QUESTION 1
Remark This question is designed to check if you understand the notion of completeness, monotonicity, application of Deduction Theorem and use of some basic tautologies.
Consider any proof system $S$, 
$$S = (\mathcal{L}_{\cap, \cup, \Rightarrow}, AX, (MP) \frac{A \Rightarrow B}{B})$$
that is complete under classical classical semantics.

Definition Let $X \subseteq F$ be any subset of the set $F$ of formulas of the language $\mathcal{L}_{\cap, \cup, \Rightarrow, \neg}$ of $S$.
We define a set $Cn(X)$ of all consequences of the set $X$ as follows
$$Cn(X) = \{ A \in F : X \vdash_A S A \},$$
i.e. $Cn(X)$ is the set of all formulas that can be proved in $S$ from the set $(AX \cup X)$.

Prove that for any $A, B \in F$,
$$Cn(\{A, B\}) = Cn(\{(A \cap B)\})$$

Hint: Use Deduction Theorem and Completeness of $S$.

Solution Assume $C \in Cn(\{A, B\})$, i.e. $\{A, B\} \vdash_S C$, what we usually write as $A, B \vdash_S C$.

By Deduction Theorem applied twice we get that
$$\vdash_S (A \Rightarrow (B \Rightarrow C)).$$

We use completeness of $S$ and the fact (proof by contradiction) that
$$\models (\models (A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C))$$
to construct the following.

$$\vdash_S (A \Rightarrow (B \Rightarrow C)) \quad (\text{assumption and Deduction Theorem}),$$
$$\vdash_S (((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C))), \quad (\text{completeness},$$
$$\vdash_S ((A \cap B) \Rightarrow C) \quad \text{MP},$$
$$\vdash_S (A \cap B) \Rightarrow C, \quad (\text{Deduction Theorem}).$$
i.e. we have proved that $C \in Cn(\{(A \cap B)\}) = Cn(A \cap B)$.

Assume now that $C \in Cn(\{(A \cap B)\})$, i.e. $\{A \cap B\} \vdash_S C$.

By Deduction Theorem, $\vdash_S (A \cap B) \Rightarrow C$.

We want to prove that $C \in Cn(\{A, B\})$, what is equivalent, by the Deduction Theorem applied twice to proving that
$$\vdash_S (A \Rightarrow (B \Rightarrow C)).$$
The proof as in the previous case. We use completeness of $S$, and the fact (proof by contradiction) that

$$|= (((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C)))$$

to get that $\vdash_S (A \Rightarrow (B \Rightarrow C))$ as follows.

$$\vdash_S ((A \cap B) \Rightarrow C) \text{ (assumption and Deduction Theorem)},$$

$$\vdash_S (((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))), \text{ (completeness)},$$

$$\vdash_S (A \Rightarrow (B \Rightarrow C)), \text{ MP},$$

what ends the proof.

**QUESTION 2**

Consider a system $\textbf{RS}_3$ obtained from $\textbf{RS}$ by changing the sequence $\Gamma'$ into $\Gamma$ in all of the rules of inference of $\textbf{RS}$.

1. Define SHORTLY Decomposition Tree for any $A$ in $\textbf{RS}_3$

**Solution**

The decomposition tree is a slight modification of definition of $\textbf{RS}$ tree; now we can decompose any decomposable formula at the decomposable node.

2. Show an example of a formula and its 2 decomposition trees

**Solution**

You can use any formula that leads to a node with at least two decomposable formulas.

item[3.] Prove Completeness Theorem for $\textbf{RS}_3$. We assume that the STRONG soundness has been proved.

**Solution**

The proof is a gain a modification of $\textbf{RS}$ proof.

Assume $\not\vdash_{\textbf{RS}_3} A$, i.e. $A$ does not have a proof in $\textbf{RS}_3$. Let $T_A$ be a set of all decomposition trees of $A$. As $\not\vdash_{\textbf{RS}_3} A$, each $T \in T_A$ has a non-axiom leaf.

We choose an arbitrary $T_A \in T_A$.

