LECTURE 5a
Short REVIEW Chapters 1 - 5
DEFINITIONS: Chapter 3 and Chapter 4

Here I repeat for you some basic **DEFINITIONS** from Chapters 3, 4 and 5

You have to prepare them for MIDTERM 1

I will ask you to **WRITE** down a full, correct text of 1-3 of them - in EXACTLY the same form as they are presented here

Knowing all basic **Definitions** is the first step to understanding the material
DEFINITIONS

Definition 1
A propositional language is a pair

\[ \mathcal{L} = (\mathcal{A}, \mathcal{F}) \]

where \( \mathcal{A}, \mathcal{F} \) are called the alphabet and a set of formulas, respectively

Definition 2
An alphabet is a set \( \mathcal{A} = \text{VAR} \cup \text{CON} \cup \text{PAR} \)
\text{VAR}, \text{CON}, \text{PAR} are all disjoint sets of propositional variables, connectives and parenthesis, respectively

We assume that
1. \( \text{PAR} = \{(, 0)\} \)
2. \( \text{VAR} \) is a countably infinite set and denote elements of \( \text{VAR} \) by \( a, b, c, d, \ldots \), with indices if necessary
2. $CON \neq \emptyset$ contains only unary and binary connectives, i.e. $CON = C_1 \cup C_2$ where
   $C_1$ is the set of all unary connectives, and
   $C_2$ is the set of all binary connectives

Language Notation
We denote the language $L$ with the set of connectives $CON$ by

$$L_{CON}$$

Metalanguage notation; we use the set union symbol $\cup$ when needed; it is clear from the context that it is not the connective $\cup$ symbol from our language $L$
DEFINITIONS

Definition 3
The set $\mathcal{F}$ of all formulas of a propositional language $\mathcal{L}_{CON}$ is build recursively from the elements of the alphabet $\mathcal{A}$ as follows:

$\mathcal{F} \subseteq \mathcal{A}^*$ and $\mathcal{F}$ is the smallest set for which the following conditions are satisfied:

1. $\text{VAR} \subseteq \mathcal{F}$
2. If $A \in \mathcal{F}$, $\downarrow \in C_1$, then $\downarrow A \in \mathcal{F}$
3. If $A, B \in \mathcal{F}$, $\circ \in C_2$ i.e. $\circ$ is a two argument connective, then $(A \circ B) \in \mathcal{F}$

Propositional variables are formulas and they are called atomic formulas.
Question Example

Question
Use **Definitions 1,2, 3** to define the language \( L\{\neg, K, \cap, \Rightarrow\} \) where \( K \) is one argument knowledge connective

Solution

\[ L\{\neg, K, \cap, \Rightarrow\} = (A, F) \]

The components \( A, F \) are defined as follows

**Alphabet** is

\[ A = \text{VAR} \cup \{\neg, K, \cap, \Rightarrow\} \cup \{(, )\} \]
The set of all formulas is defined as follows:

$F \subseteq A^*$ and $F$ is the smallest set for which the following conditions are satisfied:

1. $\text{VAR} \subseteq F$
2. If $A \in F$, then $\neg A$, $KA \in F$
3. If $A, B \in F$, then $(A \cap B)$, $(A \Rightarrow B) \in F$
DEFINITIONS: Extension

Definition 4
Given the truth assignment (in classical semantics) \( v : \text{VAR} \rightarrow \{T, F\} \)
We define its extension \( v^* \) to the set \( \mathcal{F} \) of all formulas of \( L \) as \( v^* : \mathcal{F} \rightarrow \{T, F\} \) such that
(i) for any \( a \in \text{VAR} \)
\[ v^*(a) = v(a) \]
(ii) and for any \( A, B \in \mathcal{F} \) we put
\[ v^*(\neg A) = \neg v^*(A); \]
\[ v^*((A \cap B)) = \cap(v^*(A), v^*(B)); \]
\[ v^*((A \cup B)) = \cup(v^*(A), v^*(B)); \]
\[ v^*((A \Rightarrow B)) = \Rightarrow(v^*(A), v^*(B)); \]
\[ v^*((A \Leftrightarrow B)) = \Leftrightarrow(v^*(A), v^*(B)). \]
DEFINITIONS: Satisfaction Relation

Definition 5  Let $v : \text{VAR} \rightarrow \{ T, F \}$
We say that
$v$ satisfies a formula $A \in F$ iff $v^*(A) = T$

