the United States ≡

  ∀ people x, if x is President of the United States then x is at least 35 years old (by contrapositive)

Only If

- Only If:

"$\forall$x, r(x) **only if** s(x)" means

"$\forall$x, if ∼s(x) then ∼r (x)" $\equiv$ "$\forall$x, if r(x) then s(x)."

- Example:

A product of two numbers is 0 only if one of the numbers is 0.

If neither of two numbers is 0, then the product of the numbers is not 0 $\equiv$

If a product of two numbers is 0, then one of the numbers is 0 (by contrapositive)

Statements with Multiple Quantifiers

- Example:

"There is a person supervising every detail of the production process"

  - What is the meaning?

"There is one single person who supervises all the details of the production process"?

OR

"For any particular production detail, there is a person who supervises that detail, but there might be different supervisors for different details"?

NATURAL LANGUAGE IS AMBIGUOUS

LOGIC IS CLEAR

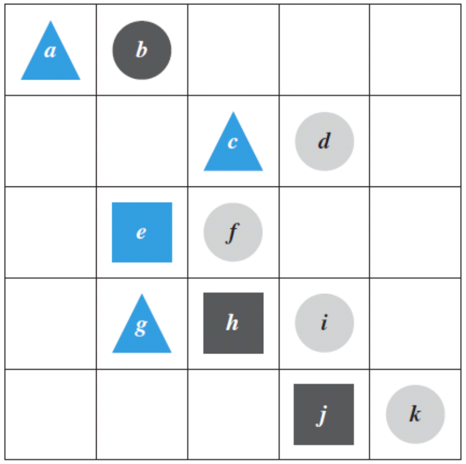
Statements with Multiple Quantifiers

- Quantifiers are performed in the order in which the quantifiers occur:
- Example:

  ∀x in set D, ∃y in set E such that x and y satisfy property P(x, y)

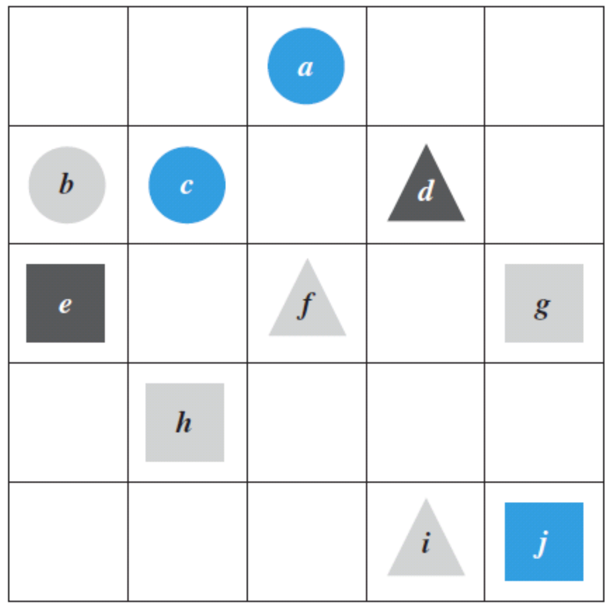
Tarski's World

- Blocks of various sizes, shapes, and colors located on a grid



- $\forall t$, Triangle(t) $\rightarrow$ Blue(t) TRUE
- $\forall x$, Blue(x) $\rightarrow$ Triangle(x). FALSE
- $\exists y$ such that Square(y) $\wedge$ RightOf(d, y). TRUE
- $\exists z$ such that Square(z) $\wedge$ Gray(z). FALSE

Statements with Multiple Quantifiers in Tarski's World



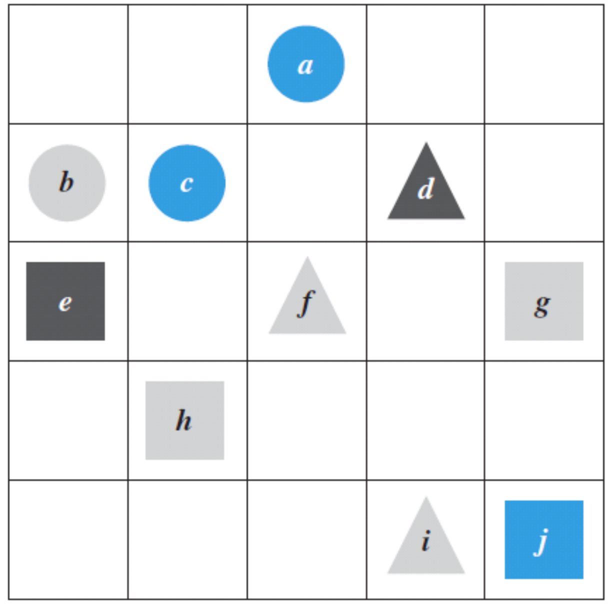
∀∃

- For all triangles x, there is a square y such that x and y have the same color

TRUE

Given $x =$	choose $y =$	and check that y is the same color as x.
d	e	yes ✓
f or i	h or g	yes ✓

