cse547
DISCRETE MATHEMATICS

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LECTURE 7

CHAPTER 2
SUMS

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SUMS

Part 3: Multiple Sums (1) - Lecture 7

Double Sum

Example 1

Double Sum - Two factors:

$$\sum_{1\leq i,\ j\leq 3} a_i b_j = a_1 b_1 + a_1 b_2 + a_1 b_3$$
$$+ a_2 b_1 + a_2 b_2 + a_2 b_3$$
$$+ a_3 b_1 + a_3 b_2 + a_3 b_3$$

Question

How can we express $\sum_{1\leq i,\ j\leq 3} a_i b_j$ in terms of single sums $\sum_i a_i$ and $\sum_j b_j$?

Double Sum Definition

We define for $1 \leq i,\ j \leq 3$

$$\sum_{1\leq i,\ j\leq 3} a_i b_j \;=\; \sum_{1\leq j\leq 3}\Big(\sum_{1\leq i\leq 3} a_i b_j\Big) \;=\; \sum_{1\leq j\leq 3}\Big(\sum_{1\leq i\leq 3} a_i b_j\Big)$$

General Definition for $i \in I,\quad j \in J$

$$\sum_{i,\ j} a_i b_j \;=\; \sum_i \sum_j a_i b_j \;=\; \sum_j \sum_i a_i b_j$$

where we write

$$\sum_{i,\ j} a_i b_j \quad \text{for} \quad \sum_{i\in I,\ j\in J} a_i b_j \quad \text{and} \quad \sum_i \sum_j a_i b_j \quad \text{for} \quad \sum_{i\in I}\Big(\sum_{j\in J} a_i b_j\Big)$$

Example 1

We evaluate the following for $i, j \in \{1, 2, 3\}$

$$\sum_{1 \le i,j \le 3} a_i b_j = a_1 b_1 + a_1 b_2 + a_1 b_3$$

$$+a_2 b_1 + a_2 b_2 + a_2 b_3$$

$$+a_3 b_1 + a_3 b_2 + a_3 b_3$$

$$= a_1(b_1 + b_2 + b_3) \quad \text{we pull out}$$

$$+a_2(b_1 + b_2 + b_3) \quad \text{the common factor}$$

$$+a_3(b_1 + b_2 + b_3)$$

$$= (b_1 + b_2 + b_3)(a_1 + a_2 + a_3)$$

$$= (a_1 + a_2 + a_3)(b_1 + b_2 + b_3)$$

Distributive Property

We have **proved** the following property

$$\sum_{1\leq i\leq 3}\sum_{1\leq j\leq 3} a_i b_j = (\sum_{1\leq i\leq 3} a_i)(\sum_{1\leq j\leq 3} b_j) = (\sum_{1\leq j\leq 3} b_j)(\sum_{1\leq i\leq 3} a_i)$$

Distributive Property for $1 \leq i, \ j \leq 3$

$$\sum_{i,j} a_i b_j = (\sum_i a_i)(\sum_j b_j)$$

Can we **generalize** it?

General Distributive Law

Now **our goal** is to prove the following

General Distributive Law

$$\sum_{i \in I,\ j \in J} a_i b_j = \Big(\sum_{i \in I} a_i\Big)\Big(\sum_{j \in J} b_j\Big)$$

In order to do so w need to bring in our **notation** and general definitions

We write

$$\sum_{i \in I} a_i = \sum_{P(i)} a_i = \sum_{i \in I} a_i [P(i)]$$

where

$\boxed{I = \{i : P(i)\}}$ $\longrightarrow$ P(i) is a predicate defining set I

and $[P(x)]$ is a **characteristic function** of P(i)

$$[P(x)] = \begin{cases} 1 & P(x) \ \ \textit{true} \\ 0 & P(x) \ \ \textit{false} \end{cases}$$

General Distributive Law

In write in a similar way

$$\sum_{j \in J} b_j = \sum_{Q(j)} b_j = \sum_{j} b_j[Q(j)]$$

where $J = \{j : Q(j)\}$ and Q(j) is a predicate defining set J of indices

We re-write the **General Distributive Law** as follows

$$\sum_{i \in I, j \in J} a_i b_j = (\sum_{i} a_i[P(i)])(\sum_{j} b_j[Q(j)])$$

Question : HOW TO RELATE LEFT SIDE TO RIGHT SIDE ?

