Regular Expressions (REs)
Expressions

- In arithmetic:
  - expressions are constructed from numbers and variables using arithmetic operations and parentheses
  - expressions represent computations with numbers
  - results of expressions evaluation are numbers

In automata theory:
- regular expressions are constructed from regular languages using regular operations and parentheses
- regular expressions represent computations with regular languages
- result of regular expressions evaluation are regular languages
Expressions

- **In arithmetic:**
  1. expressions are constructed from numbers and variables using arithmetic operations and parentheses
  2. expressions represent computations with numbers
  3. results of expressions evaluation are numbers
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Regular Expressions (REs) – p.2/3
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Example

• $(5 + 3) \times 4$ is an arithmetic expression constructed with operations $+$ and $\times$; its value is the number $32$
Example

- $(5 + 3) \times 4$ is an arithmetic expression constructed with operations $+$ and $\times$; its value is the number $32$
- $(0 \cup 1)0^*$ is a regular expression constructed from the languages $0$ and $1$ using regular operations;
Example

- \((5 + 3) \times 4\) is an arithmetic expression constructed with operations \(+\) and \(\times\); its value is the number 32.
- \((0 \cup 1)0^*\) is a regular expression constructed from the languages 0 and 1 using regular operations;
- it evaluates to the language that contains all strings of zero-s and 1 followed by zero-s.
Regular expression evaluation

Similar to the evaluation of arithmetic expression
Regular expression evaluation

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- Identify first the constants (languages) involved.
Regular expression evaluation

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- Example: \((0 \cup 1)\) represents the language \{0, 1\}, 0* represents the language \{\epsilon, 0, 00, \ldots\} and \((0 \cup 1)0^*\) represents the language \{0, 1, 00, 10, 000, 100, \ldots\}
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Note: by convention, the symbol \(\circ\) of concatenation op. is not written in regular expressions
Applications

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3. The lexicon of programming languages is specified by regular expressions. Hence, lexical analysis is done using regular expressions.

4. Other examples?
For a given alphabet $\Sigma$

- The regular expression $\Sigma$ describes the language consisting of all strings of length 1 over $\Sigma$
For a given alphabet \( \Sigma \)

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- \( \Sigma^* \) describes the language of all strings over the alphabet \( \Sigma \).
For a given alphabet $\Sigma$

- The regular expression $\Sigma$ describes the language consisting of all strings of length 1 over $\Sigma$.
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- $\Sigma^*1$ is the language of all strings over $\Sigma$ that ends in 1.
For a given alphabet $\Sigma$

- The regular expression $\Sigma$ describes the language consisting of all strings of length 1 over $\Sigma$
- $\Sigma^*$ describes the language of all strings over the alphabet $\Sigma$
- $\Sigma^*1$ is the language of all strings over $\Sigma$ that ends in 1
- Language $(0\Sigma^*) \cup (\Sigma^*1)$ describes the language that consists of all strings that either start with 0 or end in 1
Precedence relation

Denoting $\prec$ the rule describing the order of ops in the evaluation of arithmetic and regular expressions we have:
Precedence relation

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Denoting $\preceq$ the rule describing the order of ops in the evaluation of arithmetic and regular expressions we have:

- **Arithmetic expressions:** $( ) \preceq \{ \times, / \} \preceq \{ +, - \}$
- **Regular expressions:** $( ) \preceq \{ * \} \preceq \{ \cup, \circ \}$
Precedence relation

Denoting ≤ the rule describing the order of ops in the evaluation of arithmetic and regular expressions we have:

- **Arithmetic expressions:** () ≤ {×, /} ≤ {+, −}
- **Regular expressions:** () ≤ {∗} ≤ {∪, ◦}

**Note:** () denotes expressions enclosed in parentheses
Consider $\Sigma$ an alphabet. A regular expression (RE) $R$ over $\Sigma$ is defined recursively by the rules:
Formal definitions