Statements with Multiple Quantifiers in Tarski's World



∃∀

- There is a triangle x such that for all circles y, x is to the right of y

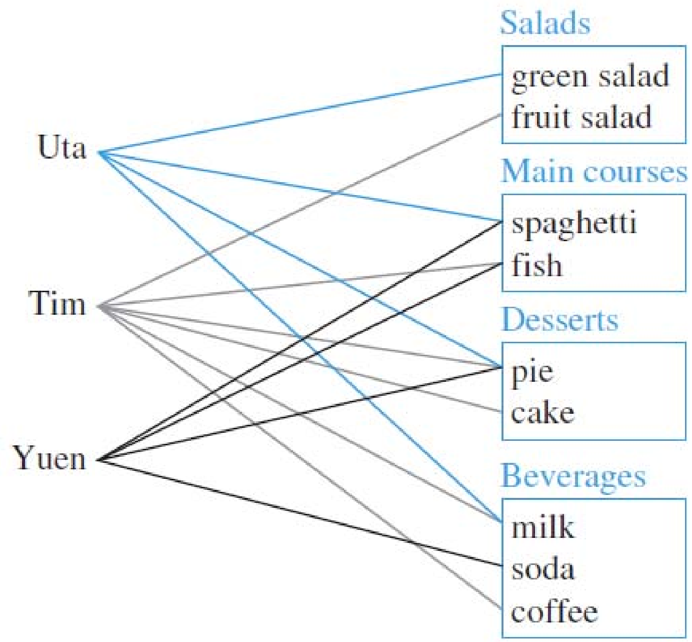
TRUE

Choose $x =$	Then, given $y =$	check that x is to the right of y.
d or i	a	yes ✓
	b	yes ✓
	c	yes ✓

Interpreting Statements with Two Different Quantifiers

- ∀x in D, ∃y in E such that P(x, y)
  - for whatever element x in D you must find an element y in E that "works" for that particular x
- ∃x in D such that ∀y in E, P(x, y)
  - find one particular x in D that will "work" no matter what y in E anyone might choose

Interpreting Statements with Two Different Quantifiers



- ∃ an item I such that ∀ students S, S chose I . TRUE
- ∃ a student S such that ∀ stations Z, ∃ an item I in Z such that S chose I TRUE
- ∀ students S and ∀ stations Z, ∃ an item I in Z such that S chose I . FALSE

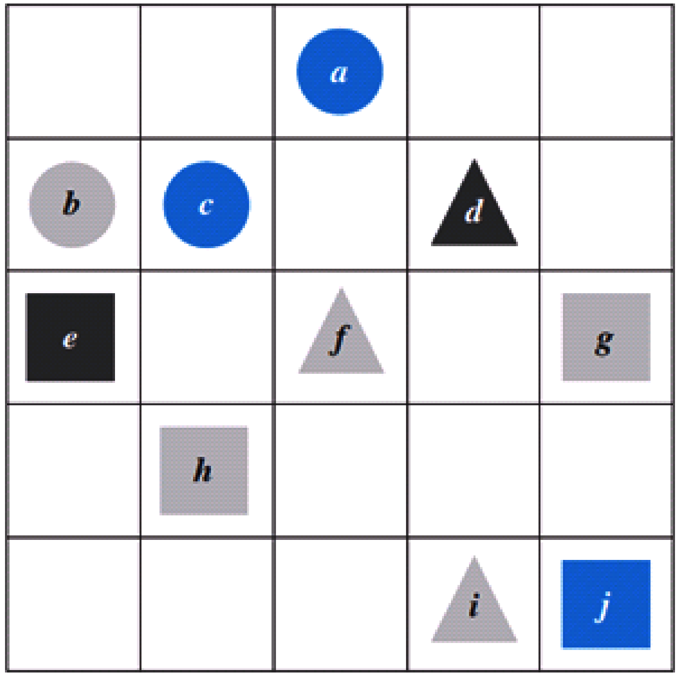
Statements with Multiple Quantifiers

- Quantifiers are performed in the order in which the quantifiers occur:
- Examples of statements with two quantifiers:
  - ∀x in D, ∃y in E such that P(x, y)

  for whatever element x in D you must find an element y in E that "works" for that particular x
  - ∃x in D such that ∀y in E, P(x, y)

  find one particular x in D that will "work" no matter what y in E anyone might choose

Statements with Multiple Quantifiers in Tarski's World



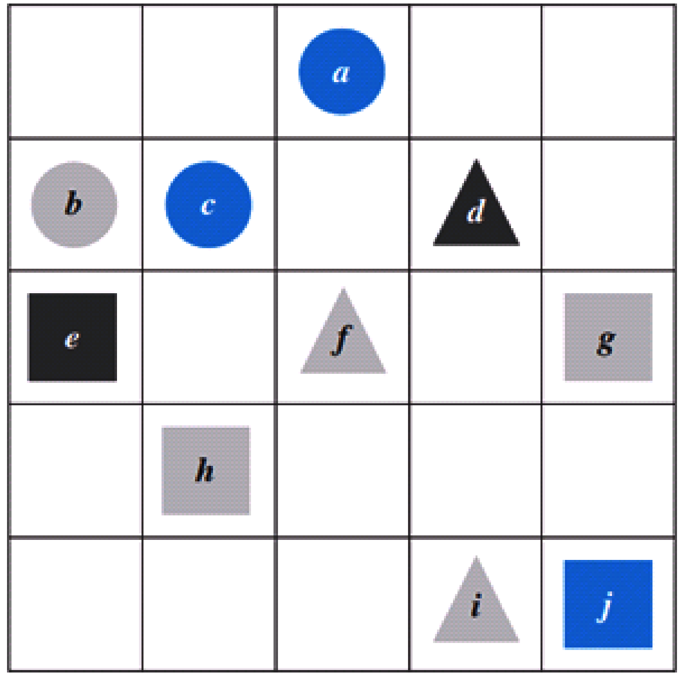
∀∃

- For all triangles x, there is a square y such that x and y have the same color

TRUE

Given $x =$	**choose $y =$**	**and check that y is the same color as x.**
d	e	yes ✓
f or i	h or g	yes ✓

Statements with Multiple Quantifiers in Tarski's World



$\exists\forall$

- There is a triangle x such that for all circles y, x is to the right of y

TRUE

Choose $x =$	Then, given $y =$	check that x is to the right of y.
d or i	a	yes ✓
	b	yes ✓
	c	yes ✓

