LECTURE 9
Chapter 9
TWO PROOFS OF COMPLETENESS THEOREM

PART 1: Introduction
PART 2: System S Definition and Proof of the Main Lemma
PART 3: Proof 1: Constructive Proof of Completeness Theorem
PART 4: Proof 2: General Proof of Completeness Theorem
PART 1: Introduction
Two Proofs

There are many proof systems that describe classical propositional logic, i.e. that are complete proof systems with the respect to the classical semantics.

We present here a Hilbert proof system for the classical propositional logic and discuss two ways of proving the Completeness Theorem for it.

Any proof of the Completeness Theorem consists always of two parts.
Two Proofs

First we have show that all formulas that have a proof are tautologies.

This implication is also called a **Soundness Theorem**, or **Soundness Part** of the **Completeness Theorem**

The **second implication** says: if a formula is a tautology then it has a proof.

This alone is sometimes called a **Completeness Theorem** (on assumption that the system is sound)

Traditionally it is called a **Completeness Part** of the **Completeness Theorem**
Two Proofs

The proof of the soundness part is standard.

We concentrate here on the Completeness Part of the Completeness Theorem and present two proofs of it.

The first proof is straightforward.

It shows how one can use the assumption that a formula $A$ is a tautology in order to construct its formal proof.

It is hence called a proof-construction method.
Two Proofs

The **second proof** shows how one can **prove** that a formula \( A \) is not a tautology **from** the fact that it does not have a proof.

It is hence called a **counter-model construction method**.

All these **proofs** and considerations are **relative to proof systems** we discuss and **their semantics**.

At this moment the semantics is, of course, that for **classical propositional logic**.

**Reminder:** we write \( \models A \) to denote that \( A \) is a **classical tautology**.
Two Proofs

As far as the proof system is concerned we define here a certain class $S$ of proof systems, instead of one proof system.

We show that the Completeness Theorem holds for any system $S$ from this class $S$.

In particular, the system $H_2$ from chapter 8 is proved to be complete, as it belongs to the class of systems $S$. 
Proof System $H_2$

Reminder: $H_2$ is the following proof system:

$$H_2 = (\mathcal{L}_{\Rightarrow,\neg}, \mathcal{F}, \{A_1, A_2, A_3\}, MP)$$

The axioms $A_1 - A_3$ are defined as follows.

A1  $(A \Rightarrow (B \Rightarrow A))$,
A2  $(((A \Rightarrow (B \Rightarrow C))) \Rightarrow (((A \Rightarrow B) \Rightarrow (A \Rightarrow C))))$,
A3  $((\neg B \Rightarrow \neg A) \Rightarrow (((\neg B \Rightarrow A) \Rightarrow B)))$

$(MP) \quad A ; (A \Rightarrow B) \quad \frac{}{B}$
Proof System $H_2$

**Obviously**, the selected axioms $A_1, A_2, A_3$ are *tautologies*, and the MP rule leads from tautologies to tautologies.

Hence our proof system $H_2$ is *sound* and the following theorem holds.

**Soundness Theorem**

For every formula $A \in \mathcal{F}$,

If $\vdash_{H_2} A$, then $\models A$
We have proved in Lecture 8 (Chapter 8) the following Lemma

Lemma
The following formulas are provable in $H_2$

1. $(A \Rightarrow A)$
2. $(\neg
\neg B \Rightarrow B)$
3. $(B \Rightarrow \neg\neg B)$
4. $(\neg A \Rightarrow (A \Rightarrow B))$
5. $((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$
6. $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$
7. $(A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B))))$
8. $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$
9. $((\neg A \Rightarrow A) \Rightarrow A)$
First Proof

The first proof of Completeness Theorem presented here is very elegant and simple, but is applicable only to the classical propositional logic.

This proof is, as was the proof of Deduction Theorem, a fully constructive.

The technique it uses, because of its specifics can’t be used even in a case of classical predicate logic, not to mention variety of non-classical logics.
Second Proof

The second proof is much more complicated.

Its strength and importance lies in a fact that the methods it uses can be applied in an extended version to the proof of completeness for classical predicate logic and some non-classical propositional and predicate logics.

The way we define a counter-model for any non-provable A is general and non-constructive.

We call it a counter-model existence method.
The two proofs of Completeness Theorem can be performed for any proof system $S$ for classical propositional logic in which the formulas 1, 3, 4, and 7-9 stated in the system $H_2$ Lemma of lecture 8 (Chapter 8) and all axioms of the system $H_2$ are provable.