Back top Example 1

Let's go back to our **Example 1**

We proved **Distributivity Property** for $1 \leq i, \ \ j \leq 3$

$$\sum_{1 \leq i,j \leq 3} a_i b_j \ = \ (\sum_{1 \leq i \leq 3} a_i)(\sum_{1 \leq j \leq 3} b_j)$$

Observe that we have here the following predicated defining the sets of indexes

P(i,j) $= (1 \leq i, \ j \leq 3) = (1 \leq i \leq 3) \cap (1 \leq j \leq 3)$

$P_1(i) = (1 \leq i \leq 3)$ and $P_2(j) = (1 \leq j \leq 3)$

Hence

$$P(i,j) = P_1(i) \cap P_2(j)$$

General Distributive Law

By definition

$$\sum_{P(i,j)} a_i b_j = \sum_{P_1(i)} \sum_{P_2(j)} a_i b_j$$

when $\boxed{P(i,j) = P_1(i) \cap P_2(j)}$

We want to prove the the following form of the

General Distributive Law

$$\boxed{\sum_{P(i,j)} a_i b_j = (\sum_{P_1(i)} a_i)(\sum_{P_2(j)} b_j)}$$

Distributive Law

Let $P(i,j) = P_1(i) \cap P_2(j)$

Observe that

$$[P_1(i) \cap P_2(j)] = [P_1(i)][P_2(j)]$$

Prove it as an exercise;

This is true for any characteristic functions

We use this fact and definitions in our calculations on the next slide

Proof of the Distributive Law

$$\sum_{P(i,j)} a_i b_j = \sum_{i,j} a_i b_j [P(i,j)]$$

$$= \sum_{i,j} a_i b_j [P_1(i)][P_2(j)]$$

$$= \sum_i \left(\sum_j a_i b_j [P_1(i)][P_2(j)] \right)$$

pull out $a_i[P_1(i)]$ independent on j

$$= \sum_i \left(a_i [P_1(i)] \sum_j b_j [P_2(j)] \right)$$

Proof of the Distributive Law

We have that

$$\sum_{P(i,j)} a_i b_j = \sum_i (a_i[P_1(i)] \sum_j b_j[P_2(j)])$$

pull out $\sum_j b_j[P_2(j)]$ independent on i

$$= (\sum_j b_j[P_2(j)])(\sum_i a_i[P_1(i)])$$

$$= (\sum_{P_1(i)} a_i)(\sum_{P_2(j)} b_j)$$

end of the **proof**

Distributive Law

We have **proved** our **General Distributive Law**

$$\sum_{P(i,\,j)} a_i b_j = \sum_{P_1(i)\cap P_2(j)} a_i b_j = (\sum_{P_1(i)} a_i)(\sum_{P_2(j)} b_j)$$

also written as

$$\sum_{i\in I, j\in J} a_i b_j = (\sum_{i\in I} a_i)(\sum_{j\in J} b_j)$$

Distributive Law Example

Example of application of the DistributivE Law

$$\sum_{i\in I, j\in J} a_i b_j = (\sum_{i\in I} a_i)(\sum_{j\in J} b_j)$$

Consider the following array (nxn)

$$A = \begin{bmatrix} a_1 a_1 & a_1 a_2 \ldots & a_1 a_n \\ a_2 a_1 & a_2 a_2 \ldots & a_2 a_n \\ \vdots & & \\ a_n a_1 & a_n a_2 \ldots & a_n a_n \end{bmatrix}$$

we have here $a_i = a_i \quad b_j = a_j$, where a_i, b_j denote sequences in the DistributivE Law

Goal : Find

$$\sum_{i,j} a_i a_j$$

Distributive Law Example

Sub-Goal : Find a simple formula for **sum** of all elements **above** or **on** main diagonal

$$\boxed{S_{\bigtriangledown} = \sum_{1 \leq i \leq j \leq n} a_i a_j}$$

OBSERVATION 1

$\boxed{a_i a_j = a_j a_i}$ for any i,j

We denote

$$\boxed{S_{\bigtriangleup} = \sum_{1 \leq j \leq i \leq n} a_i a_j}$$

sum of all elements **below** or **on** main diagonal

Distributive Law Example

We will now **prove** that

$$\boxed{S_{\bigtriangledown} = S_{\bigtriangleup}}$$

We now evaluate

$$\boxed{S_{\bigtriangledown}} = \sum_{1 \leq i \leq n,\ 1 \leq j \leq n,\ i \leq j} a_i a_j = \boxed{\sum_{P(i,j),\ i \leq j} a_i a_j}$$

for

$$P(i,j) = (1 \leq i \leq n) \cap (1 \leq j \leq n) = Q(i) \cap Q(j) = P(j,i)$$

Distributive Law Example

$S_{\triangle}$ becomes now

$$\boxed{S_{\triangle}} = \sum_{1\leq i\leq n,\ 1\leq j\leq n,\ j\leq i} a_i a_j = \boxed{\sum_{P(i,j),\ j\leq i} a_i a_j}$$

