1 YES/NO questions

1. For any function \( f \) from \( A \neq \emptyset \) onto \( A \), \( f \) has property \( f(a) \neq a \) for certain \( a \in A \).
   \textbf{Justify}: \( f(x) = x \) is always "onto". \( n \)

2. All infinite sets have the same cardinality.
   \textbf{Justify}: \( |N| \neq |R| \) and \( N \) (natural numbers) and \( R \) (real numbers) are infinite sets. \( n \)

3. \( \{\{a, b\}\} \subseteq 2\{a, b, \{a, b\}\} \)
   \textbf{Justify}: \( \{\{a, b\}\} \subseteq \{a, b, \{a, b\}\} \). \( y \)

4. For any binary relation \( R \subseteq A \times A \), \( R^{-1} \) exists.
   \textbf{Justify}: The set \( R^{-1} = \{(b, a) : (a, b) \in R\} \) always exists. \( y \)

5. Regular language is a regular expression.
   \textbf{Justify}: Regular language is a language defined by a regular expression. \( n \)

6. \( L^+ = \{w_1...w_n : w_i \in L, i = 1, 2, ..n, n \geq 1\} \)
   \textbf{Justify}: definition \( y \)

7. \( L^* = \{e\} \)
   \textbf{Justify}: only when \( e \not\in L \). When \( e \in L \) we get that \( e \in L^+ \) and \( e \not\in L^* - \{e\} \). \( n \)

8. For any languages \( L_1, L_2 \), \( (L_1 \cap L_2) \cup L_2 = L_2 \).
   \textbf{Justify}: \( L_1 \cap L_2 \subseteq L_2 \) and languages are sets. \( y \)

9. \( (\emptyset \cap b^*) \cup \emptyset^* \) describes a language with only one element.
   \textbf{Justify}: \( (\{e\} \cap \{b\}^* \) \cup \{e\} = \{e\} \) \( y \)

10. For any \( M, L(M) \neq \emptyset \) iff the set \( F \) of its final states is non-empty.
    \textbf{Justify}: Let \( M \) be such that \( \Sigma = \emptyset, F \neq \emptyset, s \not\in F \), we get \( L(M) = \emptyset \). \( n \)

11. A configuration of any finite automaton \( M = (K, \Sigma, \Delta, s, F) \) is any element of \( K \times \Sigma^* \times K \).
    \textbf{Justify}: it is element of \( K \times \Sigma^* \) (lecture definition) \( n \)

12. If \( M = (K, \Sigma, \Delta, s, F) \) is a non-deterministic as defined in the book, then \( M \) is also non-deterministic, as defined in the lecture.
    \textbf{Justify}: \( \Sigma \cup \{e\} \subseteq \Sigma^* \) \( y \)

13. Let \( M \) be a finite state automaton, \( L(M) = \{\omega \in \Sigma^* : (s, \omega) \xrightarrow{s,M} (q, e)\} \).
    \textbf{Justify}: only when \( q \in F \) \( n \)

14. \( L(M_1) = L(M_2) \) iff \( M_1, M_2 \) are finite automata.
    \textbf{Justify}: one can have 2 automata that accept different languages. \( n \)

15. DFA and NDFA recognize the same class of languages.
    \textbf{Justify}: theorem proved in class \( y \)
2 Two definitions of a non-deterministic automaton

BOOK DEFINITION: $M = (K, \Sigma, \Delta, s, F)$ is non-deterministic when

$$\Delta \subseteq K \times (\Sigma \cup \{e\}) \times K$$

OBSERVE that $\Delta$ is always finite because $K, \Sigma$ are finite sets.

LECTURE DEFINITION: $M = (K, \Sigma, \Delta, s, F)$ is non-deterministic when $\Delta$ is finite and

$$\Delta \subseteq K \times \Sigma^* \times K$$

OBSERVE that we have to say in this case that $\Delta$ is finite because $\Sigma^*$ is infinite.

SOLVING PROBLEMS you can use any of these definitions.