Notation: $v \models A$

Definition: We say that
$v$ does not satisfy a formula $A \in F$ iff $v^*(A) \neq T$

Notation: $v \not\models A$
DEFINITIONS: Model, Counter-Model, Tautology

Definition 6
Given a formula \( A \in \mathcal{F} \) and \( v : \text{VAR} \rightarrow \{T,F\} \)
We say that
\( v \) is a model for \( A \) iff \( v \models A \)
\( v \) is a counter-model for \( A \) iff \( v \not\models A \)

Definition 7
\( A \) is a tautology iff for any \( v : \text{VAR} \rightarrow \{T,F\} \) we have that \( v \models A \)

Notation
We write symbolically \( \models A \)
DEFINITIONS: Restricted Truth Assignments

**Notation:** for any formula $A$, we denote by $\text{VAR}_A$ a set of all variables that appear in $A$

**Definition 8** Given a formula $A \in \mathcal{F}$, any function

$$v_A : \text{VAR}_A \rightarrow \{T, F\}$$

is called a truth assignment restricted to $A$
DEFINITIONS: Models for Sets of Formulas

Consider $L = \mathcal{L}_{\{\neg, \cup, \cap, \Rightarrow\}}$ and let $S \neq \emptyset$ be any non empty set of formulas of $L$, i.e.

$$S \subseteq \mathcal{F}$$

Definition 9

A truth truth assignment $v : \text{VAR} \rightarrow \{T, F\}$ is a model for the set $S$ of formulas if and only if

$$v \models A \text{ for all formulas } A \in S$$

We write

$$v \models S$$

to denote that $v$ is a model for the set $S$ of formulas.
DEFINITIONS: Consistent Sets of Formulas

Definition 10
A set \( G \subseteq \mathcal{F} \) of formulas is called **consistent** if and only if \( G \) has a model, i.e. we have that

\[ G \subseteq \mathcal{F} \text{ is consistent if and only if there is } v \text{ such that } v \models G \]

Otherwise \( G \) is called **inconsistent**
DEFINITIONS: Independent Statements

Definition 11

A formula $A$ is called **independent** from a set $G \subseteq \mathcal{F}$ if and only if there are truth assignments $v_1, v_2$ such that

$$v_1 \models G \cup \{A\} \text{ and } v_2 \models G \cup \{\neg A\}$$

i.e. we say that a formula $A$ is **independent** if and only if

$$G \cup \{A\} \text{ and } G \cup \{\neg A\} \text{ are consistent}$$
The extensional semantics $\mathcal{M}$ is defined for a non-empty set of $V$ of logical values of any cardinality. We only assume that the set $V$ of logical values of $\mathcal{M}$ always has a special, distinguished logical value which serves to define a notion of tautology.

We denote this distinguished value as $T$.

Formal definition of many valued extensional semantics $\mathcal{M}$ for the language $\mathcal{L}_{\text{CON}}$ consists of giving definitions of the following main components:

1. Logical Connectives under semantics $\mathcal{M}$
2. Truth Assignment for $\mathcal{M}$
3. Satisfaction Relation, Model, Counter-Model under semantics $\mathcal{M}$
4. Tautology under semantics $\mathcal{M}$
Definition of $\mathcal{M}$ - Extensional Connectives

Given a propositional language $\mathcal{L}_{CON}$ for the set $CON = C_1 \cup C_2$, where $C_1$ is the set of all unary connectives, and $C_2$ is the set of all binary connectives.

Let $V$ be a non-empty set of logical values adopted by the semantics $\mathcal{M}$.

**Definition**

Connectives $\nabla \in C_1$, $\circ \in C_2$ are called $\mathcal{M}$ -extensional iff their semantics $\mathcal{M}$ is defined by respective functions

$$\nabla : V \rightarrow V \quad \text{and} \quad \circ : V \times V \rightarrow V$$
DEFINITION: Definability of Connectives under a semantics $M$

Given a propositional language $\mathcal{L}_{CON}$ and its extensional semantics $M$

We adopt the following definition

Definition

A connective $\circ \in CON$ is **definable** in terms of some connectives $\circ_1, \circ_2, ... \circ_n \in CON$ for $n \geq 1$ under the semantics $M$ if and only if the connective $\circ$ is a certain function composition of functions $\circ_1, \circ_2, ... \circ_n$ as they are defined by the semantics $M$
DEFINITION: M Truth Assignment Extension $v^*$ to $F$