We assume provability of these formulas as they are the only formulas used in the proof of Deduction Theorem, and in both proofs of the Completeness Theorem.

It means that both proofs are valid for any proof system $S$ defined on the next slide.
PART 2: SYSTEM S DEFINITION
PROOF OF THE MAIN LEMMA
The System S Definition

We define the system $S$ as follows

$$S = (\mathcal{L}_{\{\Rightarrow, \neg\}}, \mathcal{F}, LA, (MP))$$

where the set of logical axioms $LA \subseteq T$ is such that the formulas listed below are provable in $S$

1. $(A \Rightarrow (B \Rightarrow A))$
2. $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$
3. $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$
4. $(A \Rightarrow A)$
5. $(B \Rightarrow \neg\neg B)$
6. $(\neg A \Rightarrow (A \Rightarrow B))$
7. $(A \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B)))$
8. $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$
9. $((\neg A \Rightarrow A) \Rightarrow A)$
Soundness and Deduction Theorem

Observation 1
We have assumed that logical axioms \( LA \subseteq T \) and we know that \( MP \) is a sound rule of inference so we have the following

**Soundness Theorem for S**
For any formula \( A \in \mathcal{F} \),

If \( \vdash_s A \), then \( \models A \)

Observation 2:
All formulas that were used in its proof of Deduction Theorem are provable in \( S \), so the following theorem holds.

**Deduction Theorem for S**
For any formulas \( A, B \in \mathcal{F} \) and \( \Gamma \subseteq \mathcal{F} \)

\[ \Gamma, A \vdash_s B \text{ if and only if } \Gamma \vdash_s (A \Rightarrow B) \]
PART 2: Proof of the MAIN LEMMA
Completeness Theorem

The proof of the Completeness Theorem presented here is similar in its structure to the proof of the Deduction Theorem and is due to Kalmar, 1935.

It is a constructive proof. It shows how one can use the assumption that a formula $A$ is a tautology in order to construct its formal proof.

We hence call it a proof construction method. It relies heavily on the Deduction Theorem.

It is possible to prove the Completeness Theorem independently from the Deduction Theorem and we will present two of such a proofs in later chapters.
Introduction

We first present one definition and prove one lemma.
We write \( \vdash A \) instead of \( \vdash_S A \) as the system \( S \) is fixed.

Let \( A \) be a formula and \( b_1, b_2, ..., b_n \) be all propositional variables that occur in \( A \), i.e.

\[
A = A(b_1, b_2, ..., b_n)
\]
Definition 1

Let \( v \) be a truth assignment \( v : \text{VAR} \rightarrow \{ T, F \} \). We define, for \( A, b_1, b_2, ..., b_n \) and truth assignment \( v \) corresponding formulas \( A', B_1, B_2, ..., B_n \) as follows:

\[
A' = \begin{cases} 
A & \text{if } v^*(A) = T \\
\neg A & \text{if } v^*(A) = F
\end{cases}
\]

\[
B_i = \begin{cases} 
b_i & \text{if } v(b_i) = T \\
\neg b_i & \text{if } v(b_i) = F
\end{cases}
\]

for \( i = 1, 2, ..., n \)
Example 1

Let $A$ be a formula $(a \Rightarrow \neg b)$

Let $v$ be such that $v(a) = T$, $v(b) = F$

In this case we have that $b_1 = a$, $b_2 = b$, and

$v^*(A) = v^*(a \Rightarrow \neg b) = v(a) \Rightarrow \neg v(b) = T \Rightarrow \neg F = T$

The corresponding $A', B_1, B_2$ are:

$A' = A$  as  $v^*(A) = T$

$B_1 = a$  as  $v(a) = T$

$B_2 = \neg b$  as  $v(b) = F$
Example 2

Let $A$ be a formula $((\neg a \Rightarrow \neg b) \Rightarrow c)$ and let $v$ be such that $v(a) = T$, $v(b) = F$, $v(c) = F$.

Evaluate $A'$, $B_1, \ldots, B_n$ as defined by the definition 1.

