Exercise 1: The formulas 1. - 9. That we assumed to be provable in S are those needed for 2 proofs of the Completeness Theorem. List the formulas that are needed for the Proof 1 only.

1. \((A \Rightarrow (B \Rightarrow A))\)
2. \(((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))\)
4. \((A \Rightarrow A)\)
5. \((B \Rightarrow \neg\neg B)\)
6. \((\neg A \Rightarrow (A \Rightarrow B))\)
7. \((A \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B)))\)
8. \(((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))\)

1: \((B1 \Rightarrow (A \Rightarrow B1))\) is used in the base step in the proof for deduction theorem and in the inductive step of the case A is \((A1 \Rightarrow A2)\) in the proof for the main lemma.

2: \(((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))\) is used in the case 2 of the inductive step in the proof for the deduction theorem.

4. \((A \Rightarrow A)\) is used in the case 2 of the base step in the proof for the deduction theorem.

5. \((B \Rightarrow \neg\neg B)\) is used in the inductive step of the case A is \(\neg A1\) in the proof for the main lemma.

6. \((\neg A \Rightarrow (A \Rightarrow B))\) is used in the inductive step of the case A is \((A1 \Rightarrow A2)\) in the proof for the main lemma too.

7. \((A \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B)))\) is used in the inductive step of the case A is \((A1 \Rightarrow A2)\) in the proof for the main lemma.

8. \(((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))\) is used in the proof 1 for the completeness theorem along with Main Lemma, the Deduction Theorem, monotonicity, and Modus Ponens.

Exercise 2: We proved Completeness Theorem for the language \(L(\Rightarrow, \neg)\). Extend this proof to the language \(L(\Rightarrow, U, \neg)\) by adding all new CASES and needed PROVABLE formulas to our list 1. - 9. or to a shorter list from solution of the Exercise 1.

We define the system S as follows \(S = (L(\Rightarrow, U, \neg), F, LA, (MP))\) where the set of logical axioms \(LA \subseteq T\) is such that the formulas listed below are provable in S

1. \((A \Rightarrow (B \Rightarrow A))\)
2. \((A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))\)
3. \(((\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B))\)
4. \((A \rightarrow A)\)
5. \((B \rightarrow \neg \neg B)\)
6. \((\neg A \rightarrow (A \rightarrow B))\)
7. \(((A \rightarrow B) \rightarrow ((\neg A \rightarrow B) \rightarrow B))\)
8. \(((A \rightarrow B) \rightarrow (((A \rightarrow B) \rightarrow (A \rightarrow C)) \rightarrow (A \rightarrow B)))\)
9. \(((\neg A \rightarrow A) \rightarrow A)\)
10. \((A \rightarrow (A \cup B))\)
11. \((\neg A \rightarrow (B \rightarrow \neg (A \cup B)))\)

Proof of \((\neg A \rightarrow (B \rightarrow \neg (A \cup B)))\) for Tautology:

Assume \((\neg A \rightarrow (B \rightarrow \neg (A \cup B)))\) is not a tautology, meaning it has a counter model \(v\). (use shorthand notation)

\((\neg A \rightarrow (B \rightarrow \neg (A \cup B))) = F\) if and only if

\(\neg A = T\) and \((B \rightarrow \neg (A \cup B)) = F\) if and only if

\(\neg B = T\) and \(\neg (A \cup B) = F\) if and only if

\(A = T\) or \(B = T\), which contradicts with \(\neg A = T\) and \(\neg B = T\). Therefore, such a \(v\) does not exist, and \((\neg A \rightarrow (B \rightarrow \neg (A \cup B)))\) is a tautology.

Proof of the Main Lemma: adding the proof for the case \(A\) is \((A_1 \cup A_2)\).

If \(A\) is \((A_1 \cup A_2)\) then \(A_1\) and \(A_2\) have less than \(n\) connectives \(A = A(b_1, \ldots, b_n)\) so there are some subsequences \(c_1, \ldots, c_k\) and \(d_1, \ldots, d_m\) for \(k, m \leq n\) of the sequence \(b_1, \ldots, b_n\) such that \(A_1 = A_1(c_1, \ldots, c_k)\) and \(A_2 = A(d_1, \ldots, d_m)\) proof the Main Lemma \(A_1\) and \(A_2\) have less than \(n\) connectives and so by the inductive assumption we have appropriate formulas \(C_1, \ldots, C_k\) and \(D_1, \ldots, D_m\) such that \(C_1, C_2, \ldots, C_k \rightarrow A_1'\) and \(D_1, D_2, \ldots, D_m \rightarrow A_2'\) and \(C_1, C_2, \ldots, C_k, D_1, D_2, \ldots, D_m\) are subsequences of formulas \(B_1, B_2, \ldots, B_n\) corresponding to the propositional variables in \(A\) By the inductive assumption and monotonicity we have \(B_1, B_2, \ldots, B_n \rightarrow A_1'\) and \(B_1, B_2, \ldots, B_n \rightarrow A_2'\).

Case: \(v \ast (A_1) = T\)

If \(v \ast (A_1) = T\) then \(A_1' = A_1\)

Observe that if \(v \ast (A_1) = T\) then \(A_1'\) is \(A_1\) and, whatever value \(v\) gives \(A_2\), we have \(v \ast (A_1 \cup A_2) = T\) So \(A'\) is \((A_1 \cup A_2)\)

By the above and the inductive assumption \(B_1, B_2, \ldots, B_n \rightarrow A_1\) and since we have assumed 10. about \(S\) and by monotonicity we have \(B_1, B_2, \ldots, B_n \rightarrow (A_1 \Rightarrow (A_1 \cup A_2))\) By above and MP we have \(B_1, B_2, \ldots, B_n \rightarrow (A_1 \cup A_2)\) that is \(B_1, B_2, \ldots, B_n \rightarrow A\).
Case: $v \ast (A2) = T$ similar to $v \ast (A1) = T$.

Case: $v \ast (A1) = F$, $v \ast (A2) = F$

If $v \ast (A1) = F$ then $A1' = \neg A1$ and if $v \ast (A2) = F$ then $A2' = \neg A2$. Also we have in this case $v \ast (A1 \cup A2) = F$ and so $A' = \neg (A1 \cup A2)$

By the above, the inductive assumption and monotonicity $B1, B2, ..., Bn |- \neg A1$ and also $B1, B2, ..., Bn |- \neg A2$.

Since we have assumed 11. about $S$ and by monotonicity we have $B1, B2, ..., Bn |- (\neg A1 \Rightarrow (\neg A2 \Rightarrow \neg (A1 \cup A2)))$ By above and MP twice we have $B1, B2, ..., Bn |- \neg (A1 \cup A2)$ that is $B1, B2, ..., Bn |- A'$.

Now, the extended language has the Proof of Completeness Theorem as in the original language because of monotonicity.