We evaluate on the next slide

Distributive Law Example

$$\text{USE: } P(i,j) = P(j,i)$$

$$S_\bigtriangledown = \sum_{P(i,j),\ i \le j} a_i a_j = \boxed{\sum_{P(j,i),\ j \le i} a_j a_i} = \sum_{P(i,j),\ j \le i} a_i a_j = S_\bigtriangleup$$

$$\text{Re-name } \ j \to i,\ \ i \to j$$

We proved

$$\boxed{S_\bigtriangledown = S_\bigtriangleup}$$

Distributive Law Example

EVALUATE (remember: our GOAL is to FIND $S_\triangle$)

$$
\begin{aligned}
2S_\bigtriangledown &= S_\bigtriangledown + S_\triangle \\
&= \sum_{P(i,j),\ i\le j} a_i a_j + \sum_{P(i,j),\ j\le i} a_i a_j
\end{aligned}
$$

$$\downarrow \qquad\qquad \downarrow$$

$$Q(i,j) \qquad R(i,j)$$

$$= \sum_{Q(i,j)} a_i a_j + \sum_{R(i,j)} a_i a_j$$

WE WANT NOW TO COMBINE DOMAINS Q(i,j) and R(i,j)

Combining Domains

Formula for **COMBINED DOMAINS**

$$\boxed{\sum_{Q(k)} a_k + \sum_{R(k)} a_k = \sum_{Q(k) \cap R(k)} a_k + \sum_{Q(k) \cup R(k)} a_k}$$

OR

$$\boxed{\sum_{k \in K} a_k + \sum_{k \in K'} a_k = \sum_{k \in K \cap K'} a_k + \sum_{k \in K \cup K'} a_k}$$

The second formula is listed **without the proof** on page 31 in our BOOK

Combined Limits

For any set A, we denote by $|A|$ the cardinality of the set A in a case when A is finite it denotes a number of elements of the set A. We obviously have the following

Fact

For any **finite sets** A, B

$$|A \cup B| = |A| + |B| - |A \cap B|$$

From the **Fact** we have that

$|K \cup K'| = |K| + |K'| - |K \cap K'|$ and hence

$$|K| + |K'| = |K \cap K'| + |K \cup K'|$$

It **justifies** but yet not formally proves that

$$\sum_{k \in K} a_k + \sum_{k \in K'} a_k = \underbrace{\sum_{k \in K \cap K'}}_{\uparrow} a_k + \underbrace{\sum_{k \in K \cup K'}}_{\uparrow} a_k$$

COMBINE LIMITS

Combining Domains

Let's put

$$K = \{k : Q(k)\} \qquad K' = \{k : R(k)\}$$

The previous formula becomes:

$$\sum_{Q(k)} a_k + \sum_{R(k)} a_k = \sum_{Q(k) \cap R(k)} a_k + \sum_{Q(k) \cup R(k)} a_k$$

COMBINE DOMAINS

Proof is based on the **Property** given on the next slide as an easy exercise to prove

Combined Domains Property

Exercise

Prove using the Truth Tables and definition of the characteristic function that the following holds

Combined Domains Property

For any predicates P(k), Q(k)

$$[P(k) \cup Q(k)] = [P(k)] + [Q(k)] - [P(k) \cap Q(k)]$$

Combined Domains Proof

Proof

We evaluate

$$\sum_{Q(k)\cup R(k)} a_k = \sum_k a_k [Q(k) \cup R(k)]$$

$$= \sum_k a_k ([Q(k)] + [R(k)] - [Q(k) \cap R(k)])$$

$$= \sum_k a_k [Q(k)] + \sum_k a_k [R(k)] - \sum_k a_k [Q(k) \cap R(k)])$$

$$= \sum_{Q(k)} a_k + \sum_{R(k)} a_k - \sum_{Q(k)\cap R(k)} a_k$$

Back to Combined Domains Example

REMINDER

$$\boxed{S_{\triangledown}} = \sum_{1\le i\le n,\ 1\le j\le n,\ i\le j} a_i a_j = \boxed{\sum_{P(i,j),\ i\le j} a_i a_j}$$

for

$$P(i,j) = (1 \le i \le n) \cap (1 \le j \le n) = Q(i) \cap Q(j)$$

$$\boxed{S_{\triangle}} = \sum_{1\le i\le n,\ 1\le j\le n,\ j\le i} a_i a_j = \boxed{\sum_{P(i,j),\ j\le i} a_i a_j}$$