In this case $n = 3$ and $b_1 = a$, $b_2 = b$, $b_3 = c$.

and we evaluate

$v^*(A) = v^*((\neg a \Rightarrow \neg b) \Rightarrow c) = ((\neg v(a) \Rightarrow \neg v(b)) \Rightarrow v(c)) = ((\neg T \Rightarrow \neg F) \Rightarrow F) = (T \Rightarrow F) = F$

The corresponding $A'$, $B_1, B_2, B_3$ are:

$A' = \neg A = \neg((\neg a \Rightarrow \neg b) \Rightarrow c)$ as $v^*(A) = F$

$B_1 = a$ as $v(a) = T$, $B_2 = \neg b$ as $v(b) = F$, and

$B_3 = \neg c$ as $v(c) = F$
The lemma stated below describes a method of transforming a semantic notion of a tautology into a syntactic notion of provability.

It defines, for any formula \( A \) and a truth assignment \( v \) a corresponding deducibility relation.

**Main Lemma**

For any formula \( A = A(b_1, b_2, ..., b_n) \) and any truth assignment \( v \)

If \( A', B_1, B_2, ..., B_n \) are corresponding formulas defined by definition 1, then

\[
B_1, B_2, ..., B_n \vdash A'
\]
Examples

Example 3
Let $A, v$ be as defined in the **Example 1**, i.e. $A' = A$, $B_1 = a$, $B_2 = \neg b$

**Main Lemma** asserts that

$$a, \neg b \vdash (a \Rightarrow \neg b)$$

Example 4
Let $A, v$ be defined as in **Example 2**, then the **Main Lemma** asserts that

$$a, \neg b, \neg c \vdash \neg((\neg a \Rightarrow \neg b) \Rightarrow c)$$
Proof of the Main Lemma

The proof is by induction on the degree of the formula \( A \).

**Base Case** \( n = 0 \)

In this case \( A \) is atomic and so consists of a single propositional variable, say \( a \).

If \( v^*(A) = T \) then we have by definition 1

\[
A' = A = a, \quad B_1 = a
\]

We obtain, by definition of provability from a set \( \Gamma \) of hypothesis for \( \Gamma = \{a\} \) that

\[
a \vdash a
\]
Proof of the Main Lemma

If \( \nu^*(A) = F \) we have by Definition 1 that

\[ A' = \neg A = \neg a \quad \text{and} \quad B_1 = \neg a \]

We obtain, by definition of provability from a set \( \Gamma \) of hypothesis for \( \Gamma = \{\neg a\} \) that

\[ \neg a \vdash \neg a \]

This proves that Main Lemma holds for \( n=0 \)
Proof of the Main Lemma

Inductive Step
Now assume that the Lemma holds for any formula with \( j < n \) connectives

Need to prove: the Lemma holds for \( A \) with \( n \) connectives
There are several sub-cases to deal with

Case: \( A \) is \( \neg A_1 \)

By the inductive assumption we have the formulas

\[ A', B_1, B_2, ..., B_n \]

corresponding to the \( A_1 \) and the propositional variables \( b_1, b_2, ..., b_n \) in \( A_1 \), such that

\[ B_1, B_2, ..., B_n \vdash A' \]

Observe that the formulas \( A \) and \( \neg A_1 \) have the same propositional variables
So the corresponding formulas \( B_1, B_2, ..., B_n \) are the same for both of them.
Proof of the Main Lemma

We are going to show that the inductive assumption allows us to prove that

\[ B_1, B_2, \ldots, B_n \vdash A' \]

There are two cases to consider.

**Case:** \( v^*(A_1) = T \)

If \( v^*(A_1) = T \) then by definition 1 \( A'_1 = A_1 \) and by the inductive assumption

\[ B_1, B_2, \ldots, B_n \vdash A_1 \]

In this case: \( v^*(A) = v^*(\neg A_1) = \neg v^*(T) = F \)

So we have that \( A' = \neg A = \neg \neg A_1 \)
Proof of the Main Lemma

Since we have assumed 5. about $S$, i.e. we have that that

$$\vdash (A_1 \Rightarrow \neg\neg A_1)$$

we obtain by the monotonicity that also

$$B_1, B_2, \ldots, B_n \vdash (A_1 \Rightarrow \neg\neg A_1)$$

By inductive assumption $B_1, B_2, \ldots, B_n \vdash A_1$ and by MP we have

$$B_1, B_2, \ldots, B_n \vdash \neg\neg A_1$$

and as $A' = \neg A = \neg\neg A_1$ we get

$$B_1, B_2, \ldots, B_n \vdash \neg A$$ and so $$B_1, B_2, \ldots, B_n \vdash A'$$
Proof of the Main Lemma

Case: \( v^*(A_1) = F \)

If \( v^*(A_1) = F \) then \( A_1' = \neg A_1 \) and \( v^*(A) = T \) so \( A' = A \)