Negations of Multiply-Quantified Statements

- Apply negation to quantified statements from left to right:

$\sim(\forall x$ in D, $\exists y$ in E such that $P(x, y))$

$\equiv \exists x$ in D such that $\sim(\exists y$ in E such that $P(x, y))$

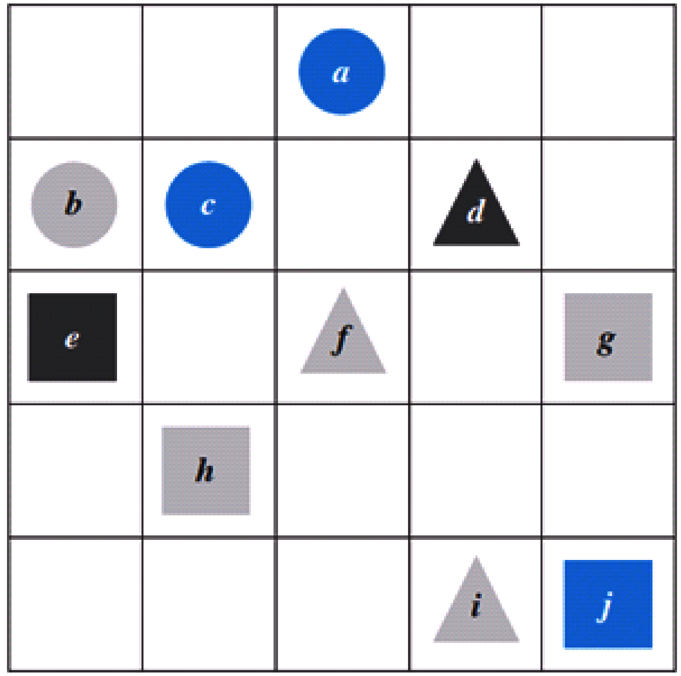
$\equiv \exists x$ in D such that $\forall y$ in E, $\sim P(x, y)$.

$\sim(\exists x$ in D such that $\forall y$ in E, $P(x, y))$

$\equiv \forall x$ in D, $\sim(\forall y$ in E, $P(x, y))$

$\equiv \forall x$ in D, $\exists y$ in E such that $\sim P(x, y)$.

Negating Statements in Tarski's World



- For all squares x, there is a circle y such that x and y have the same color

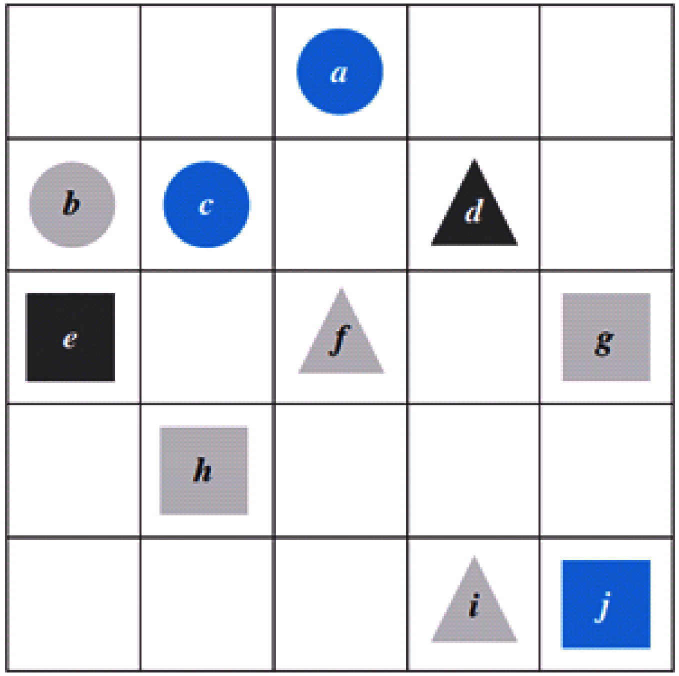
Negation:

∃ a square x such that ∼(∃ a circle y such that x and y have the same color)

≡ ∃ a square x such that ∀ circles y, x and y do not have the same color

TRUE: Square e is black and no circle is black.

Negating Statements in Tarski's World



- There is a triangle x such that for all squares y, x is to the right of y

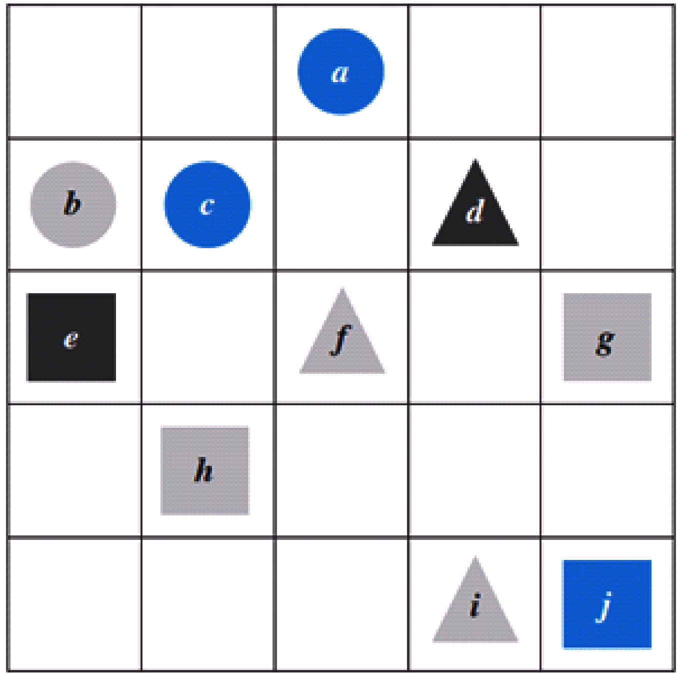
Negation:

