1. Prove that \( \{a^n ba^m ba^{m+n} : n, m \geq 1 \} \) is not regular.

**Solution:** Suppose this language were regular. Then, by the conditions of the pumping theorem, there would be some constant \( k \) for this language. Consider then the string \( a^k ba^k ba^{2k} \). Now there should be a decomposition \( w = xyz \) guaranteed by the pumping theorem. There are several cases for the decomposition, each of which leads to a contradiction. First, because each string in \( L \) contains exactly two bs, it cannot be the case that \( y \) contains any instances of \( b \), because then \( xy^n = xz \) would have too few bs and would thus not be in \( L \). Thus \( y \) contains only as. Let \( |y| = p \). If \( y \) falls before the first \( b \), then \( xy^2z = a^k + pba^kba^{2k} \), and \( 2k + p \neq 2k \) unless \( p = 0 \), which is ruled out by \( y \neq \epsilon \). Similarly, if \( y \) falls between the bs, \( xy^2z = a^k ba^{k+p}ba^{2k} \), and \( 2k + p \neq 2k \). Finally, if \( y \) falls after the last \( b \), then \( xy^2z = a^k ba^k a^{2k+p} \), and once again \( 2k \neq 2k + p \). This exhausts all the possibilities for the decomposition, so the contradiction is forced, and \( L \) is not regular.

2. Prove that \( \{ww^R : w \in \{a, b\}^*\} \) is not regular.

**Solution:** Assume \( L \) is regular, and let \( k \) be the constant whose existence the pumping theorem guarantees. Choose string \( a^k ba^k \). Clearly this string is of length at least \( k \), and so the strong version of the Pumping Theorem must hold. If \( |xy| \leq k \), then \( y = a^i \), where \( i > 0 \). But then \( xy^iz = a^k + (n-1)i bba^k \), which is clearly asymmetric for any \( n \neq 1 \). The theorem fails, and thus the assumption that \( L \) is regular is wrong.

3. If \( L \) is regular, then is \( \{xy : x \in L \text{ and } y \notin L \} \) regular or not? Explain your answer in one or two sentences.

**Solution:** True. This language is equivalent to \( L \overline{L} \). Since \( L \) is regular, so is its complement \( \overline{L} \), and thus their concatenation is the concatenation of two regular languages and is itself regular.

4. Prove that \( \{a^{k^2} : k = 0, 1, 2, \ldots \} \) is regular or not regular.

**Solution:** Assume that \( L = \{a^{k^2} : k = 0, 1, 2, \ldots \} \) is regular. The using \( N \) as provided by the Pumping Theorem, we choose the string \( w = a^2(n+1)^2 \in L \) of length \( (n+1)^2 \geq n \). If it is represented as \( xyz \) with \( |xy| \leq n \), then \( y = a^i \) for some \( 0 < i \leq n \). Thus \( a^{(n+1)^2-i} \) must be in \( L \). Now note that \( n^2 \), the square closes to \( (n+1)^2 \) is smaller than \( (n+1)^2 - i : (n+1)^2 = n^2 + 2n + 1 \) and \( i \leq n \); therefore \( (n+1)^2 - i \geq n^2 + n + 1 > n^2 \). Thus the assumption that \( L \) is regular is wrong.

5. Prove that \( \{ww : w \in \{a, b\}^*\} \) is regular or not regular.

**Solution:** Assume that \( L \) is regular, and let \( k \) be the constant from the Pumping Theorem. Choose the string \( a^k ba^k b \). This string has length \( 2k + 2 \), which is definitely at least \( k \). If \( |xy| \leq k \), then \( y = a^i \) for some \( i > 0 \). Thus \( xy^2z = a^{k+i}ba^k b \), which is clearly asymmetric. Thus the assumption that \( L \) is regular must be wrong.