An Impossible Counting Problem

Suppose you went to a grocery store to buy some fruits. There are some constraints though:

A. The store has only two **apples** left: one red and one green. So you cannot take more than 2 apples.

B. All but 3 **bananas** are rotten. You do not like rotten bananas.

F. **Figs** are sold 6 per pack. You can take as many packs as you want.

M. **Mangoes** are sold in pairs. But you must not take more than a pair of pairs.

P. They sell 4 **peaches** per pack. Take as many packs as you want.

Now the question is: in how many ways can you buy $n$ fruits from the store?
Generating Functions

Generating functions represent sequences by coding the terms of a sequence as coefficients of powers of a variable in a formal power series.

For example, one can represent a sequence $s_0, s_1, s_2, \ldots$ as:

$$S(z) = s_0 + s_1 z + s_2 z^2 + s_3 z^3 + \cdots + s_n z^n + \cdots$$

So $s_n$ is the coefficient of $z^n$ in $S(z)$. 
An Impossible Counting Problem

A. The store has only two **apples** left: one red and one green. So you cannot take more than 2 apples.

\[
A(z) = 1 + 2z + z^2 = (1 + z)^2
\]

B. All but 3 **bananas** are rotten. You do not like rotten bananas.

\[
B(z) = 1 + z + z^2 + z^3 = \frac{1 - z^4}{1 - z}
\]

F. **Figs** are sold 6 per pack. You can take as many packs as you want.

\[
F(z) = 1 + z^6 + z^{12} + z^{18} + \cdots = \frac{1}{1 - z^6}
\]

M. **Mangoes** are sold in pairs. But you must not take more than a pair of pairs.

\[
M(z) = 1 + z^2 + z^4 = \frac{1 - z^6}{1 - z^2}
\]

P. They sell 4 **peaches** per pack. Take as many packs as you want.

\[
P(z) = 1 + z^4 + z^8 + z^{12} + \cdots = \frac{1}{1 - z^4}
\]
An Impossible Counting Problem

Suppose you can choose \( n \) fruits in \( s_n \) different ways.

Then the generating function for \( s_n \) is:

\[
S(z) = A(z)B(z)F(z)M(z)P(z) = (1 + z)^2 \times \frac{1 - z^4}{1 - z} \times \frac{1}{1 - z^6} \times \frac{1}{1 - z^2} \times \frac{1}{1 - z^4}
\]

\[
= \frac{1 + z}{(1 - z)^2}
\]

\[
= (1 + z) \sum_{n=0}^{\infty} (n + 1)z^n
\]

\[
= \sum_{n=0}^{\infty} (2n + 1)z^n
\]

Equating the coefficients of \( z^n \) from both sides:

\[ s_n = 2n + 1 \]
Fibonacci Numbers

Recurrence for Fibonacci numbers:

\[ f_n = \begin{cases} 
0 & \text{if } n = 0, \\
1 & \text{if } n = 1, \\
& \text{otherwise.} 
\end{cases} \]

\[ \Rightarrow f_n = f_{n-1} + f_{n-2} + [n = 1] \]

Generating function: \[ F(z) = f_0 + f_1 z + f_2 z^2 + f_3 z^3 + \ldots \]

\[ F(z) = \sum_n f_n z^n = \sum_n f_{n-1} z^n + \sum_n f_{n-2} z^n + \sum_n [n = 1] z^n \]

\[ = \sum_n f_n z^{n+1} + \sum_n f_n z^{n+2} + z \]

\[ = z F(z) + z^2 F(z) + z \]
Fibonacci Numbers

\[ F(z) = zF(z) + z^2F(z) + z \]

\[ \Rightarrow F(z) = \frac{z}{1 - z - z^2} \]

\[ = \frac{z}{(1 - \varphi z)(1 - \hat{\varphi} z)} , \text{ where } \varphi = \frac{1 + \sqrt{5}}{2} \text{ and } \hat{\varphi} = \frac{1 - \sqrt{5}}{2} \]

\[ = \frac{1}{\sqrt{5}} \left( \frac{1}{1 - \varphi z} - \frac{1}{1 - \hat{\varphi} z} \right) \]

\[ = \frac{1}{\sqrt{5}} \sum_{n} (\varphi^n - \hat{\varphi}^n)z^n \]

Equating the coefficients of \( z^n \) from both sides:

\[ f_n = \frac{1}{\sqrt{5}} (\varphi^n - \hat{\varphi}^n) = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n \]
Average Case Analysis of Quicksort
Quicksort

**Input:** An array $A[1: n]$ of $n$ distinct numbers.

**Output:** Numbers of $A[1: n]$ rearranged in increasing order of value.

**Steps:**

1. **Pivot Selection:** Select pivot $x = A[1]$.

2. **Partition:** Use a stable partitioning algorithm to rearrange the numbers of $A[1: n]$ such that $A[k] = x$ for some $k \in [1, n]$, each number in $A[1: k - 1]$ is smaller than $x$, and each in $A[k + 1: n]$ is larger than $x$.