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2. $\epsilon$ is a RE describing the language $L(\epsilon) = \{\epsilon\}$
3. $\emptyset$ is a RE describing the empty language $L(\emptyset) = \emptyset$. 
Recursion body: If $R_1$ and $R_2$ are REs evaluating to the languages $L(R_1)$ and $L(R_2)$ then:
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1. $(R_1 \cup R_2)$ is a RE evaluating to $L(R_1) \cup L(R_2)$
Recursion body: If $R_1$ and $R_2$ are REs evaluating to the languages $L(R_1)$ and $L(R_2)$ then:

1. $(R_1 \cup R_2)$ is a RE evaluating to $L(R_1) \cup L(R_2)$
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2. $(R_1 \circ R_2)$ is a RE evaluating to $L(R_1) \circ L(R_2)$
3. $R_1^*$ is a RE evaluating to $\bigcup_{i \geq 0} L(R_1)^i$
   where $L(R_1)^i = L(R_1)^{i-1} \circ L(R_1)$ and $L(R_1)^0 = \epsilon$. 
Note

A definition of the form expressed by rule (4) above is called an *inductive definition*
Potential confusions

- Do not confuse the REs $\epsilon$ and $\emptyset$;
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Potential confusions

- **Do not confuse the REs** $\epsilon$ and $\emptyset$;
  1. $\epsilon$ represents the language containing just the empty string
  2. $\emptyset$ is the RE that represents the empty language

- **Note the distinction between** $R$ and $L(R)$. $R$ is an expression and $L(R)$ is the set of strings specified by $R$
Example REs

1. $0^*10^*$,
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1. $0^*10^*$, $L(0^*10^*) = \{ w \mid w \text{ contains exactly a single 1} \}$
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5. $(\Sigma\Sigma\Sigma)^*$, $L(\Sigma\Sigma\Sigma)^* = \{ w \mid \text{the length of } w \text{ is a multiple of three} \}$
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Example REs

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6. $01 \cup 01$, $L(01 \cup 01) = \{01, 01\}$
7. $0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1,$
Example REs

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5. $(\Sigma\Sigma\Sigma)^*$, $L(\Sigma\Sigma\Sigma)^* = \{w \mid \text{the length of } w \text{ is a multiple of three}\}$
6. $01 \cup 01$, $L(01 \cup 01) = \{01, 01\}$
7. $0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1$, $L(0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1) = \{w \mid w \text{ starts and ends with the same symbol}\}$
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4. \( (\Sigma\Sigma)^* \), \( L(\Sigma\Sigma)^* = \{w \mid w \text{ is a string of even length}\} \)
5. \( (\Sigma\Sigma\Sigma)^* \), \( L(\Sigma\Sigma\Sigma)^* = \{w \mid \text{the length of } w \text{ is a multiple of three}\} \)
6. \( 01 \cup 01 \), \( L(01 \cup 01) = \{01, 01\} \)
7. \( 0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1 \), \( L(0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1) = \{w \mid w \text{ starts and ends with the same symbol}\} \)
8. \( (0 \cup \epsilon)1^* \),
Example REs

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4. \((\Sigma\Sigma)^*, \ L(\Sigma\Sigma)^* = \{w \mid w \text{ is a string of even length}\}\)
5. \((\Sigma\Sigma\Sigma)^*, \ L(\Sigma\Sigma\Sigma)^* = \{w \mid \text{the length of } w \text{ is a multiple of three}\}\)
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8. \((0 \cup \epsilon)1^*, \ L((0 \cup \epsilon)1^*) = \{01^* \cup 1^*\}\)
1. $0^*10^*$, $L(0^*10^*) = \{w \mid w \text{ contains exactly a single 1}\}$
2. $\Sigma^*1\Sigma^*$, $L(\Sigma^*1\Sigma^*) = \{w \mid w \text{ contains at least one 1}\}$
3. $\Sigma^*001\Sigma^*$, $L(\Sigma^*001\Sigma^*) = \{w \mid w \text{ contains 001 as a substring}\}$
4. $(\Sigma\Sigma)^*$, $L(\Sigma\Sigma)^* = \{w \mid w \text{ is a string of even length}\}$
5. $(\Sigma\Sigma\Sigma)^*$, $L(\Sigma\Sigma\Sigma)^* = \{w \mid \text{the length of } w \text{ is a multiple of three}\}$
6. $01 \cup 01$, $L(01 \cup 01) = \{01, 01\}$
7. $0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1$, $L(0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1) = \{w \mid w \text{ starts and ends with the same symbol}\}$
8. $(0 \cup \epsilon)1^*$, $L((0 \cup \epsilon)1^*) = \{01^* \cup 1^*\}$
9. $(0 \cup \epsilon)(1 \cup \epsilon)$,
Example REs

