An *argument* (form) is a (finite) sequence of statements (forms), usually written as follows:

\[
\alpha_1 \\
\ldots \\
\alpha_n \\
\therefore \beta
\]

We call \(\alpha_1, \ldots, \alpha_n\) the *premises* (or *assumptions* or *hypotheses*) and \(\beta\) the *conclusion*, of the argument.

Argument forms are also called *inference rules* and written as

\[
\frac{\alpha_1 \cdots \alpha_n}{\beta}
\]

An inference rule is said to be *valid*, or *(logically) sound*, if it is the case that, for each truth valuation, if all the premises true, then the conclusion is also true.

An argument is valid if it is based on a valid argument form.
Modus Ponens

The following argument is valid:

If Socrates is a man, then Socrates is mortal.
Socrates is a man.
\[ \therefore \text{Socrates is mortal.} \]

Its form corresponds to a well-known inference rule:

**Modus Ponens**

\[
\begin{array}{c}
\alpha \rightarrow \beta \\
\alpha \\
\hline
\beta \\
\end{array}
\]

A similar inference rule is

**Modus Tollens**

\[
\begin{array}{c}
\alpha \rightarrow \beta \\
\neg \beta \\
\hline
\neg \alpha \\
\end{array}
\]

The following argument is of this form:

If Zeus is a man, then Zeus is mortal.
Zeus is not mortal.
\[ \therefore \text{Zeus is not a man.} \]
Valid and Invalid Arguments

A valid argument may have a false conclusion, e.g.,

If John Lennon was a rock star, then John Lennon had red hair.
John Lennon was a rock star.
∴ John Lennon had red hair.

An argument with a true conclusion may be invalid:

If New York is a big city, then New York has tall buildings.
New York has tall buildings.
∴ New York is a big city.

The validity of an argument form can be verified by constructing a suitable truth table and checking that the conclusion is true whenever all premises are true.

The question of validity can also be expressed in terms of a tautology problem:

**Theorem**

An inference rule

\[
\frac{\alpha_1 \cdot \cdot \cdot \alpha_n}{\beta}
\]

is valid if, and only if, the conditional

\[\alpha_1 \land \cdot \cdot \cdot \land \alpha_n \rightarrow \beta\]

is a tautology.
Other Inference Rules

Generalization

$$\frac{\alpha}{\alpha \lor \beta} \quad \text{and} \quad \frac{\beta}{\alpha \lor \beta}$$

Specialization

$$\frac{\alpha \land \beta}{\alpha} \quad \text{and} \quad \frac{\alpha \land \beta}{\beta}$$

Conjunction

$$\frac{\alpha \quad \beta}{\alpha \land \beta}$$

Elimination

$$\frac{\alpha \lor \beta \quad \sim \beta}{\alpha} \quad \text{and} \quad \frac{\alpha \lor \beta \quad \sim \alpha}{\beta}$$

Transitivity

$$\frac{\alpha \rightarrow \beta \quad \beta \rightarrow \gamma}{\alpha \rightarrow \gamma}$$
Proof Techniques

The following inference rules represent proof techniques that are often used in practice.

Proof by Division into Cases

\[
\begin{array}{ccc}
\alpha \lor \beta & \alpha \to \gamma & \beta \to \gamma \\
\hline
\gamma
\end{array}
\]

For instance, if a disjunction \( p \lor q \) has been derived and the goal is to prove \( r \), then according to this inference rule it would be sufficient to derive \( p \to r \) and \( q \to r \).

Proof by Contradiction

\[
\frac{\neg \alpha \to \bot}{\alpha}
\]

This rule formalizes the idea of proof by contradiction.

The usual way to derive a conditional \( \neg \alpha \to \bot \) is to assume \( \neg \alpha \) and then derive \( \bot \) (i.e., a contradiction). Thus, if one can derive a contradiction from \( \neg \alpha \), then one may conclude that \( \alpha \) is true.
Quine’s Method

The following method can be used to determine whether a given propositional formula is a tautology, a contradiction, or a contingency.

Let $\alpha$ be a propositional formula.

- If $\alpha$ contains no variables, it can be simplified to $\top$ or $\bot$, and hence is either a tautology or a contradiction.

- If $\alpha$ contains a variable, then (i) select a variable, say $p$, (ii) simplify both $\alpha[p \mapsto \top]$ and $\alpha[p \mapsto \bot]$, denoting the simplified formulas by $\alpha_1$ and $\alpha_2$, respectively, and (iii) apply the method recursively to $\alpha_1$ and $\alpha_2$.

  If $\alpha_1$ and $\alpha_2$ are both tautologies, so is $\alpha$. If $\alpha_1$ and $\alpha_2$ are both contradictions, so is $\alpha$. In all other cases, $\alpha$ is a contingency.
Example

Let $\alpha$ be the formula

$$(p \land \neg q \to r) \land (r \to p \lor q) \land (p \to \neg r) \land (p \lor q \lor r) \to q.$$ 

We first select a variable, say $q$, and then consider the two cases, $q = \top$ and $q = \bot$.

1) The formula $\alpha[q \mapsto \top]$ is of the form $\beta \to \top$, and hence can be simplified to $\top$.

2) We next simplify $\alpha[q \mapsto \bot]$:

$$(p \land \neg \bot \to r) \land (r \to p \lor \bot) \land (p \to \neg r) \land (p \lor \bot \lor r) \to \bot$$

$\equiv (p \land \top \to r) \land (r \to p) \land (p \to \neg r) \land (p \lor r) \to \bot$

$\equiv (p \to r) \land (r \to p) \land (p \to \neg r) \land (p \lor r) \to \bot$

$\equiv \neg[(p \to r) \land (r \to p) \land (p \to \neg r) \land (p \lor r)]$

Denote the simplified formula by $\alpha_1$.

We select the variable $p$ in $\alpha_1$ and consider the resulting two cases.

2.1) The formula $\alpha_1[p \mapsto \top]$ can be simplified as follows:

$$\neg[(\top \to r) \land (r \to \top) \land (\top \to \neg r) \land (\top \lor r)]$$

$\equiv \neg[r \land \top \land \neg r \land \top]$

$\equiv \neg[r \land \neg r]$

$\equiv \neg \bot$

$\equiv \top$
2.2) The formula $\alpha_1[p \rightarrow \bot]$ can be simplified as follows:

\[
\sim[(\bot \rightarrow r) \land (r \rightarrow \bot) \land (\bot \rightarrow \sim r) \land (\bot \lor r)] \\
\equiv \sim[\top \land \sim r \land \top \land r] \\
\equiv \sim[\sim r \land r] \\
\equiv \sim \bot \\
\equiv \top
\]

This completes the process. All formulas considered, including the original formula $\alpha$, are tautologies.

There are different ways of simplifying a formula. We assume that simplification eliminates all occurrences of $\top$ and $\bot$ from a formula. Additional simplification steps, such as replacing a subformula $p \land \sim p$ by $\bot$, are optional.