∀ triangles x, ~ (∀ squares y, x is to the right of y)

≡ ∀ triangles x, ∃ a square y such that x is not to the right of y

TRUE

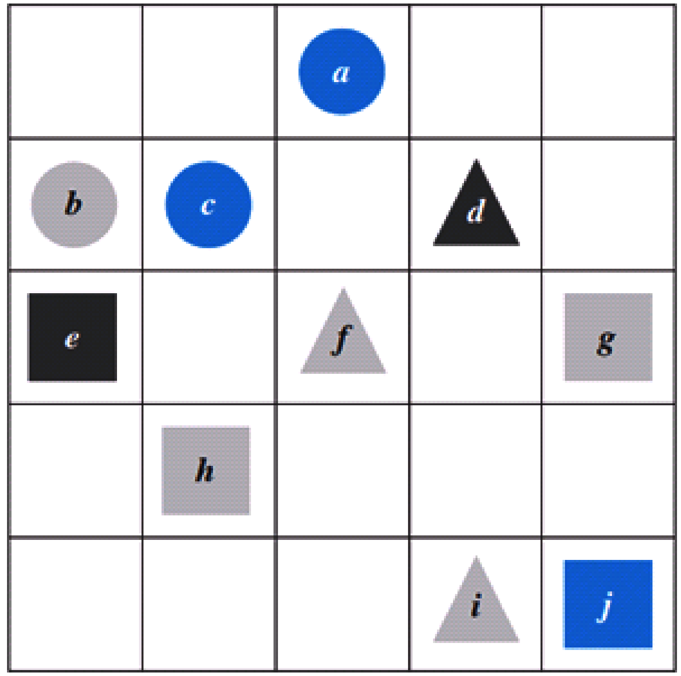
Quantifier Order in Tarski's World



- For every square x there is a triangle y such that x and y have different colors

- There exists a triangle y such that for every square x, x and y have different colors

Quantifier Order in Tarski's World



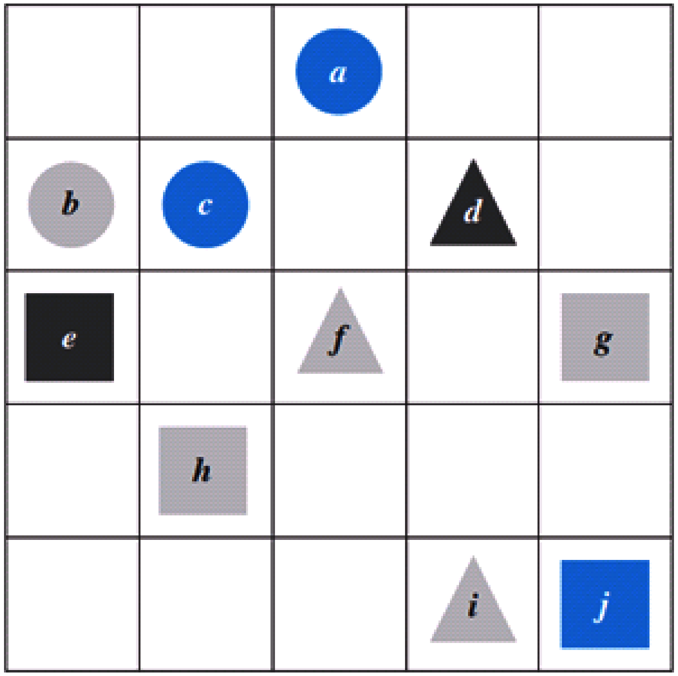
- For every square x there is a triangle y such that x and y have different colors

TRUE

- There exists a triangle y such that for every square x, x and y have different colors

FALSE

Formalizing Statements in Tarski's World



- Triangle(x) means "x is a triangle"
- Circle(x) means "x is a circle"
- Square(x) means "x is a square"
- Blue(x) means "x is blue"
- Gray(x) means "x is gray"
- Black(x) means "x is black"
- RightOf(x, y) means "x is to the right of y"
- Above(x, y) means "x is above y"
- SameColorAs(x, y) means "x has the same color as y"
- x = y denotes the predicate "x is equal to y"

Formalizing Statements in Tarski's World

- For all circles x, x is above f

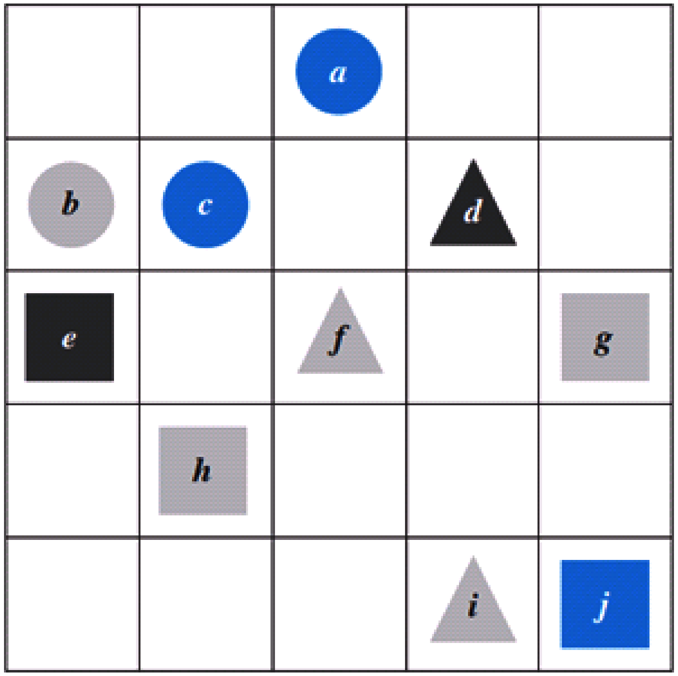
$\forall x(\text{Circle}(x) \rightarrow \text{Above}(x, f))$