Therefore by the **inductive assumption** we have that

\[
B_1, B_2, ..., B_n \vdash \neg A_1
\]

that is as \( A = \neg A_1 \)

\[
B_1, B_2, ..., B_n \vdash A'
\]
Proof of the Main Lemma

Case: $A$ is $(A_1 \Rightarrow A_2)$

If $A$ is $(A_1 \Rightarrow A_2)$ then $A_1$ and $A_2$ have less than $n$ connectives

$A = A(b_1, ... b_n)$ so there are some subsequences $c_1, ..., c_k$ and $d_1, ... d_m$ for $k, m \leq n$ of the sequence $b_1, ..., b_n$ such that

$A_1 = A_1(c_1, ..., c_k)$ and $A_2 = A(d_1, ... d_m)$
Proof of the Main Lemma

$A_1$ and $A_2$ have less than $n$ connectives and so by the inductive assumption we have appropriate formulas $C_1, ..., C_k$ and $D_1, ..., D_m$ such that

$$C_1, C_2, \ldots, C_k \vdash A_1' \quad \text{and} \quad D_1, D_2, \ldots, D_m \vdash A_2'$$

and $C_1, C_2, ..., C_k$, $D_1, D_2, ..., D_m$ are subsequences of formulas $B_1, B_2, ..., B_n$ corresponding to the propositional variables in $A$

By the inductive assumption and monotonicity we have

$$B_1, B_2, ..., B_n \vdash A_1' \quad \text{and} \quad B_1, B_2, ..., B_n \vdash A_2'$$

Now we have the following sub-cases to consider
Proof of the Main Lemma

Case: \( v^*(A_1) = v^*(A_2) = T \)

If \( v^*(A_1) = T \) then \( A_1' = A_1 \) and
if \( v^*(A_2) = T \) then \( A_2' = A_2 \)

We also have \( v^*(A_1 \Rightarrow A_2) = T \) and so \( A' = (A_1 \Rightarrow A_2) \)

By the above and the inductive assumption

\[
B_1, B_2, ..., B_n \vdash A_2
\]

and since we have assumed 1. about \( S \) and by monotonicity we have

\[
B_1, B_2, ..., B_n \vdash (A_2 \Rightarrow (A_1 \Rightarrow A_2))
\]

By above and MP we have \( B_1, B_2, ..., B_n \vdash (A_1 \Rightarrow A_2) \) that is

\[
B_1, B_2, ..., B_n \vdash A'
\]
Proof of the Main Lemma

Case: $\nu^*(A_1) = T, \nu^*(A_2) = F$

If $\nu^*(A_1) = T$ then $A_1' = A_1$ and
if $\nu^*(A_2) = F$ then $A_2' = \neg A_2$

Also we have in this case $\nu^*(A_1 \Rightarrow A_2) = F$ and so $A' = \neg (A_1 \Rightarrow A_2)$

By the above, the inductive assumption and monotonicity $B_1, B_2, ..., B_n \vdash \neg A_2$

Since we have assumed 7. about $S$ and by monotonicity we have

$$B_1, B_2, ..., B_n \vdash (A_1 \Rightarrow (\neg A_2 \Rightarrow \neg (A_1 \Rightarrow A_2)))$$

By above and MP twice we have $B_1, B_2, ..., B_n \vdash \neg (A_1 \Rightarrow A_2)$ that is

$$B_1, B_2, ..., B_n \vdash A'$$
Proof of the Main Lemma

Case: \( v^*(A_1) = F \)

Observe that if \( v^*(A_1) = F \) then \( A_1' \) is \( \neg A_1 \) and, whatever value \( v \) gives \( A_2 \), we have

\[
v^*(A_1 \Rightarrow A_2) = T
\]

So \( A' \) is \( (A_1 \Rightarrow A_2) \)

Therefore

\[
B_1, B_2, \ldots, B_n \vdash \neg A_1
\]

Since by formula 6. is provable in \( S \), we have by monotonicity

\[
B_1, B_2, \ldots, B_n \vdash (\neg A_1 \Rightarrow (A_1 \Rightarrow A_2))
\]
Proof of the Main Lemma

By Modus Ponens we get that

\[ B_1, B_2, ..., B_n \vdash (A_1 \Rightarrow A_2) \]

that is

\[ B_1, B_2, ..., B_n \vdash A' \]

We have covered all cases and, by mathematical induction on the degree of the formula \( A \) we got

\[ B_1, B_2, ..., B_n \vdash A' \]