Definition
Given the M truth assignment $v : \text{VAR} \rightarrow V$
We define its M extension $v^*$ to the set $F$ of all formulas of $\mathcal{L}$ as any function $v^* : F \rightarrow V$, such that the following conditions are satisfied

(i) for any $a \in \text{VAR}$

$$v^*(a) = v(a);$$

(ii) For any connectives $\nabla \in C_1$, $\circ \in C_2$ and for any formulas $A, B \in F$ we put

$$v^*(\nabla A) = \nabla v^*(A)$$

$$v^*((A \circ B)) = \circ(v^*(A), v^*(B))$$
**Definition:** Let $v : VAR \rightarrow V$

Let $T \in V$ be the distinguished logical value.

We say that $v$ **M satisfies** a formula $A \in \mathcal{F}$ ($v \models_M A$) if and only if $v^*(A) = T$.

**Definition:**

Given a formula $A \in \mathcal{F}$ and $v : VAR \rightarrow V$.

Any $v$ such that $v \models_M A$ is called a **M model** for $A$.

Any $v$ such that $v \not\models_M A$ is called a **M counter model** for $A$.

$A$ is a **M tautology** ($\models_M A$) if and only if $v \models_M A$, for all $v : VAR \rightarrow V$. 

Chapter 5: Challenge Exercise

1. Define your own propositional language $\mathcal{L}_{CON}$ that contains also different connectives that the standard connectives $\neg, \cup, \cap, \implies$

Your language $\mathcal{L}_{CON}$ does not need to include all (if any!) of the standard connectives $\neg, \cup, \cap, \implies$

2. Describe intuitive meaning of the new connectives of your language

3. Give some motivation for your own semantic

4. Define formally your own extensional semantics $M$ for your language $\mathcal{L}_{CON}$ - it means

write carefully all Steps 1-4 of the definition of your $M$
Challenge Problems

Work on Challenge Problems posted in Lectures 3-5
Chapter 3: Question 1

Question 1  Write the following natural language statement:

*From the fact that it is not necessary that a red flower is not a bird we deduce that:*

*it is not possible that the red flower is a bird or, if it is possible that the red flower is a bird, then it is not necessary that a bird flies*

as a formulas of two languages

1. $A_1 \in F_1$ of a language $L_{\{\neg,\cap,\cup,\Rightarrow\}}$
2. $A_2 \in F_2$ of a language $L_{\{\neg,\cap,\cup,\Rightarrow\}}$
Chapter 3: Question 1

Solution  The statement is:

*From the fact that it is not necessary that a red flower is not a bird we deduce that:*

it is not possible that the red flower is a bird or, if it is possible that the red flower is a bird, then it is not necessary that a bird flies

1. We translate our statement into a formula $A_1 \in F$ of a language $L_{\{\neg, c, i, \land, \lor, \rightarrow\}}$ as follows.

Propositional Variables: a, b, where

a denotes statement: *red flower is a bird*,
b denotes statement: *a bird flies*

Propositional Modal Connectives: C, I

C denotes statement: *it is possible that*, I denotes statement: *it is necessary that*
Chapter 3: Question 1

Solution  The statement is :

*From the fact that it is not necessary that a red flower is not a bird we deduce that:*

*it is not possible that the red flower is a bird or, if it is possible that the red flower is a bird, then it is not necessary that a bird flies*

Translation  for the language $\mathcal{L}_{\neg, \land, \lor, \Rightarrow}$ is

$$A_1 = (\neg \neg a \Rightarrow (\neg Ca \lor (Ca \Rightarrow \neg b)))$$

Observe that you could also use symbols $\Box$ for necessity and $\Diamond$ for possibility but in this case the formula would belong to the language $\mathcal{L}_{\neg, \Diamond, \Box, \land, \lor, \Rightarrow}$ and hence not to the language $\mathcal{L}_{\neg, c, i, \land, \lor, \Rightarrow}$ as stated in the Question
Chapter 3: Question 1

The statement is:

*From the fact that it is not necessary that a red flower is not a bird we deduce that:*

*it is not possible that the red flower is a bird or, if it is possible that the red flower is a bird, then it is not necessary that a bird flies*