- Negation:

$\sim(\forall x(\text{Circle}(x) \rightarrow \text{Above}(x, f)))$

$\equiv \exists x \sim (\text{Circle}(x) \rightarrow \text{Above}(x, f))$

$\equiv \exists x(\text{Circle}(x) \wedge \sim\text{Above}(x, f))$



Formalizing Statements in Tarski's World

- There is a square x such that x is black

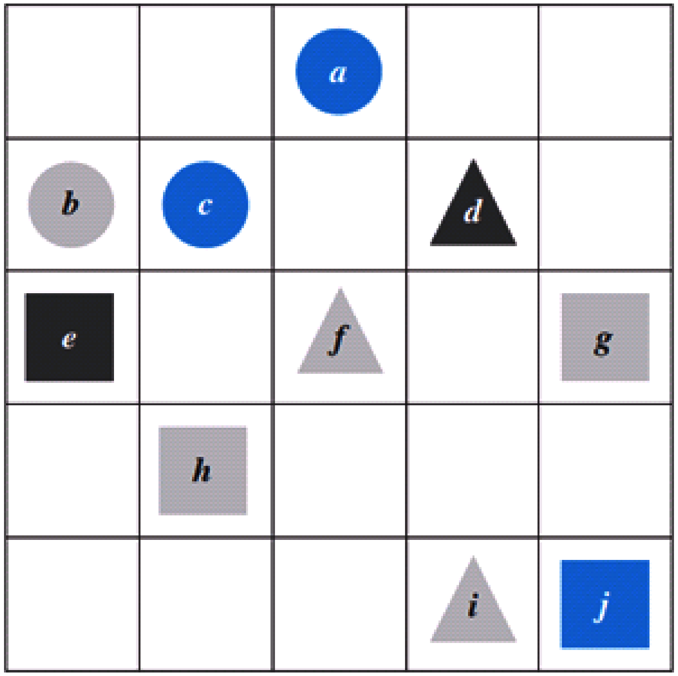
$\exists x(\text{Square}(x) \wedge \text{Black}(x))$

- Negation:

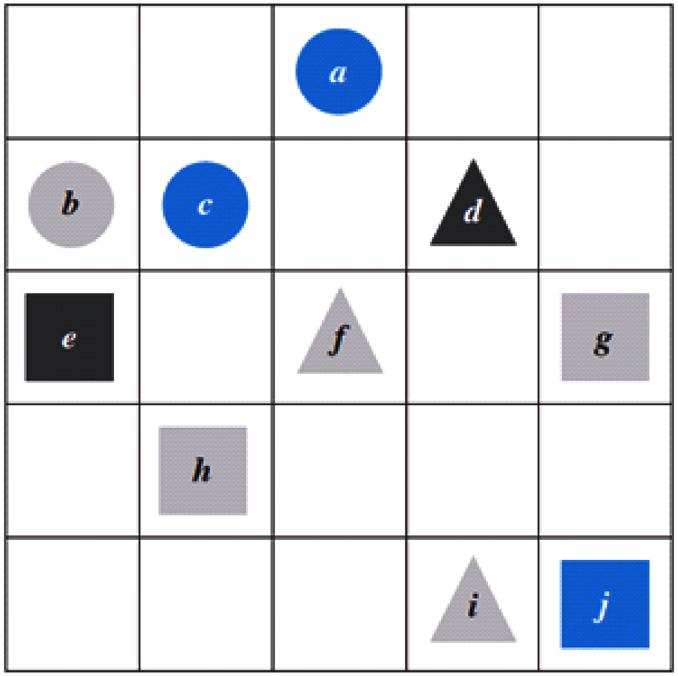
$\sim(\exists x(\text{Square}(x) \wedge \text{Black}(x)))$

$\equiv \forall x \sim (\text{Square}(x) \wedge \text{Black}(x))$

$\equiv \forall x(\sim\text{Square}(x) \vee \sim\text{Black}(x))$



Formalizing Statements in Tarski's World



- For all circles x, there is a square y such that x and y have the same color

$\forall x(\text{Circle}(x) \rightarrow \exists y(\text{Square}(y) \wedge \text{SameColor}(x, y)))$

- Negation:

$\sim(\forall x(\text{Circle}(x) \rightarrow \exists y(\text{Square}(y) \wedge \text{SameColor}(x, y))))$

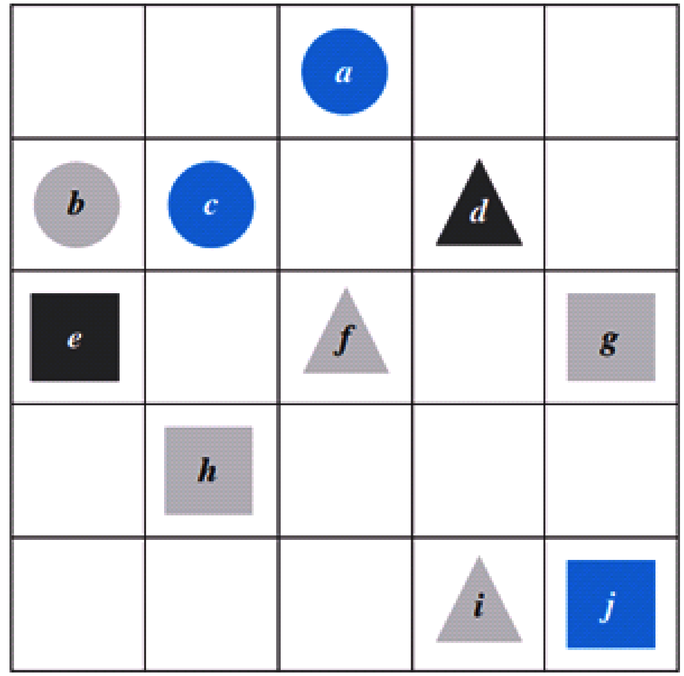
$\equiv \exists x \sim (\text{Circle}(x) \rightarrow \exists y(\text{Square}(y) \wedge \text{SameColor}(x, y)))$