Distributivity Law Example

Our **goal** is to **find** $S_\triangle$

$$2S_\bigtriangledown = S_\bigtriangledown + S_\triangle$$

$$= \sum_{P(i,j),\ i \leq j} a_i a_j + \sum_{P(i,j),\ j \leq i} a_i a_j$$

$$\downarrow \qquad \downarrow$$

$$Q = Q(i,j) \qquad R = R(i,j)$$

$$= \sum_Q a_i a_j + \sum_R a_i a_j$$

Now we **know** how to COMBINE DOMAINS Q(i,j) and R(i,j)

Distributivity Law Example

$$\boxed{2S_{\bigtriangledown}} = S_{\bigtriangledown} + S_{\bigtriangleup}$$

$$= \sum_{Q} a_i a_j + \sum_{R} a_i a_j$$

$$\boxed{= \sum_{Q \cap R} a_i a_j + \sum_{Q \cup R} a_i a_j}$$

We have to **evaluate** $Q \cap R$ and $Q \cup R$

Distributivity Law Example

We know that $Q = P(i,j) \cap (i \leq j)$ and $R = P(i,j) \cap (j \leq i)$

We now **evaluate** $Q \cap R$ and $Q \cup R$ as follows

$$Q \cap R = (P(i,j) \cap (i \leq j)) \cap (P(i,j) \cap (j \leq i))$$

$$= P(i,j) \cap P(i,j) \cap (i \leq j) \cap (j \leq i) = \boxed{P(i,j) \cap (i = j)}$$

$$Q \cup R = (P(i,j) \cap (i \leq j)) \cup (P(i,j) \cap (j \leq i))$$

$$= P(i,j) \cap ((i \leq j) \cup (j \leq i)) = P(i,j) \cap True = \boxed{\text{P(i,j)}}$$

Distributivity Law Example

Reminder: $P(i,j) = 1 \leq i \leq n \cap 1 \leq j \leq n$ and we **put it all together** as follows

$$2S_{\bigtriangledown} = \sum_{Q \cap R} a_i a_j + \sum_{Q \cup R} a_i a_j$$

$$= \sum_{1 \leq i \leq n,\ 1 \leq j \leq n} a_i a_j + \sum_{1 \leq i \leq n,\ 1 \leq j \leq n,\ i=j} a_i a_j$$

$$=^{DLaw} \left(\sum_{1 \leq i \leq n} a_i\right)\left(\sum_{1 \leq j \leq n} a_j\right) + \left(\sum_{1 \leq i \leq n} a_i^2\right)$$

Distributivity Law Example

$$2S_{\triangledown} =^{DLaw} \Big(\sum_{1 \le i \le n} a_i\Big)\Big(\sum_{1 \le j \le n} a_j\Big) + \Big(\sum_{1 \le i \le n} a_i^2\Big)$$

we rename $j \to i$

$$= \Big(\sum_{1 \le i \le n} a_i\Big)^2 + \Big(\sum_{1 \le i \le n} a_i^2\Big)$$

Finally, we get:

$$\boxed{S_{\triangledown} = \frac{1}{2}\Big(\Big(\sum_{1 \le i \le n} a_i\Big)^2 + \Big(\sum_{1 \le i \le n} a_i^2\Big)\Big)}$$

$S_\bigtriangledown$ Short Solution

Find $S_\bigtriangledown = \sum_{1 \le i,j \le n} a_i a_j$

Step1: EVALUATE

$$S = \sum_{1 \le i \le n, 1 \le j \le n} a_i a_j =^{DLaw} \left(\sum_{1 \le i \le n} a_i\right)\left(\sum_{1 \le j \le n} a_j\right)$$

Step 2 PROVE : $S_\bigtriangledown = S_\bigtriangleup$

$S_{\bigtriangledown}$ Short Solution

Step 3 OBSERVE:

$$S = S_{\bigtriangledown} + S_{\bigtriangleup} - \sum_{1 \le j \le n} (a_j)^2 = 2S_{\bigtriangledown} - \sum_{1 \le i \le n} (a_i)^2$$