The proof of the **Main Lemma** is complete
PART3

Proof 1: Constructive Proof of Completeness Theorem
Proof of Completeness Theorem

Now we use the **Main Lemma** to prove the **Completeness Theorem** i.e. to prove the following implication

For any formula $A \in \mathcal{F}$

$$\text{if } \models A \text{ then } \vdash A$$

**Proof**

Assume that $\models A$

Let $b_1, b_2, ..., b_n$ be all propositional variables that occur in the formula $A$, i.e.

$$A = A(b_1, b_2, ..., b_n)$$

By the **Main Lemma** we know that, for any truth assignment $v$, the corresponding formulas $A', B_1, B_2, ..., B_n$ can be found such that

$$B_1, B_2, ..., B_n \vdash A'$$
Proof

**Note that** $A'$ in this case is $A$ for any $v$ since $\models A$

Hence, if $v$ is such that $v(b_n) = T$, then $B_n = b_n$ and

$$B_1, B_2, ..., b_n \vdash A$$

If $v$ is such that $v(b_n) = F$, then $B_n = \neg b_n$ and by the **Main Lemma**

$$B_1, B_2, ..., \neg b_n \vdash A$$

So, by the **Deduction Theorem** we have

$$B_1, B_2, ..., B_{n-1} \vdash (b_n \Rightarrow A)$$

And

$$B_1, B_2, ..., B_{n-1} \vdash (\neg b_n \Rightarrow A)$$
Proof of Completeness Theorem

By assumed formula 8.

\[ \vdash ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B)) \]

and by monotonicity we have that

\[ B_1, B_2, ..., B_{n-1} \vdash ((b_n \Rightarrow A) \Rightarrow ((\neg b_n \Rightarrow A) \Rightarrow A)) \]

Applying Modus Ponens twice we get that

\[ B_1, B_2, ..., B_{n-1} \vdash A \]

Similarly, \( v^*(B_{n-1}) \) may be T or F

Applying the Main Lemma, the Deduction Theorem, monotonicity, formula 8. and Modus Ponens twice we can eliminate \( B_{n-1} \) just as we have eliminated \( B_n \)

After \( n \) steps, we finally obtain proof of \( A \) in \( S \), i.e. we have that

\[ \vdash A \]
Constructiveness of the Proof

Observe that our proof of the Completeness Theorem is a constructive one.

Moreover, we have used in it only Main Lemma and Deduction Theorem which both have a constructive proofs.

We can hence reconstruct proofs in each case when we apply these theorems back to the original axioms of the system $S$, and in particular to the original axioms $A_1 – A_3$ of $H_2$.

The same applies to the proofs in $H_2$ of all formulas 1. - 9. of the system $S$.

It means that for any $A$, such that $\models A$, the set $V_A$ of all $v$ restricted to $A$ provides us a method of a construction of the formal proof of $A$ in $H_2$, or in any system $S$ in which formulas 1. - 9. are provable.
Example

Example

The proof of **Completeness Theorem** defines a method of efficiently combining \( v \in V_A \) while **constructing** the proof of \( A \).

Let's consider the following **tautology** \( A = A(a, b, c) \):

\[
((\neg a \Rightarrow b) \Rightarrow (\neg(\neg a \Rightarrow b) \Rightarrow c)
\]

We present on the next slides all steps of the **Proof 1** as applied to \( A \).
Example

Given

\[ A(a, b, c) = ((\neg a \Rightarrow b) \Rightarrow (\neg(\neg a \Rightarrow b) \Rightarrow c) \]

By the **Main Lemma** and the assumption that 

\[ \models A(a, b, c) \]

any \( v \in V_A \) **defines** formulas \( B_a, B_b, B_c \) such that 

\[ B_a, B_b, B_c \vdash A \]

**The proof** is based on a method of using all \( v \in V_A \) (there is 8 of them) to **define** a process of **elimination** of all hypothesis \( B_a, B_b, B_c \) to **construct** the proof of \( A \), i.e. to prove that 

\[ \vdash A \]
Example

Step 1: elimination of $B_c$

Observe that by definition, $B_c$ is $c$ or $\neg c$ depending on the choice of $v \in V_A$

We choose two truth assignments $v_1 \neq v_2 \in V_A$ such that

$$v_1 | \{a, b\} = v_2 | \{a, b\} \quad \text{and} \quad v_1(c) = T, \quad v_2(c) = F$$