$\equiv \exists x(\text{Circle}(x) \wedge \sim(\exists y(\text{Square}(y) \wedge \text{SameColor}(x, y))))$

$\equiv \exists x(\text{Circle}(x) \wedge \forall y(\sim(\text{Square}(y) \wedge \text{SameColor}(x, y))))$

$\equiv \exists x(\text{Circle}(x) \wedge \forall y(\sim\text{Square}(y) \vee \sim\text{SameColor}(x, y)))$

Formalizing Statements in Tarski's World



- There is a square x such that for all triangles y, x is to right of y

$\exists x(\text{Square}(x) \wedge \forall y(\text{Triangle}(y) \rightarrow \text{RightOf}(x, y)))$

- Negation:

$\sim(\exists x(\text{Square}(x) \wedge \forall y(\text{Triangle}(y) \rightarrow \text{RightOf}(x, y))))$

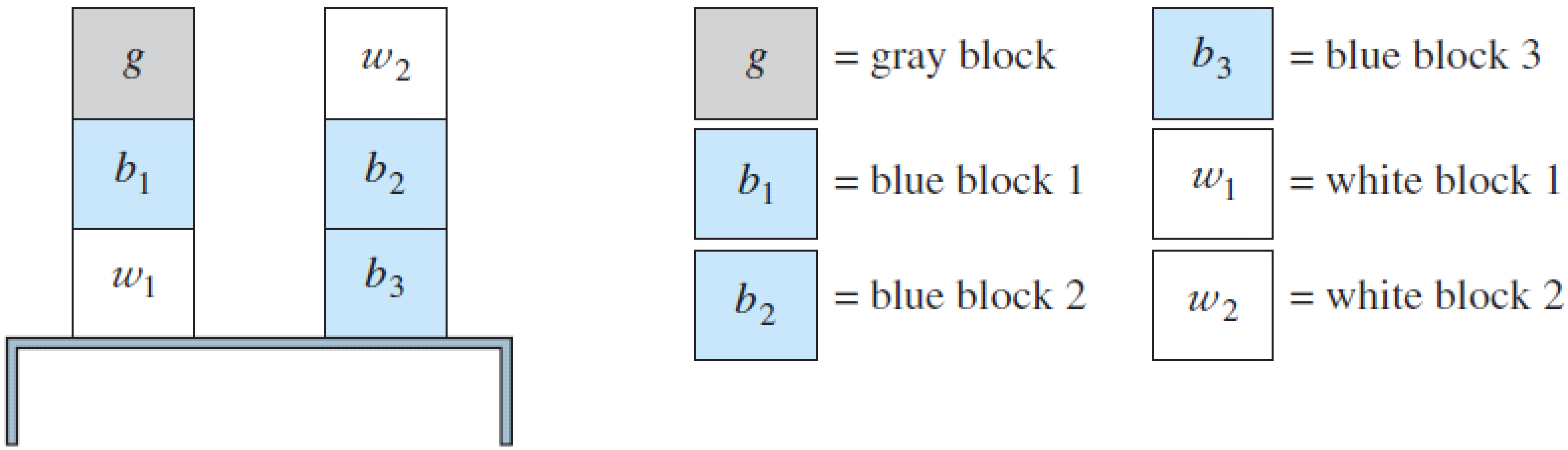
$\equiv \forall x \sim (\text{Square}(x) \wedge \forall y(\text{Triangle}(x) \rightarrow \text{RightOf}(x, y)))$

$\equiv \forall x(\sim\text{Square}(x) \vee \sim(\forall y(\text{Triangle}(y) \rightarrow \text{RightOf}(x, y))))$

$\equiv \forall x(\sim\text{Square}(x) \vee \exists y(\sim(\text{Triangle}(y) \rightarrow \text{RightOf}(x, y))))$

$\equiv \forall x(\sim\text{Square}(x) \vee \exists y(\text{Triangle}(y) \wedge \sim\text{RightOf}(x, y)))$

Prolog (Programming in logic)



- Prolog statements:

isabove(g, b_1). color(g, gray). color(b_3, blue). isabove(b_1,w_1).

color(b_1, blue). color(w_1, white). isabove(w_2, b_2). color(b_2, blue).

color(w_2, white). isabove(b_2, b_3).

isabove(X, Z) :- isabove(X,Y), isabove(Y, Z).

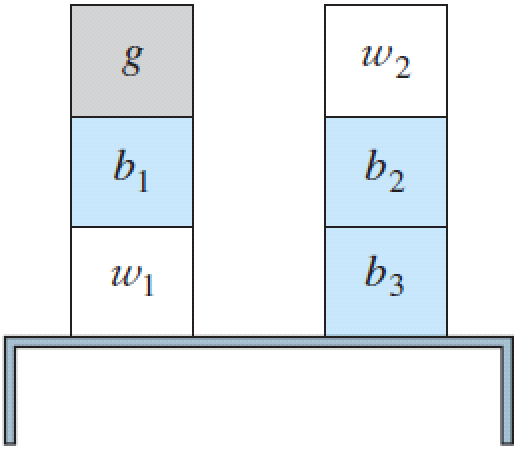
?- color(b_1, blue).