**Stable Partitioning:** If two numbers $p$ and $q$ end up in the same partition and $p$ appears before $q$ in the input, then $p$ must also appear before $q$ in the resulting partition.
Average Number of Comparisons by Quicksort

We will average the number of comparisons performed by Quicksort on all possible arrangements of the numbers in the input array.

Let \( t_n = \text{average } \#\text{comparisons performed by Quicksort on } n \text{ numbers.} \)

Then

\[
 t_n = \begin{cases} 
 0 & \text{if } n < 1, \\
 n - 1 + \frac{1}{n} \sum_{k=1}^{n} (t_{k-1} + t_{n-k}) & \text{otherwise.}
\end{cases}
\]

The recurrence can be rewritten as follows.

\[
 t_n = \begin{cases} 
 0 & \text{if } n < 1, \\
 n - 1 + \frac{2}{n} \sum_{k=0}^{n-1} t_k & \text{otherwise.}
\end{cases}
\]
**Average Number of Comparisons by Quicksort**

The recurrence: \[ t_n = \begin{cases} 
0 & \text{if } n < 1, \\
n - 1 + \frac{2}{n} \sum_{k=0}^{n-1} t_k & \text{otherwise}.
\end{cases} \]

Let \( T(z) \) be an ordinary generating function for \( t_n \)'s:

\[
T(z) = t_0 + t_1 z + t_2 z^2 + \cdots + t_n z^n + \cdots
\]

\[
= t_0 + \sum_{n=1}^{\infty} t_n z^n
\]

\[
= t_0 + \sum_{n=1}^{\infty} \left( n - 1 + \frac{2}{n} \sum_{k=0}^{n-1} t_k \right) z^n
\]
We have:  \[ T(z) = t_0 + \sum_{n=1}^{\infty} \left( n - 1 + \frac{2}{n} \sum_{k=0}^{n-1} t_k \right) z^n \]

Differentiating:

\[ T'(z) = \sum_{n=1}^{\infty} \left( n(n-1) + 2 \sum_{k=0}^{n-1} t_k \right) z^{n-1} \]

\[ = z \sum_{n=2}^{\infty} n(n-1)z^{n-2} + 2 \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} t_k \right) z^n \]

\[ = z \frac{d^2}{dz^2} \left( \left( \sum_{n=0}^{\infty} z^n \right) - 1 - z \right) + 2 \sum_{n=0}^{\infty} \left( t_n z^n \left( \sum_{k=0}^{\infty} z^k \right) \right) \]
Average Number of Comparisons by Quicksort

\[ T'(z) = z \frac{d^2}{dz^2} \left( \left( \sum_{n=0}^{\infty} z^n \right) - 1 - z \right) + 2 \sum_{n=0}^{\infty} \left( t_n z^n \left( \sum_{k=0}^{\infty} z^k \right) \right) \]

\[ = z \frac{d^2}{dz^2} \left( (1 - z)^{-1} - 1 - z \right) + 2(1 - z)^{-1} \sum_{n=0}^{\infty} t_n z^n \]

\[ = \frac{2z}{(1 - z)^3} + \frac{2}{1 - z} T(z) \]

Rearranging: \((1 - z)^2 T'(z) - 2(1 - z)T(z) = \frac{2z}{1 - z}\)

\[ \Rightarrow \frac{d}{dz} \left( (1 - z)^2 T(z) \right) = \frac{d}{dz} (-2 \ln(1 - z) - 2z) \]

Integrating: \((1 - z)^2 T(z) = -2 \ln(1 - z) - 2z + c \quad (c \text{ is a constant})\)
Average Number of Comparisons by Quicksort

We have, \((1 - z)^2 T(z) = -2 \ln(1 - z) - 2z + c\) (\(c\) is a constant)

Putting \(z = 0\), \(T(0) = c \Rightarrow t_0 = c \Rightarrow c = 0\)

Hence, \((1 - z)^2 T(z) = -2 \ln(1 - z) - 2z\)

\[ \Rightarrow T(z) = 2(-\ln(1 - z) - z)(1 - z)^{-2} \]

\[ = 2 \left( \sum_{j=2}^{\infty} \frac{z^j}{j} \right) \left( \sum_{k=0}^{\infty} (k + 1)z^k \right) \]

Equating coefficients of \(z^n\) from both sides,

\[ t_n = 2 \left( \sum_{k=2}^{n} \frac{n + 1 - k}{k} \right) = 2(n + 1) \sum_{k=1}^{n} \frac{1}{k} - 4n = 2(n + 1)H_n - 4n, \]

where \(H_n = \sum_{k=1}^{n} \frac{1}{k}\) is the \(n^{th}\) harmonic number.
Average Number of Comparisons by Quicksort

We have, \( t_n = 2(n + 1)H_n - 4n, \)

where \( H_n = \sum_{k=1}^{n} \left( \frac{1}{k} \right) \) is the \( n^{th} \) harmonic number.

But we know, \( H_n = \ln n + O(1) \) \hspace{1cm} (prove it)

Hence, \( t_n = 2(n + 1)(\ln n + O(1)) - 4n = \Theta(n \log n). \)