Case 1: $v_1(c) = T$

By by definition $B_c = c$

By our choice, the assumption that $\models A$ and the Main Lemma applied to $v_1$

$$B_a, B_b, c \vdash A$$

By Deduction Theorem we have that

$$B_a, B_b \vdash (c \Rightarrow A)$$
Example

Case 2: $v_2(c) = F$

By definition $B_c = \neg c$

By our choice, assumption that $\models A$, and the Main Lemma applied to $v_2$

$$B_a, B_b, \neg c \vdash A$$

By the Deduction Theorem we have that

$$B_a, B_b \vdash (\neg c \Rightarrow A)$$
Example

By the assumed provability of the formula 8. for $A = c, \ B = A$ we have that

$$\vdash ((c \Rightarrow A) \Rightarrow ((\neg c \Rightarrow A) \Rightarrow A))$$

By monotonicity we have that

$$B_a, B_b \vdash ((c \Rightarrow A) \Rightarrow ((\neg c \Rightarrow A) \Rightarrow A))$$

Applying Modus Ponens twice to the above property and properties on the previous slide we get that

$$B_a, B_b \vdash A$$

We have eliminated $B_c$
Example

Step 2: elimination of $B_b$ from $B_a, B_b \vdash A$

We repeat the Step 1

As before we have 2 cases to consider: $B_b = b$ or $B_b = \neg b$

We choose two truth assignments $w_1 \neq w_2 \in V_A$ such that

$$w_1|\{a\} = w_2|\{a\} = v_1|\{a\} = v_2|\{a\} \text{ and } w_1(b) = T, \; w_2(b) = F$$

Case 1: $w_1(b) = T$ and by definition $B_b = b$

By our choice, assumption that $\models A$ and the Main Lemma applied to $w_1$

$$B_a, b \vdash A$$

By Deduction Theorem we have that

$$B_a \vdash (b \Rightarrow A)$$
Example

Case 2: \( w_2(b) = F \) and by definition \( B_b = \neg b \)

By choice, assumption that \( \models A \) and the **Main Lemma** applied to \( w_2 \)

\[ B_a, \neg b \vdash A \]

By the **Deduction Theorem** we have that

\[ B_a \vdash (\neg b \Rightarrow A) \]
Example

By the assumed provability of the formula 8. for $A = b, B = A$ we have that

$$\vdash ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A))$$

By monotonicity

$$B_a \vdash ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A))$$

Applying **Modus Ponens** twice to the above property and properties from the previous slide we get that

$$B_a \vdash A$$

We have **eliminated** $B_b$
Example

Step 3: elimination] of $B_a$ from $B_a \vdash A$

We repeat the Step 2

As before we have 2 cases to consider: $B_a = a$ or $B_a = \neg a$

We choose two truth assignments $g_1 \neq g_2 \in V_A$ such that

$$g_1(a) = T \quad \text{and} \quad g_2(a) = F$$

Case 1: $g_1(a) = T$, and by definition $B_a = a$

By the choice, assumption that $\models A$, and the Main Lemma applied to $g_1$

$$a \vdash A$$

By Deduction Theorem we have that

$$\vdash (a \Rightarrow A)$$
Example

Case 2: \( g_2(a) = F \) and by definition \( B_a = \neg a \)

By the choice, assumption that \( \models A \), and the Main Lemma applied to \( g_2 \)

\[ \neg a \vdash A \]

By the Deduction Theorem we have that

\[ \vdash (\neg a \Rightarrow A) \]
Example

By the assumed provability of the formula 8. for $A = a$, $B = A$ we have that

$$
\vdash ((a \Rightarrow A) \Rightarrow ((\neg a \Rightarrow A) \Rightarrow A))
$$

Applying Modus Ponens twice to the above property and properties from previous slides we get that

$$
\vdash A
$$

We have eliminated $B_a$, $B_b$, $B_c$ and constructed the proof of $A$ in $S$.
**EXERCISES**

**Exercise 1**

The formulas 1. - 9. that we assumed to be provable in $S$ are those needed for 2 proofs of the **Completeness Theorem**. **List** the formulas that are needed for the **Proof 1** only.

**Exercise 2**

We proved **Completeness Theorem** for the language $L_{\Rightarrow,\neg}$ **Extend this proof** to the language $L_{\Rightarrow,\cup,\neg}$ by **adding** all new CASES and needed PROVABLE formulas to our list 1. - 9. or to a shorter list from solution of the **Exercise 1**