Solve on $S_{\bigtriangledown}$

$$\boxed{S_{\bigtriangledown} = \frac{1}{2}\Big(\big(\sum_{1 \le i \le n} a_i\big)^2 + \big(\sum_{1 \le i \le n} a_i\big)^2\Big)}$$

New Problem

Given sequences $\{a_n\}_{n \in N}$, $\{b_n\}_{n \in N}$
EVALUATE the SUM

$$S = \sum_{1 \le j < k \le n} (a_k - a_j)(b_k - b_j)$$

Observe that

$$1 \le j < k \le n = (1 \le j \le n) \cap (1 \le k \le n) \cap (j < k)$$

denote $P(j,k) = (1 \le j \le n) \cap (1 \le k \le n)$ observe that $P(j,k) = P(k,j)$
and we re-write the limits of our S as follows

$$1 \le j < k \le n = P(j,k) \cap (j < k) = P(k,j) \cap (j < k)$$

New Problem

We write now out SUM as

Equation 1

$$\boxed{S = \sum_{P(j,k),\ j<k} (a_k - a_j)(b_k - b_j)}$$

Now we EXCHANGE j and k (re-name) in S and use $P(j,k) = P(k,j)$ and we get

$$S = \sum_{P(j,k),\ k<j} (a_j - a_k)(b_j - b_k)$$

New Problem

We evaluate

$$(a_j - a_k)(b_j - b_k) = -(a_k - a_j)(-(b_k - b_j)) = (a_k - a_j)(b_k - b_j)$$

and S becomes now

Equation 2

$$\boxed{S = \sum_{P(j,k),k<j} (a_k - a_j)(b_k - b_j)}$$

We ADD now **Equation 1** and **Equation 2** and get

Equation 3

$$2S = \sum_{P(j,k),\ j\leq k} (a_k - a_j)(b_k - b_j) + \sum_{P(j,k),k\leq j} (a_k - a_j)(b_k - b_j)$$

Observations

Observation 1 We could change the original limits of summation in both sums from $j < k, \ k < j$ to $j \leq k, \ \ k \leq j$, respectively because the condition k =j gives in both sum the term equal 0, i.e we have $(a_k - a_k) = (b_k - b_k) = 0$

Observation 2 To evaluate 2S we need to use the formula for **combining domains**

$$\boxed{\sum_{Q} a_k + \sum_{R} a_k = \sum_{Q \cap R} a_k + \sum_{Q \cup R} a_k}$$

for $Q = P(j,k) \cap (j \leq k)$ and $R = P(j,k) \cap (k \leq j)$

Observations

EVALUATE $Q \cap R$

$$Q \cap R = P(j,k) \cap (j \leq k) \cap P(j,k) \cap (k \leq j) = P(j,k) \cap (k = j)$$

EVALUATE $Q \cup R$

$$Q \cup R = (P(j,k) \cap (j \leq k)) \cup (P(j,k) \cap (k \leq j)$$
$$= P(j,k) \cap (j \leq k \cup k \leq j) = P(j,k) \cap True = P(j,k)$$

We re-write our 2S from **Equation 3** as follows

Back to the Problem

$$2S = \underset{\substack{\uparrow \\ Q}}{\sum_{P(j,k),\ j \le k}} (a_k - a_j)(b_k - b_j) + \underset{\substack{\uparrow \\ R}}{\sum_{P(j,k),\ k \le j}} (a_k - a_j)(b_k - b_j)$$

$$= \sum_{Q \cap R} (a_k - a_j)(b_k - b_j) + \sum_{Q \cup R} (a_k - a_j)(b_k - b_j)$$

We have that $Q \cap R = P(j,k) \cap (k = j)$, $Q \cup R = P(j,k)$, and

$$\sum_{P(j,k) \cap (k=j)} (a_k - a_j)(b_k - b_j) = 0,$$ so

$$\boxed{2S = \sum_{P(j,k)} (a_k - a_j)(b_k - b_j)}$$

Back to the Problem

Let's expend now

$(a_k - a_j)(b_k - b_j) = a_k b_k - a_j b_k - a_k b_j + a_j b_j$

Back to our sum

$$2S = \sum_{P(j,k)} a_k b_k + \sum_{P(j,k)} a_j b_j - 2 \sum_{P(j,k)} a_k b_j$$

re-name $j \to k$, in second sum and get

$2S = 2 \sum_{P(j,k)} a_k b_k - 2 \sum_{P(j,k)} a_k b_j$ and so

$$\boxed{S = \sum_{P(j,k)} a_k b_k - \sum_{P(j,k)} a_k b_j}$$

Back to the Problem

Use distributivity for for the second sum

$$\sum_{P(j,k)} a_k b_j = \sum_{1\le j\le n,\ 1\le k\le n} a_k b_j$$

$$= (\sum_{1\le k\le n} a_k)(\sum_{1\le j\le n} b_j) \quad \text{re-name} \quad j \to k$$