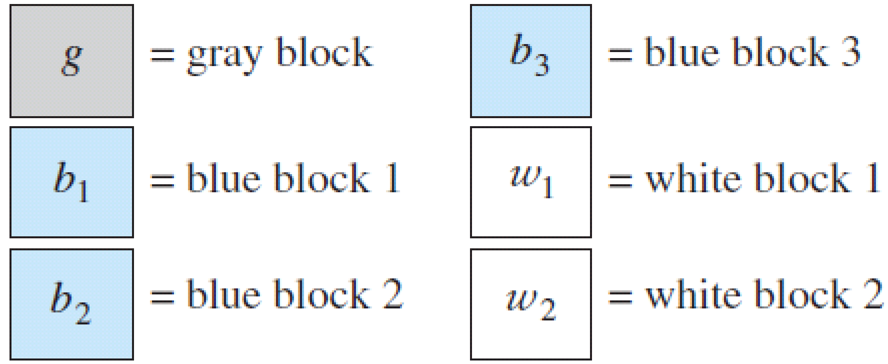
TRUE

?- isabove(X,w_1).

X=b_1; X=g

Prolog (Programming in logic)





?- isabove(b_2,w_1).

TRUE

?- color(w_1, X) .

X = white

?- color(X, blue).

X = b1;   X = b2;   X = b3.

Arguments with Quantified Statements

- Universal instantiation: if some property is true of everything in a set, then it is true of any particular thing in the set.
- Example:

  All men are mortal.

  Socrates is a man.

  ∴ Socrates is mortal.

Universal Modus Ponens

Formal Version	*Informal Version*
$\forall x$, if P(x) then Q(x).	If x makes P(x) true, then x makes Q(x) true.
P(a) for a particular a.	a makes P(x) true.
∴ Q(a).	∴ a makes Q(x) true.

- Example:

$\forall x$, if E(x) then S(x).	If an integer is even, then its square is even.
E(k),for a particular k.	k is a particular integer that is even.
∴ S(k).	∴ k^2 is even.

Universal Modus Tollens

Formal Version	*Informal Version*
∀x, if P(x) then Q(x).	If x makes P(x) true, then x makes Q(x) true.
~Q(a), for a particular a.	a does not make Q(x) true.
∴ ~P(a).	∴ a does not make P(x) true.

- Example:

∀x, if H(x) then M(x)	All human beings are mortal.
~M(Z)	Zeus is not mortal.
∴ ~H(Z).	∴ Zeus is not human.

Validity of Arguments with Quantified Statements

- An argument form is **valid**, if and only if, for any particular predicates substituted for the predicate symbols in the premises **if the resulting premise statements are all true, then the conclusion is also true**
- Using Diagrams to Test for Validity:

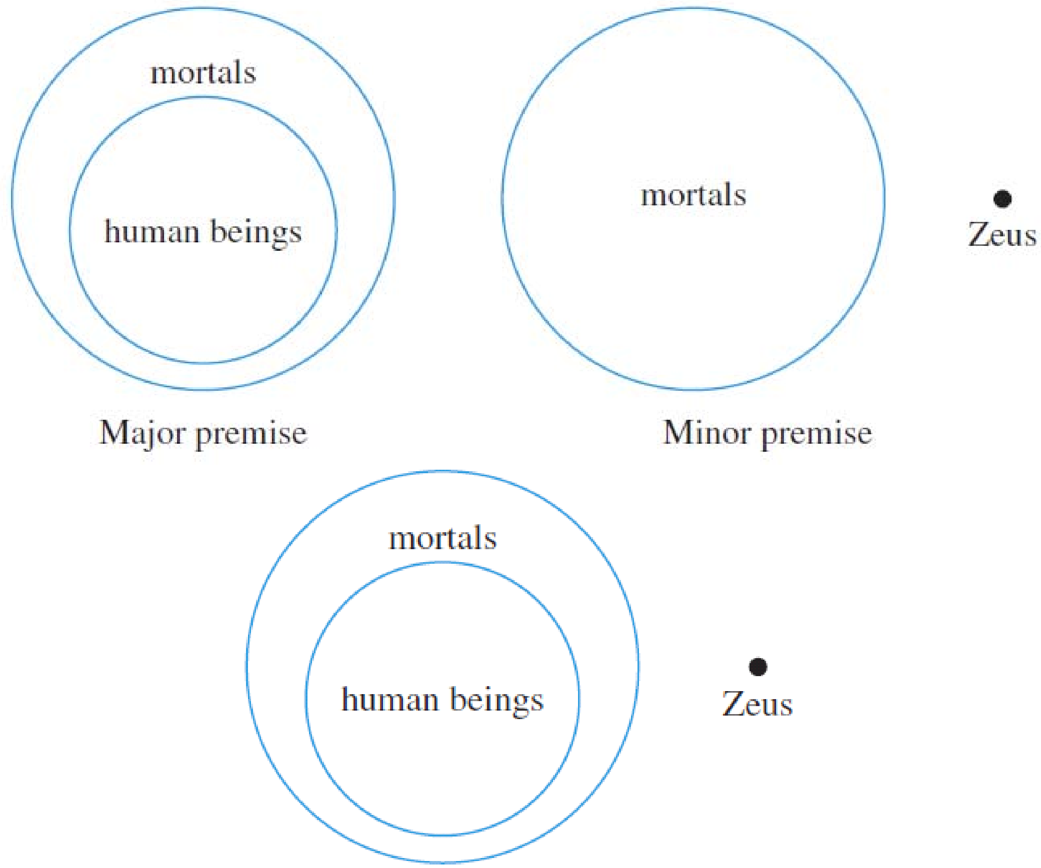
$$\forall \textit{ integers } n, n \textit{ is a rational number}$$

Using Diagrams to Test for Validity

All human beings are mortal.

Zeus is not mortal.

∴ Zeus is not a human being.