2. Now we translate our statement into a formula \( A_2 \in \mathcal{F}_2 \) of a language \( \mathcal{L}\{\neg,\cap,\cup,\Rightarrow\} \) as follows

Propositional Variables: a, b, c

- a denotes statement: *it is necessary that a red flower is not a bird*
- b denotes statement: *it is possible that a red flower is a bird*
- c denotes a statement: *it is necessary that a bird flies*

Translation

\[
A_2 = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))
\]
Chapter 3: Question 2

Question 2

1. Determine the main connectives and degrees of the formulas from Q4, i.e. of

\[ A_1 = (\neg I \neg a \Rightarrow (\neg C a \cup (C a \Rightarrow \neg I b))) \],

\[ A_2 = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))) \]

Solution

Main connective of the formula \( A_1 \) is: \( \Rightarrow \)
Main connective of the formula \( A_2 \) is also: \( \Rightarrow \)
Degree of the formula \( A_1 \) is: 11
Degree of the formula \( A_2 \) is: 6
Chapter 3: Question 2

2. **Determine** all proper, non-atomic sub-formulas of $A_1$, and non-atomic sub-formulas of $A_2$

**Solution**

All proper, non-atomic sub-formulas of $A_1$ are:

$\neg I \neg a, (\neg C a \cup (C a \Rightarrow \neg I b)), I \neg a, \neg a, \neg C a, (C a \Rightarrow \neg I b), C a, \neg I b, I b$

All non-atomic sub-formulas of $A_2$ are:

$(\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)), \neg a, (\neg b \cup (b \Rightarrow \neg c)), \neg b, (b \Rightarrow \neg c), \neg c$
CHAPTER 4: Question 3

Question 3
1. Find a restricted model for formula A, where

\[ A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))) \]

You can’t use short-hand notation
Show each step of solution

Solution
For any formula A, we denote by \( \text{VAR}_A \) a set of all variables that appear in A.
In our case we have \( \text{VAR}_A = \{ a, b, c \} \).
Any function \( v_A : \text{VAR}_A \rightarrow \{ T, F \} \) is called a truth assignment restricted to A.
Let $v : \text{VAR} \rightarrow \{T, F\}$ be any truth assignment such that $v(a) = v_A(a) = T$, $v(b) = v_A(b) = T$, $v(c) = v_A(c) = F$.

We evaluate the value of the extension $v^*$ of $v$ on the formula $A$ as follows:

$v^*(A) = v^*((\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))))$

$= v^*(\neg a) \Rightarrow v^*((\neg b \cup (b \Rightarrow \neg c)))$

$= \neg v^*(a) \Rightarrow (v^*(\neg b) \cup v^*((b \Rightarrow \neg c)))$

$= \neg v(a) \Rightarrow (\neg v(b) \cup (v(b) \Rightarrow \neg v(c)))$

$= \neg v_A(a) \Rightarrow (\neg v_A(b) \cup (v_A(b) \Rightarrow \neg v_A(c)))$

$(\neg T \Rightarrow (\neg T \cup (T \Rightarrow \neg F))) = F \Rightarrow (F \cup T) = F \Rightarrow T = T$, i.e.

$v_A \models A$ and $v \models A$.
Chapter 4: Question 4

Question 4
1. Find a restricted model and a restricted counter-model for $A$, where

$$A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$$

You can use short-hand notation. Show work

Solution
Notation: for any formula $A$, we denote by $\text{VAR}_A$ a set of all variables that appear in $A$

In our case we have $\text{VAR}_A = \{a, b, c\}$

Any function $v_A : \text{VAR}_A \rightarrow \{T, F\}$ is called a truth assignment restricted to $A$

We define now $v_A(a) = T$, $v_A(b) = T$, $v_A(c) = F$, in shorthand: $a = T, b = T, c = F$ and evaluate

$$(\neg T \Rightarrow (\neg T \cup (T \Rightarrow \neg F))) = F \Rightarrow (F \cup T) = F \Rightarrow T = T$$

i.e.

$$v_A \models A$$
Chapter 4: Question 4

Observe that

\((\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))) = T\) when \(a = T\) and \(b, c\) any truth values as by definition of implication we have that \(F \Rightarrow \text{anything} = T\).