3 Very short questions (25pts)

For all state diagrams below do the following.

1. Determine whether it defines a finite state automaton.
2. Determine whether it is a deterministic / non-deterministic automaton.
3. Write full definition of $M$ by listing all its components.
4. Describe the language by writing a regular expression or a property that defines it.

Q1 Solution: $M = (K, \Sigma, s, \Delta, F)$ for $K = \{q_0\} = F$, $s = q_0$, $\Sigma = \emptyset, \Delta = \emptyset$. $M$ is deterministic and

$L(M) = \{e\} \neq \emptyset$

Q2 Solution: $M = (K, \Sigma, s, \Delta, F)$ for $\Sigma = \{a, b\}, K = \{q_0, q_1\}, s = q_0, F = \{q_0\}, \Delta = \{(q_0, a, q_1), (q_1, b, q_0)\}$. $M$ is non-deterministic; $\Delta$ is not a function on $K \times \Sigma$.

$L(M) = (ab)^*$

Q3 Solution: $M = (K, \Sigma, s, \Delta, F)$ for $\Sigma = \{a, b\}, K = \{q_0, q_1, q_2, q_3\}, F = \{q_1\}$,

$\Delta = \{(q_0, a, q_1), (q_0, b, q_1), (q_1, a, q_1), (q_1, e, q_2), (q_2, ab, q_2)\}$. It is NOT an automaton. It has no initial state.

Q4 $M = (K, \Sigma, s, \Delta, F)$ for $\Sigma = \{a, b\}, K = \{q_0, q_1, q_2, q_3\}, s = q_0, F = \emptyset$,

$\Delta = \{(q_0, a, q_1), (q_1, b, q_2), (q_2, a, q_1), (q_0, e, q_3), (q_2, a, q_3)\}$. $M$ is non-deterministic; $\Delta \subseteq K \times \Sigma \cup \{e\} \times K$.

$L(M) = \emptyset$

Q5 $M = (K, \Sigma, s, \Delta, F)$ for $\Sigma = \{a, b\}, K = \{q_0, q_1, q_2, q_3\}, s = q_0, F = \{q_1\}$,

$\Delta = \{(q_0, ab, q_1), (q_1, e, q_0), (q_1, a, q_2), (q_1, ba, q_2), (q_2, a, q_2), (q_0, e, q_3), (q_1, a, q_3)\}$. $M$ is non-deterministic; $\Delta \subseteq K \times \Sigma^* \times K$, $q_2, q_3$ are trap states.

$L(M) = (ab)^+$
4 Problems

Problem 1 Let $L$ be a language defined as follows

\[ L = \{ w \in \{a, b\}^* : \text{between any two } a's \text{ in } w \text{ there is an even number of consecutive } b's. \}. \]

1. Describe a regular expression $r$ such that $L(M) = L$.

Solution Remark that 0 is an even number, hence $a^* \in L$,

\[ r = b^* \cup b^*ab^* \cup b^*(a(bb)^*a)^*b^* = b^*ab^* \cup b^* \]

2. Construct a finite state automata $M$, such that $L(M) = L$.

Solution 1 Components of $M$ are:

\[ \Sigma = \{a, b\}, \quad K = \{q_0, q_1, q_2, q_3\}, \quad s = q_0, \quad F = \{q_0, q_2, q_3\} \]

\[ \Delta = \{(q_0, b, q_0), (q_0, a, q_3), (q_0, a, q_1), (q_1, bb, q_1), (q_1, a, q_2), (q_2, c, q_0), (q_3, b, q_3)\} \]

Some elements of $L(M)$ as defined by the state diagram are:

\[ a, aaa, bbb, aaaaabbb, bbbaaaa, abba, abbabbbba, abbbbbbabba, \ldots \]