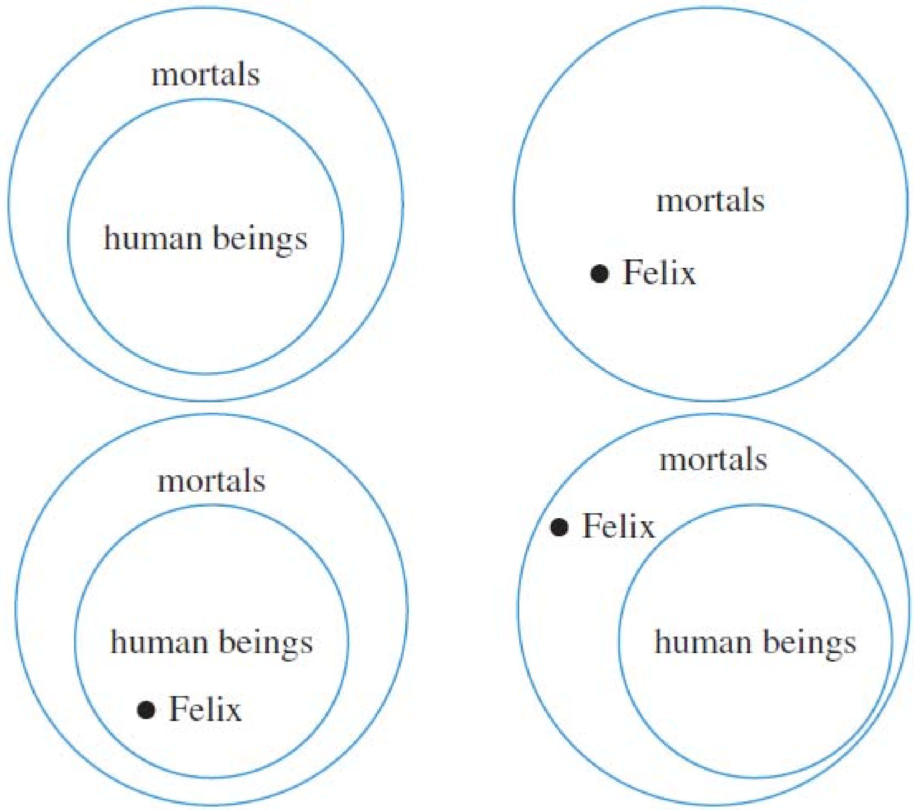


Using Diagrams to Show Invalidity

All human beings are mortal.

Felix is mortal.

∴ Felix is a human being.



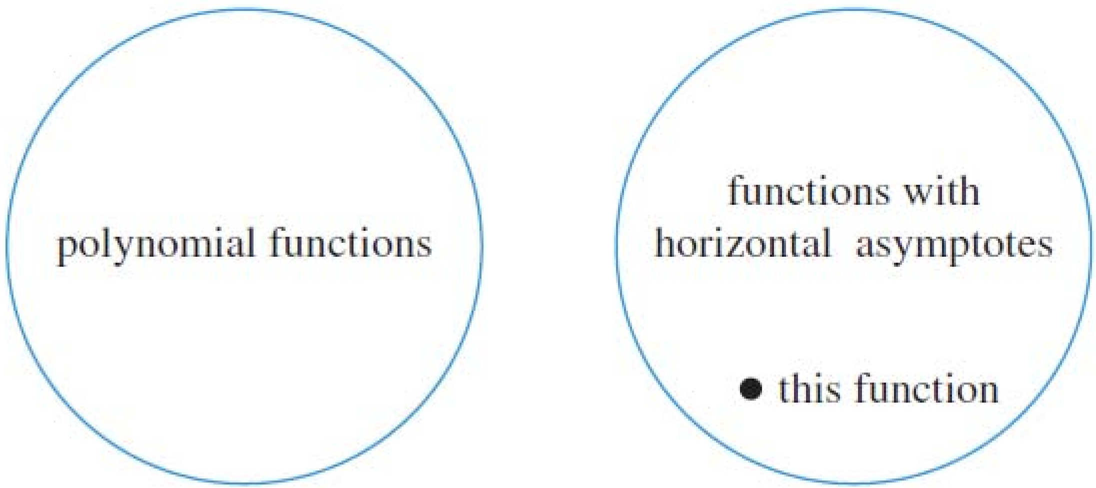
Using Diagrams to Test for Validity

- Universal modus tollens Example:

No polynomial functions have horizontal asymptotes.

This function has a horizontal asymptote.

∴ This function is not a polynomial function



Universal Transitivity

Formal Version	*Informal Version*
$\forall x\ P(x) \rightarrow Q(x)$.	Any x that makes P(x) true makes Q(x) true.
$\forall x Q(x) \rightarrow R(x)$.	Any x that makes Q(x) true makes R(x) true.
$\therefore \forall x\ P(x) \rightarrow R(x)$.	∴ Any x that makes P(x) true makes R(x) true.

- Example from Tarski's World:

∀x, if x is a triangle, then x is blue.

∀x, if x is blue, then x is to the right of all the squares.

∴ ∀x, if x is a triangle, then x is to the right of all the squares

Converse Error (Quantified Form)

Formal Version	*Informal Version*
∀x, if P(x) then Q(x).	If x makes P(x) true, then x makes Q(x) true.
Q(a) for a particular a.	a makes Q(x) true.
∴ P(a).	∴ a makes P(x) true.

invalid conclusion

Inverse Error (Quantified Form)

Formal Version	*Informal Version*
$\forall x$, if P(x) then Q(x).	If x makes P(x) true, then x makes Q(x) true.
~P(a), for a particular a.	a does not make P(x) true.
∴ ~Q(a).	∴ a does not make Q(x) true.

invalid conclusion