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8. $(0 \cup \epsilon)1^*$, \( L((0 \cup \epsilon)1^*) = \{01^* \cup 1^*\} \)
9. $(0 \cup \epsilon)(1 \cup \epsilon)$, \( L((0 \cup \epsilon)(1 \cup \epsilon)) = \{\epsilon, 0, 1, 01\} \)
10. $1^* \emptyset, L(1^* \emptyset) = \emptyset$;

Note: concatenating $\emptyset$ to any set yields $\emptyset$
10. \(1^* \emptyset, L(1^* \emptyset) = \emptyset\);

Note: concatenating \(\emptyset\) to any set yields \(\emptyset\)

11. \(\emptyset^*, L(\emptyset^*) = \{\epsilon\}\); by definition,

\(\epsilon\) is in the star operation applied on any language;
if the language is empty \(\epsilon\) becomes the one element
If $R$ is a RE then the following identities take place:
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- $L(R \cup \emptyset) = L(R)$; adding the empty language to any other language will not change that language.
If $R$ is a RE then the following identities take place:

- $L(R \cup \emptyset) = L(R)$; adding the empty language to any other language will not change that language.
- $L(R \circ \epsilon) = L(R)$; concatenating any string with the empty string does not change that string.
Note

- \( L(R \cup \epsilon) \neq L(R) \). Example: if \( R = 0 \) then \( L(R) = 0 \);
  \[ L(R \cup \epsilon) = \{0, \epsilon\} \]
Note

- \( L(R \cup \varepsilon) \neq L(R) \). Example: if \( R = 0 \) then \( L(R) = 0 \);
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- \( L(R \circ \emptyset) \neq L(R) \). Example: if \( R = 0 \) then \( L(R) = \{0\} \)
  but \( L(R \circ \emptyset) = \emptyset \)
Application

- REs are useful tools for the design of compilers
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• Language lexicon is described by REs.
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- **Language lexicon** is described by **REs**.

**Example:** numerical constants can be described by:

\[ \{+, -, \epsilon\}(DD^* \cup DD^* \cdot D^* \cup D. DD^*) \]

where

\[ D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \]
Application

- **REs are useful tools** for the design of compilers
- **Language lexicon** is described by **REs**.

  **Example:** numerical constants can be described by:

  \[
  \{+, -, \epsilon\}(DD^* \cup DD^*.D^* \cup D.DD^*) \text{ where} \\
  D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}
  \]

- **From the lexicon description by REs** one can generate automatically lexical analyzers
A RE $R$ that evaluates to the language $L(R)$ is further said that specifies the language $L(R)$.
Equivalence with FA

- **REs and finite automata are equivalent** in their descriptive power
Equivalence with FA

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- Any RE $E$ can be converted into a finite automaton, $A_E$, that recognizes the language specified by $E$. 
Equivalence with FA

- **REs and finite automata are equivalent** in their descriptive power.
- **Any RE** $E$ can be converted into a finite automaton, $A_E$, that recognizes the language specified by $E$.
- **Vice-versa**, any finite automaton recognizing a language $A$ can be converted into a RE $E_A$ that specifies the language $A$. 
Theorem 1.54

Language is regular iff some RE specifies it
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Proof idea: this proof has two parts:
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- First part: we show that a language specified by a RE is regular, i.e., there is a finite automaton that recognizes it.
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Language is regular iff some RE specifies it

Proof idea: this proof has two parts:

- **First part:** we show that a language specified by a RE is regular, i.e., there is a finite automaton that recognizes it.
- **Second part:** we show that if a language is regular then there is a RE that specifies it
Lemma 1.55

If a language is specified by a RE, then it is regular.
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**Proof idea:** Assume that we have a RE $\mathcal{R}$ that evaluate to the language $\mathcal{A}$. 
Lemma 1.55

If a language is specified by a RE, then it is regular.