Hence \(a = T\) gives us 4 models as we have \(2^2\) possible values on \(b\) and \(c\).
Chapter 4: Question 4

We take as a restricted counter-model: $a=F$, $b=T$ and $c=T$

**Evaluation:** observe that

$$(\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))) = F$$ if and only if $\neg a = T$ and $(\neg b \cup (b \Rightarrow \neg c)) = F$ if and only if $a = F$, $\neg b = F$ and $(b \Rightarrow \neg c) = F$ if and only if $a = F$, $b = T$ and $(T \Rightarrow \neg c) = F$ if and only if $a = F$, $b = T$ and $\neg c = F$ if and only if $a = F$, $b = T$ and $c = T$

The above proves also that $a=F$, $b=T$ and $c=T$ is the only restricted counter-model for $A$
Question 5  Justify whether the following statements true or false

S1  There are more than 3 possible restricted counter-models for $A$

S2  There are more than 2 possible restricted models of $A$

Solution

Statement: There are more than 3 possible restricted counter-models for $A$ is false

We have just proved that there is only one possible restricted counter-model for $A$

Statement: There are more than 2 possible restricted models of $A$ is true

There are 7 possible restricted models for $A$

Justification: $2^3 - 1 = 7$
Chapter 4: Question 6

Question 6

1. List 3 models and 2 counter-models for $A$ from Question 3, i.e.

$$A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$$

that are extensions to the set $VAR$ of all variables of one the restricted models and of one of the restricted counter-models that you have found in Questions 3, 4
Chapter 4: Question 6

Solution
One of the restricted models is, for example a function
\[ v_A : \{a, b, c\} \rightarrow \{T, F\} \] such that
\[ v_A(a) = T, \; v_A(b) = T, \; v_A(c) = F \]
We extend \( v_A \) to the set of all propositional variables \( \text{VAR} \)
to obtain a (non restricted) models as follows
Chapter 4: Question 6

Model \( w_1 \) is a function

\[
\begin{align*}
w_1 : \text{VAR} & \rightarrow \{T, F\} \quad \text{such that} \\
w_1(a) &= v_A(a) = T, \quad w_1(b) = v_A(b) = T, \\
w_1(c) &= v_A(c) = F, \quad \text{and} \quad w_1(x) = T, \quad \text{for all} \\
x &\in \text{VAR} - \{a, b, c\}
\end{align*}
\]

Model \( w_2 \) is defined by a formula

\[
\begin{align*}
w_2(a) &= v_A(a) = T, \quad w_2(b) = v_A(b) = T, \\
w_2(c) &= v_A(c) = F, \quad \text{and} \quad w_2(x) = F, \quad \text{for all} \\
x &\in \text{VAR} - \{a, b, c\}
\end{align*}
\]
Chapter 4: Question 6

Model  \( w_3 \) is defined by a formula

\[
\begin{align*}
  w_3(a) &= v_A(a) = T, \\
  w_3(b) &= v_A(b) = T, \\
  w_3(c) &= v(c) = F, \\
  w_3(d) &= F 	ext{ and } w_3(x) = T \text{ for all } x \in VAR - \{a, b, c, d\}
\end{align*}
\]

There is as many of such models, as extensions of \( v_A \) to the set \( VAR \), i.e. as many as real numbers.
A counter-model for a formula $A$, by definition, is any function
\[ v : \text{VAR} \rightarrow \{ T, F \} \]
such that \[ v^*(A) = F \]
A restricted counter-model for $A$ (only one as proved in question 5) is a function
\[ v_A : \{a, b\} \rightarrow \{ T, F \} \]
such that
\[ v_A(a) = F, \quad v_A(b) = T, \quad v_A(c) = T \]
We extend \( v_A \) to the set of all propositional variables \( \text{VAR} \) to obtain (non restricted) some counter-models.

Here are two of such extensions

**Counter-model** \( w_1 \):

\[
\begin{align*}
    w_1(a) &= v_A(a) = F, & w_1(b) &= v_A(b) = T, \\
    w_1(c) &= v(c) = T, & \text{and } w_1(x) &= F, \text{ for all } x \in \text{VAR} - \{a, b, c\}
\end{align*}
\]

**Counter-model** \( w_2 \):

\[
\begin{align*}
    w_2(a) &= v_A(a) = T, & w_2(b) &= v_A(b) = T, \\
    w_2(c) &= v(c) = T, & \text{and } w_2(x) &= T \text{ for all } x \in \text{VAR} - \{a, b, c\}
\end{align*}
\]

There is as many of such counter-models, as extensions of \( v_A \) to the set \( \text{VAR} \), i.e. as many as real numbers.
Chapter 4: Models for Sets of Formulas

Definition
A truth assignment $v$ is a model for a set $G \subseteq \mathcal{F}$ of formulas of a given language $\mathcal{L} = \mathcal{L}\{\neg, \Rightarrow, \cup, \cap\}$ if and only if $v \models B$ for all $B \in G$.