The non-axiom leaf $L_A$ defines a truth assignment $v$ which falsifies $A$, as follows:

$$v(a) = \begin{cases} 
F & \text{if } a \text{ appears in } L_A \\
T & \text{if } \neg a \text{ appears in } L_A \\
\text{any value} & \text{if } a \text{ does not appear in } L_A
\end{cases}$$
QUESTION 3

We know that a classical tautology \((\neg(a \cap b) \Rightarrow \neg(a \cup \neg b))\) is NOT Intuitionistic tautology and we know by Tarski Theorem that \(\neg\neg(\neg(a \cap b) \Rightarrow \neg(a \cup \neg b))\) is intuitionistically PROVABLE.

**FIND** the proof of the formula \(\neg\neg(\neg(a \cap b) \Rightarrow \neg(a \cup \neg b))\) in the Gentzen system LI for Intuitionistic Logic.

**Solution**

\[ T \rightarrow A \]

\[ \rightarrow \neg(\neg(a \cap b) \Rightarrow \neg(a \cup \neg b)) \]

\[ \mid (\rightarrow \neg) \]

\[ \neg(\neg(a \cap b) \Rightarrow \neg(a \cup \neg b)) \rightarrow \]

\[ \mid (\text{contr } \rightarrow) \]

\[ \neg(\neg(a \cap b) \Rightarrow \neg(a \cup \neg b)), \neg(\neg(a \cap b) \Rightarrow \neg(a \cup \neg b)) \rightarrow \]

\[ \mid (\neg \rightarrow) \]

\[ \neg(\neg(a \cap b) \Rightarrow \neg(a \cup \neg b)) \rightarrow (\neg(a \cap b) \Rightarrow \neg(a \cup \neg b)) \]

\[ \mid (\rightarrow \Rightarrow) \]

\[ \neg(a \cap b), \neg(\neg(a \cap b) \Rightarrow \neg(a \cup \neg b)) \rightarrow \neg(a \cup \neg b) \]

\[ \mid (\rightarrow \cup)_1 \]

\[ \neg(a \cap b), \neg(\neg(a \cap b) \Rightarrow \neg(a \cup \neg b)) \rightarrow \neg a \]

\[ \mid (\rightarrow \neg) \]

\[ a, \neg(a \cap b), \neg(\neg(a \cap b) \Rightarrow \neg(a \cup \neg b)) \rightarrow \]

\[ \mid (\text{exch } \rightarrow) \]

\[ \neg(a \cap b), a, \neg(\neg(a \cap b) \Rightarrow \neg(a \cup \neg b)) \rightarrow \]

\[ \mid (\neg \rightarrow) \]

\[ a, \neg(\neg(a \cap b) \Rightarrow \neg(a \cup \neg b)) \rightarrow (a \cap b) \]

\[ \bigwedge(\neg \cap) \]
\[ a, \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \rightarrow a \]

\text{axiom}

\[ a, \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \rightarrow b \]
\[ | (\rightarrow \text{weak}) \]
\[ a, \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \rightarrow \]
\[ | (\text{exch} \rightarrow) \]
\[ \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)), a \rightarrow \]
\[ | (\neg \rightarrow) \]
\[ a \rightarrow (\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \]
\[ | (\rightarrow \Rightarrow) \]
\[ \neg(a \cap b), a \rightarrow (\neg a \cup \neg b) \]
\[ | (\rightarrow \lor)_2 \]
\[ \neg(a \cap b), a \rightarrow \neg b \]
\[ | (\rightarrow \neg) \]
\[ b, \neg(a \cap b), a \rightarrow \]
\[ | (\text{exch} \rightarrow) \]
\[ \neg(a \cap b), b, a \rightarrow \]
\[ | (\neg \rightarrow) \]
\[ b, a \rightarrow (a \cap b) \]
\[ \bigwedge(\rightarrow \cap) \]

\[ b, a \rightarrow a \quad \text{axiom} \]
\[ b, a \rightarrow b \quad \text{axiom} \]

All leaves are axioms, the tree is a proof of \( A \) in \( \mathbf{LI} \).