Proof idea: Assume that we have a RE \( R \) that evaluates to the language \( A \).

1. We will show how to convert \( R \) into an NFA that recognizes \( A \).
Lemma 1.55

If a language is specified by a RE, then it is regular.

Proof idea: Assume that we have a RE $R$ that evaluates to the language $A$.

1. We will show how to convert $R$ into an NFA that recognizes $A$.
2. Then by corollary 1.40, if an NFA recognizes $A$ then $A$ is regular.
Proof

Convert $R$ into an NFA $N$ by the following six-steps procedure:
Step 1:

If \( R = a \in \Sigma \), then \( L(R) = \{a\} \) and the NFA \( N \) recognizing \( L(R) \) is in Figure 1.

\[
\begin{array}{c}
q_1 \xrightarrow{a} q_2
\end{array}
\]

Figure 1: NFA \( N \) recognizing \( \{a\} \)
Step 1:

If $R = a \in \Sigma$, then $L(R) = \{a\}$ and the NFA $N$ recognizing $L(R)$ is in Figure 1.

![Diagram](image)

**Figure 1:** NFA $N$ recognizing $\{a\}$

**Note:** this is an NFA but not a DFA because it has states with no exiting arrow for each possible input symbol.
Formally \( N = (\{q_1, q_2\}, \Sigma, \delta, q_1, \{q_2\}) \) where:
\[
\delta(q_1, a) = \{q_2\}, \\
\delta(r, b) = \emptyset, \text{ for } r \neq q_1 \text{ and } b \neq a
\]
Step 2:

If $R = \epsilon$ then $L(R) = \{\epsilon\}$ and the NFA $N$ that recognizes $L(R)$ is in Figure 2.

---

Figure 2: The NFA $N$ recognizing $\{\epsilon\}$
Formally $N = (\{q_1\}, \Sigma, \delta, q_1, \{q_1\})$ where:

$\delta(r, b) = \emptyset$ for any $r$ and $b \in \Sigma$
Step 3:

If $R = \emptyset$ then $L(R) = \emptyset$, and the NFA $N$ that recognizes $L(R)$ is in Figure 3.

![Diagram of an NFA recognizing an empty language]

Figure 3: The NFA $N$ recognizing $\emptyset$
Formally $N = (\{q\}, \Sigma, \delta, q, \emptyset)$ where:

$\delta(r, b) = \emptyset$ for any $r$ and $b$
Step 4:

If $R = R_1 \cup R_2$ then $L(R) = L(R_1) \cup L(R_2)$. 

Note: in view with the inductive nature of we may assume that:
1. $L(NFA_1)$ is an NFA recognizing
2. $L(NFA_2)$ is an NFA recognizing

The NFA recognizing $L(R)$ is given in Figure 4.
Step 4:

If $R = R_1 \cup R_2$ then $L(R) = L(R_1) \cup L(R_2)$.

Note: in view with the inductive nature of $R$ we may assume that:
Step 4:

If $R = R_1 \cup R_2$ then $L(R) = L(R_1) \cup L(R_2)$.

Note: in view with the inductive nature of $R$ we may assume that:

1. $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ is an NFA recognizing $L(R_1)$
Step 4:

If $R = R_1 \cup R_2$ then $L(R) = L(R_1) \cup L(R_2)$.