We denote it by $v \models G$.

Observe that the set $G \subseteq \mathcal{F}$ can be finite or infinite.
Chapter 4: Consistent Sets of Formulas

Definition
A set $G \subseteq F$ of formulas is called consistent if and only if $G$ has a model, i.e. we have that

$G \subseteq F$ is consistent if and only if there is $v$ such that $v \models G$

Otherwise $G$ is called inconsistent
Chapter 4: Independent Statements

Definition
A formula $A$ is called independent from a set $G \subseteq \mathcal{F}$ if and only if there are truth assignments $v_1, v_2$ such that

$$v_1 \models G \cup \{A\} \quad \text{and} \quad v_2 \models G \cup \{\neg A\}$$

i.e. we say that a formula $A$ is independent if and only if

$$G \cup \{A\} \quad \text{and} \quad G \cup \{\neg A\}$$

are consistent.
Chapter 4: Question 7

Question 7
Given a set

\[ G = \{((a \cap b) \Rightarrow b), \ (a \cup b), \neg a\} \]

Show that \( G \) is consistent

Solution
We have to find \( v : VAR \rightarrow \{T, F\} \) such that

\[ v \models G \]

It means that we need to find \( v \) such that

\[ v^*((a \cap b) \Rightarrow b) = T, \ v^*(a \cup b) = T, \ v^*(\neg a) = T \]
Chapter 4: Question 7

Observe that \( \models ((a \land b) \Rightarrow b) \), hence we have that

1. \( v^*((a \land b) \Rightarrow b) = T \) for any \( v \)

\( v^*(\neg a) = \neg v^*(a) = \neg v(a) = T \) only when \( v(a) = F \) hence

2. \( v(a) = F \)

\( v^*(a \cup b) = v^*(a) \cup v^*(b) = v(a) \cup v(b) = F \cup v(b) = T \) only when \( v(b) = T \) so we get

3. \( v(b) = T \)

This means that for any \( v : \text{VAR} \rightarrow \{T, F\} \) such that \( v(a) = F, \ v(b) = T \)

\[ v \models G \]

and we proved that \( G \) is consistent
Chapter 4: Question 8

Question 8
Show that a formula $A = (\neg a \cap b)$ is not independent of

$$G = \{((a \cap b) \Rightarrow b), (a \cup b), \neg a\}$$

Solution
We have to show that it is impossible to construct $v_1, v_2$ such that

$$v_1 \models G \cup \{A\} \quad \text{and} \quad v_2 \models G \cup \{\neg A\}$$

Observe that we have just proved that any $v$ such that $v(a) = F$, and $v(b) = T$ is the only model restricted to the set of variables $\{a, b\}$ for $G$ so we have to check now if it is possible that $v \models A$ and $v \models \neg A$
Chapter 4: Question 8

We have to evaluate $v^*(A)$ and $v^*(\neg A)$ for $v(a) = F$, and $v(b) = T$

$v^*(A) = v^*((\neg a \cap b)) = \neg v(a) \cap v(b) = \neg F \cap T = T \cap T = T$
and so $v \models A$

$v^*(\neg A) = \neg v^*(A) = \neg T = F$
and so $v \not\models \neg A$

This end the proof that $A$ is not independent of $G$
Chapter 4: Question 9

Question 9
2. Find an infinite number of formulas that are independent of \( G = \{((a \land b) \Rightarrow b), (a \lor b), \neg a\} \)

This my solution - there are many others- this one seemed to me the most simple

Solution

We just proved that any \( v \) such that \( v(a) = F, \ v(b) = T \) is the only model restricted to the set of variables \( \{a, b\} \) and so all other possible models for \( G \) must be extensions of \( v \)
Chapter 4: Question 9

We define a countably infinite set of formulas (and their negations) and corresponding extensions of \( \nu \) (restricted to to the set of variables \( \{a, b\} \)) such that \( \nu \models G \) as follows.