$$= (\sum_{1\le k\le n} a_k)(\sum_{1\le k\le n} b_k)$$

Back to the Problem

We evaluate the first sum $\sum_{P(j,k)} a_k b_k$ separately

$$\sum_{P(j,k)} a_k b_k =^{def} \sum_{1 \le j \le n,\ 1 \le k \le n} a_k b_k$$

$$=^{def} \sum_{1 \le k \le n} \left(\sum_{1 \le j \le n} a_k b_k \right)$$

$\uparrow$

$a_k b_k$ constant with respect to j

$$= \sum_{1 \le k \le n} \left(a_k b_k \sum_{1 \le j \le n} 1 \right)$$

$$= \sum_{1 \le k \le n} a_k b_k n \quad \leftarrow \quad \text{n is constant with respect to k}$$

$$= n \sum_{1 \le k \le n} a_k b_k$$

Solution

We put evaluated components into

$$S = \sum_{P(j,k)} a_k b_k - \sum_{P(j,k)} a_k b_j$$

and get

$$\boxed{S = n \sum_{1 \le k \le n} a_k b_k - \Big(\sum_{1 \le k \le n} a_k\Big)\Big(\sum_{1 \le k \le n} b_k\Big)}$$

FORMULA Multiple Sum $\rightarrow$ SingleSums is

$$\boxed{\sum_{1 \le j < k \le n} (a_k - a_j)(b_k - b_j) = n \sum_{1 \le k \le n} a_k b_k - \Big(\sum_{1 \le k \le n} a_k\Big)\Big(\sum_{1 \le k \le n} b_k\Big)}$$

Formula Application

We use the **FORMULA** to evaluate relationships between

$$\boxed{\sum_{1\leq k\leq n} a_k b_k} \quad \text{and} \quad \boxed{(\sum_{1\leq k\leq n} a_k)(\sum_{1\leq k\leq n} b_k)}$$

Obtained relationships are called

CHEBYSHEV'S INEQUALITIES

Chebyshev's Inequalities

We re-write the **FORMULA** as follows

$$\boxed{\Big(\sum_{1\le k\le n} a_k\Big)\Big(\sum_{1\le k\le n} b_k\Big) = n\sum_{1\le k\le n} a_k b_k - \Big(\sum_{P(k,j),\ j<k} (a_k - a_j)(b_k - b_j)\Big)}$$

ASSUME

$\boxed{\text{C1}}$:

$$a_1 \le a_2 \le a_3 \le \dots \quad \le a_n$$

$$b_1 \le b_2 \le b_3 \le \dots \quad \le b_n$$

Chebyshev's Inequalities

Observe that under the condition $\boxed{C1}$

$(a_k - a_j)$, $(b_k - b_j)$ and hence the sum

$\sum (a_k - a_j)(b_k - b_j)$ are all POSITIVE for $j < k$

Hence we get that

Chebyshev Inequality 1

$$\Big(\sum_{1 \le k \le n} a_k\Big)\Big(\sum_{1 \le k \le n} b_k\Big) \le n \sum_{1 \le k \le n} a_k b_k$$

holds for

$$a_1 \le a_2 \le a_3 \le \ldots \quad \le a_n$$

$$b_1 \le b_2 \le b_3 \le \ldots \quad \le b_n$$

Chebyshev's Inequalities

Assume now conditions $\boxed{C2:}$

$a_1 \leq a_2 \leq a_3 \leq .... \quad \leq a_n$

$b_1 \geq b_2 \geq b_3 \geq ..... \quad \geq b_n$

Observe that under the conditions $\boxed{C2}$

$\sum (a_k - a_j)(b_k - b_j)$ is NEGATIVE as it has all negative terms and hence

$-\sum (a_k - a_j)(b_k - b_j)$ is POSITIVE and we get

Chebyshev Inequality 2

$$\boxed{\Big(\sum_{1 \leq k \leq n} a_k\Big)\Big(\sum_{1 \leq k \leq n} b_k\Big) \geq n \sum_{1 \leq k \leq n} a_k b_k}$$