Note: in view with the inductive nature of $R$ we may assume that:

1. $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ is an NFA recognizing $L(R_1)$
2. $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ is an NFA recognizing $L(R_2)$
Step 4:

If $R = R_1 \cup R_2$ then $L(R) = L(R_1) \cup L(R_2)$.

Note: in view with the inductive nature of $R$ we may assume that:

1. $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ is an NFA recognizing $L(R_1)$

2. $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ is an NFA recognizing $L(R_2)$

The NFA $N$ recognizing $L(R_1 \cup R_2)$ is given in Figure 4
NFA recognizing $L(R_1) \cup L(R_2)$

Figure 4: Construction of $N$ to recognize $L(R_1 \cup R_2)$
Construction procedure

1. \( Q = \{q_0\} \cup Q_1 \cup Q_2 \): That is, the states of \( N \) are all states on \( N_1 \) and \( N_2 \) with the addition of a new state \( q_0 \)
Construction procedure

1. \( Q = \{q_0\} \cup Q_1 \cup Q_2 \): That is, the states of \( N \) are all states on \( N_1 \) and \( N_2 \) with the addition of a new state \( q_0 \).

2. The start state of \( N \) is \( q_0 \).
Construction procedure

1. \( Q = \{q_0\} \cup Q_1 \cup Q_2 \): That is, the states of \( N \) are all states on \( N_1 \) and \( N_2 \) with the addition of a new state \( q_0 \).

2. The start state of \( N \) is \( q_0 \).

3. The accept states of \( N \) are \( F = F_1 \cup F_2 \): That is, the accept states of \( N \) are all the accept states of \( N_1 \) and \( N_2 \).
Construction procedure

1. $Q = \{q_0\} \cup Q_1 \cup Q_2$: That is, the states of $N$ are all states on $N_1$ and $N_2$ with the addition of a new state $q_0$

2. The start state of $N$ is $q_0$

3. The accept states of $N$ are $F = F_1 \cup F_2$: That is, the accept states of $N$ are all the accept states of $N_1$ and $N_2$

4. Define $\delta$ so that for any $q \in Q$ and any $a \in \Sigma_\epsilon$:

$$\delta(q, a) = \begin{cases} 
\delta_1(q, a), & \text{if } q \in Q_1 \\
\delta_2(q, a), & \text{if } q \in Q_2 \\
\{q_1, q_2\}, & \text{if } q = q_0 \text{ and } a = \epsilon \\
\emptyset, & \text{if } q = q_0 \text{ and } a \neq \epsilon.
\end{cases}$$
Step 5:

If $R = R_1 \circ R_2$ then $L(R) = L(R_1) \circ L(R_2)$. 
Step 5:

If $R = R_1 \circ R_2$ then $L(R) = L(R_1) \circ L(R_2)$.

Note: in view with the inductive nature of $R$ we may assume that:
Step 5:

If \( R = R_1 \circ R_2 \) then \( L(R) = L(R_1) \circ L(R_2) \).

Note: in view with the inductive nature of \( R \) we may assume that:

1. \( N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1) \) is an NFA recognizing \( L(R_1) \)
Step 5:

If $R = R_1 \circ R_2$ then $L(R) = L(R_1) \circ L(R_2)$.

Note: in view with the inductive nature of $R$ we may assume that:

1. $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ is an NFA recognizing $L(R_1)$
2. $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ is an NFA recognizing $L(R_2)$. 
NFA recognizing \( L(R_1) \circ L(R_2) \)

Figure 5: Construction of \( N \) to recognize \( L(R_1 \circ R_2) \)
Construction procedure

1. \( Q = Q_1 \cup Q_2 \). The states of \( N \) are all states of \( N_1 \) and \( N_2 \)
Construction procedure