Observe that all extensions of \( \nu \) restricted to the set of variables \( \{a, b\} \) have as domain the infinitely countable set

\[
VAR - \{a, b\} = \{a_1, a_2, \ldots, a_n, \ldots\}
\]

We take as a set of formulas (to be proved to be independent) the set of atomic formulas

\[
\mathcal{F}_0 = \{a_1, a_2, \ldots, a_n, \ldots\}
\]
Chapter 4: Question 9

We define now two sequences 
\{v_i\}_{n \geq 1} \text{ and } \{w_i\}_{n \geq 1} \text{ of extensions of } v \text{ as follows}

\(v_i: \text{VAR} - \{a, b\} \rightarrow \{T, F\}\) is such that \(v_i(a_i) = T\)

\(w_i: \text{VAR} - \{a, b\} \rightarrow \{T, F\}\) is such that \(w_i(a_i) = F\)

By definition of the extension we have that

\(v_i \models G\) \quad \(w_i \models G\) \quad \text{for all } i \geq 1 \quad \text{and}

\[v_i \models G \cup \{a_i\} \quad \text{and} \quad w_i \models G \cup \{\neg a_i\}\]

This proves that each formula \(a_i \in F_0\) is independent of the set \(G\)
CHAPTER 5
Some Extensional Many Valued Semantics
Chapter 5: Question 10

Question 10
We define a 4 valued $H_4$ logic semantics as follows

The language is $\mathcal{L} = \mathcal{L}\{\neg, \Rightarrow, \cup, \cap\}$

The logical connectives $\neg, \Rightarrow, \cup, \cap$ of $H_4$ are operations in the set $\{F, \bot_1, \bot_2, T\}$, where $\{F < \bot_1 < \bot_2 < T\}$ and are defined as follows

Conjunction $\cap$ is a function

$\cap : \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \rightarrow \{F, \bot_1, \bot_2, , T\}$, such that for any $a, b \in \{F, \bot_1, \bot_2, T\}$

$$a \cap b = \min\{a, b\}$$
Chapter 5: Many Valued Semantics

**Disjunction** $\cup$ is a function

$\cup : \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \rightarrow \{F, \bot_1, \bot_2, T\}$, such that for any $a, b \in \{F, \bot_1, \bot_2, T\}$

$$a \cup b = \max\{a, b\}$$

**Implication** $\Rightarrow$ is a function

$\Rightarrow : \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \rightarrow \{F, \bot_1, \bot_2, T\}$, such that for any $a, b \in \{F, \bot_1, \bot_2, T\}$,

$$a \Rightarrow b = \begin{cases} T & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$$

**Negation:**

$$\neg a = a \Rightarrow F$$
Chapter 5: Question 10

Part 1  Write Truth Tables for IMPLICATION and NEGATION in $H_4$

Solution

$H_4$ Implication

<table>
<thead>
<tr>
<th>$\Rightarrow$</th>
<th>F</th>
<th>$\perp_1$</th>
<th>$\perp_2$</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>$\perp_1$</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>$\perp_2$</td>
<td>F</td>
<td>$\perp_1$</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>$\perp_1$</td>
<td>$\perp_2$</td>
<td>T</td>
</tr>
</tbody>
</table>

$H_4$ Negation

<table>
<thead>
<tr>
<th>$\neg$</th>
<th>F</th>
<th>$\perp_1$</th>
<th>$\perp_2$</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
Chapter 5: Question 10

Part 2  Verify whether

\[ \models_{H_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b)) \]

Solution
Take any \( v \) such that
\[ v(a) = \bot_1, \quad v(b) = \bot_2 \]
Evaluate
\[ v \star ((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = (\bot_1 \Rightarrow \bot_2) \Rightarrow (\neg \bot_1 \cup \bot_2) = T \Rightarrow (F \cup \bot_2)) = T \Rightarrow \bot_2 = \bot_2 \]
This proves that our \( v \) is a counter-model and hence

\[ \not\models_{H_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b)) \]
Question 11
Show that (can't use TTables!)

\[ \models ((\neg a \cup b) \Rightarrow (((c \cap d) \Rightarrow \neg d) \Rightarrow (\neg a \cup b))) \]

Solution
Denote \( A = (\neg a \cup b) \), and \( B = ((c \cap d) \Rightarrow \neg d) \)

Our formula becomes a substitution of a basic tautology

\( (A \Rightarrow (B \Rightarrow A)) \)

and hence is a tautology