Solution 2 Components of $M$ are:

\[ \Sigma = \{a, b\}, \quad K = \{q_0, q_1, q_2\}, \quad s = q_0, \quad F = \{q_0, q_1, q_2\} \]

\[ \Delta = \{(q_0, b, q_0), (q_0, a, q_1), (q_1, bb, q_1), (q_1, a, q_2), (q_1, c, q_2), (q_2, b, q_2)\} \]

Problem 2 Let

\[ M = (K, \Sigma, s, \Delta, F) \]

for \( K = \{q_0\}, \quad s = q_0, \Sigma = \{a, b\}, \quad F = \{q_0\} \) and

\[ \Delta = \{(q_0, aba, q_0), (q_0, ab, q_0)\} \]

1. List some elements of $L(M)$.

Solution

\[ e, ab, abab, ababa, ababaaba, \ldots \]

2. Write a regular expression for the language accepted by $M$.

Solution

\[ L = (ab \cup aba)^* \]
**Problem 3** We know that for any deterministic finite automaton \( M = (K, \Sigma, s, \delta, F) \) the following is true:

\[ e \in L(M) \iff s \in F. \]

Show that the above is not true for all non-deterministic automata.

**Solution** Let \( M = (K, \Sigma, s, \Delta, F) \) for \( K = \{q_0, q_1\}, s = q_0, \Sigma = \emptyset, F = \{q_1\} \), and \( \Delta = \{(q_0, e, q_1)\} \).

\[ L(M) = \{e\} \text{ and } s \notin F. \]

**Problem 4** For \( M \) defined as follows

\[ M = (K, \Sigma, s, \Delta, F) \]

for \( K = \{q_0, q_1, q_2, q_3\}, s = q_0 \)

\[ \Sigma = \{a, b\}, F = \{q_2, q_3\} \text{ and } \Delta = \{(q_0, a, q_1), (q_0, e, q_1), (q_0, b, q_2), (q_1, b, q_3), (q_1, e, q_3), (q_2, b, q_2), (q_2, e, q_3), (q_3, a, q_3)\} \]

Write a regular expression describing \( L(M) \).

**Solution**

\[ aa^* \cup a^* \cup aba^* \cup bb^* \cup bb\ast a^* \]

Write 4 steps of the general method of transformation the NDFA \( M \), into an equivalent deterministic \( M' \).

**Reminder:** \( E(q) = \{p \in K : (q, e) \xrightarrow{q, \sigma} (p, e)\} \) and

\[ \delta(Q, \sigma) = \bigcup\{E(p) : \exists q \in Q, (q, \sigma, p) \in \Delta\}. \]

**Solution Step 1:**

\[ E(q_0) = \{q_0, q_1, q_3\}, \ E(q_1) = \{q_1, q_3\}, \ E(q_2) = \{q_2, q_3\}, \ E(q_3) = \{q_3\}. \]

**Solution Step 2:**

\[ \delta(E(q_0), a) = \delta(\{q_0, q_1, q_3\}, a) = E(q_1) \cup E(q_3) = \{q_1, q_3\} \in F, \]

\[ \delta(E(q_0), b) = \delta(\{q_0, q_1, q_3\}, b) = E(q_2) \cup E(q_3) \cup \emptyset = \{q_2, q_3\} \in F, \]

**Solution Step 3:**

\[ \delta(\{q_1, q_3\}, a) = \emptyset \cup E(q_3) = \{q_3\} \in F, \]

\[ \delta(\{q_1, q_3\}, b) = E(q_3) \cup \emptyset = \{q_3\} \in F, \]

\[ \delta(\{q_2, q_3\}, a) = \emptyset \cup E(q_3) = \{q_3\} \in F, \]

\[ \delta(\{q_2, q_3\}, b) = \emptyset \cup \emptyset = \emptyset \]

**Solution Step 4:**

\[ \delta(\{q_3\}, a) = E(q_3) = \{q_3\} \in F, \ \delta(\{q_3\}, b) = \emptyset, \]

\[ \delta(\emptyset, a) = \emptyset, \ \delta(\emptyset, b) = \emptyset. \]

**End** of the construction.