1. $Q = Q_1 \cup Q_2$. The states of $N$ are all states of $N_1$ and $N_2$

2. The start state is the state $q_1$ of $N_1$
Construction procedure

1. $Q = Q_1 \cup Q_2$. The states of $N$ are all states of $N_1$ and $N_2$

2. The start state is the state $q_1$ of $N_1$

3. The accept states is the set $F_2$ of the accept states of $N_2$
Construction procedure

1. $Q = Q_1 \cup Q_2$. The states of $N$ are all states of $N_1$ and $N_2$
2. The start state is the state $q_1$ of $N_1$
3. The accept states is the set $F_2$ of the accept states of $N_2$
4. Define $\delta$ so that for any $q \in Q$ and any $a \in \Sigma_e$:

$$\delta(q, a) = \begin{cases} 
\delta_1(q, a), & \text{if } q \in Q_1 \text{ and } q \notin F_1 \\
\delta_1(q, a), & \text{if } q \in F_1 \text{ and } a \neq \epsilon \\
\delta_1(q, a) \cup \{q_2\}, & \text{if } q \in F_1 \text{ and } a = \epsilon \\
\delta_2(q, a), & \text{if } q \in Q_2.
\end{cases}$$
Step 6:

If $R = R_1^*$ then $L(R) = \bigcup_{i \geq 0} L(R_1)^i$. 

Note: in view with the inductive nature of we may assume that:

The NFA recognizing $L(R)$ is given in Figure 6

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Step 6:

If $R = R_1^*$ then $L(R) = \bigcup_{i \geq 0} L(R_1)^i$.

Note: in view with the inductive nature of $R$ we may assume that:
Step 6:

If $R = R_1^*$ then $L(R) = \bigcup_{i \geq 0} L(R_1)^i$.

**Note:** in view with the inductive nature of $R$ we may assume that:

1. $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ is an NFA recognizing $L(R_1)$
Step 6:

If $R = R_1^*$ then $L(R) = \bigcup_{i \geq 0} L(R_1)^i$.

**Note:** in view with the inductive nature of $R$ we may assume that:

1. $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ is an NFA recognizing $L(R_1)$

The NFA $N$ recognizing $L(R_1^*)$ is given in Figure 6.
NFA recognizing $L(R_1^*)$

Figure 6: Construction of $N$ to recognize $L(R_1^*)$
Construction procedure

1. \( Q = \{q_0\} \cup Q_1; \) that is, states of \( N \) are the states of \( N_1 \) plus a new state \( q_0 \)
Construction procedure

1. \( Q = \{ q_0 \} \cup Q_1 \); that is, states of \( N \) are the states of \( N_1 \) plus a new state \( q_0 \)

2. Start state of \( N \) is \( q_0 \)
Construction procedure

1. \( Q = \{q_0\} \cup Q_1 \); that is, states of \( N \) are the states of \( N_1 \) plus a new state \( q_0 \)

2. **Start state** of \( N \) is \( q_0 \)

3. \( F = \{q_0\} \cup F_1 \); that is, the accept states of \( N \) are the accept states of \( N_1 \) plus the new start state
Construction procedure

1. \( Q = \{q_0\} \cup Q_1 \); that is, states of \( N \) are the states of \( N_1 \) plus a new state \( q_0 \)
2. Start state of \( N \) is \( q_0 \)
3. \( F = \{q_0\} \cup F_1 \); that is, the accept states of \( N \) are the accept states of \( N_1 \) plus the new start state
4. Define \( \delta \) so that for any \( q \in Q \) and \( a \in \Sigma_\epsilon \):

\[
\delta(q, a) = \begin{cases} 
\delta_1(q, a), & \text{if } q \in Q_1 \text{ and } q \not\in F_1 \\
\delta_1(q, a), & \text{if } q \in F_1 \text{ and } a \neq \epsilon \\
\delta_1(q, a) \cup \{q_1\}, & \text{if } q \in F_1 \text{ and } a = \epsilon \\
\{q_1\}, & \text{if } q = q_0 \text{ and } a = \epsilon \\
\emptyset, & \text{if } q = q_0 \text{ and } a \neq \epsilon.
\end{cases}
\]
Examples conversion

Convert the following REs into NFA following the procedure presented above

1. \((ab \cup a)^*\) to an NFA
2. \((a \cup b)^* aba\)