YES/NO questions Circle the correct answer. Write SHORT justification.

1. For any language $L \subseteq \Sigma^*, \Sigma \neq \emptyset$ there is a deterministic automata $M$, such that $L = L(M)$.
   Justify: only when $L$ is regular

2. Any regular language has a finite representation.
   Justify: definition; regular expression is a finite string

3. Any finite language is regular.
   Justify: any finite language is a finite union of one element regular languages

4. Given $L_1, L_2$ languages over $\Sigma$, then $((L_1 \cap (\Sigma^* - L_2)) \cup L_2)L_1$ is regular.
   Justify: only when both are regular languages

5. For any deterministic automata $M$, $L(M) = \bigcup \{ R(1, j, n) : q_j \in F \}$, where $R(1, j, n)$ is the set of all strings in $\Sigma^*$ that may drive $M$ from state initial state to state $q_j$ without passing through any intermediate state numbered $n + 1$ or greater, where $n$ is the number of states of $M$.
   Justify: basic fact and definition

6. $\Sigma$ in any Generalized Finite Automaton includes some regular expressions.
   Justify: GFA recognizes regular expressions over $\Sigma$

7. For any finite automata $M$, there is a regular expression $r$, such that $L(M) = r$ (short hand notation).
   Justify: main theorem

8. Pumping Lemma says that we can always prove that a language is not regular.
   Justify: PL gives a certain characterization of infinite regular languages

9. Pumping Lemma serves as a tool for proving that a language is not regular.
   Justify: when the language is infinite and we can get contradiction

10. $L = \{ w \in \{a, b\}^* : w = w^R \}$ is regular.
    Justify: not regular, proof by PL

11. $L = \{a^n a^n : n \geq 0 \}$ is not regular.
    Justify: $L = (aa)^*$ and hence regular
12. \( L = \{a^n c^n : n \geq 0 \} \) is regular.  
\textbf{Justify:} not regular, proof by PL

13. Let \( L \) be a regular language, and \( L_1 \subseteq L \), then \( L_1 \) is regular.  
\textbf{Justify:} \( L_1 = \{a^n b^n : n \geq 0 \} \) is a non-regular subset of regular \( L = a^* b^* \)

14. Let \( L \) be a regular language. The language \( L^R = \{w^R : w \in L\} \) is regular.  
\textbf{Justify:} \( L^R \) is accepted by finite automata \( M^R \) constructed from \( M \) such that \( L(M) = L \)

PROBLEMS

QUESTION 1  Using the construction in the proof of theorem

\textit{A language is regular iff it is accepted by a finite automata}

construct a a finite automata \( M \) accepting

\[ L_1 = L = ((ab)^* \cup (bc)^*)ba \]

You can just draw a diagrams.

1. Diagrams for \( M_1, M_2, M_3 \) such that \( L(M_1) = ab, L(M_2) = bc, L(M_3) = ba \)

\textbf{Solution}

\( M_1 \) components:

\[ K = \{q_1, q_2\}, \Sigma = \{a, b, c\}, s = q_1, F = \{q_2\}, \]
\[ \Delta_{M_1} = \{(q_1, ab, q_2)\} \]

\( M_2 \) components:

\[ K = \{q_3, q_4\}, \Sigma = \{a, b, c\}, s = q_3, F = \{q_4\}, \]
\[ \Delta_{M_2} = \{(q_2, bc, q_4)\} \]

\( M_3 \) components:

\[ K = \{q_5, q_6\}, \Sigma = \{a, b, c\}, s = q_5, F = \{q_6\}, \]
\[ \Delta_{M_3} = \{(q_5, ba, q_6)\} \]
2. Diagrams for $M_4, M_5$ such that $L(M_4) = L(M_1)^*, L(M_5) = L(M_2)^*

Solution

$M_4$ components:

$K = \{q_1, q_2, q_7\}, \Sigma = \{a, b, c\}, s = q_7, F = \{q_2, q_7\},$

$\Delta_{M_4} = \{(q_7, e, q_1), (q_1, ab, q_2), (q_2, e, q_1)\}$

$M_5$ components:

$K = \{q_3, q_4, q_8\}, \Sigma = \{a, b, c\}, s = q_8, F = \{q_4, q_8\},$

$\Delta_{M_4} = \{(q_8, e, q_3), (q_3, bc, q_4), (q_4, e, q_3)\}$

3. Diagram for $M$ such that $L(M_5) = L(M_4) \cup L(M_5)$

Solution

$M_5$ components:

$K = \{q_1, q_2, q_3, q_4, q_7, q_8, q_9\}, \Sigma = \{a, b, c\}, s = q_9, F = \{q_2, q_4, q_7, q_8\},$

$\Delta_{M_5} = \{(q_9, e, q_7), (q_9, e, q_8), (q_7, e, q_1), (q_1, ab, q_2), (q_2, e, q_1), (q_8, e, q_3), (q_3, bc, q_4), (q_4, e, q_3)\}$

4. Diagram for $M = M_5M_3$, i.e. $M$ is such that $L(M) = L(M_5)L(M_3)$.

$M$ components:

$K = \{q_1, q_2, q_3, q_4, q_7, q_8, q_9\}, \Sigma = \{a, b, c\}, s = q_9, F = \{q_6\},$

$\Delta_{M_5} = \Delta_{M_4} \cup \{(q_7, e, q_5), (q_8, e, q_5), (q_2, e, q_5), (q_4, e, q_5), (q_5, ba, q_6)\}$

$= \{(q_9, e, q_7), (q_9, e, q_8), (q_7, e, q_1), (q_1, ab, q_2), (q_2, e, q_1), (q_8, e, q_3), (q_3, bc, q_4), (q_4, e, q_3),

(q_7, e, q_5), (q_8, e, q_5), (q_2, e, q_5), (q_4, e, q_5), (q_5, ba, q_6)\}$

**QUESTION 2** For the automaton $M$

$M = (\{q_1, q_2\}, \{a, b\}, s = q_1,$

$\Delta = \{(q_1, a, q_1), (q_1, b, q_2), (q_2, a, q_2), (q_2, b, q_1)\}, F = \{q_2\})$

Q2(a) (2pts) Evaluate 4 steps, in which you must include at least one $R(i, j, 0)$, in the construction of regular expression that defines $L(M)$ that uses the formulas:

$L(M) = \bigcup\{R(1, j, n) : q_j \in F\}$

$R(i, j, k) = R(i, j, k - 1) \cup R(i, k, k - 1)R(k, k, k - 1)^*R(k, j, k - 1)$

3
where \( n \) is the number of states of \( M \), \( k = 1, \ldots, n \) and

\[
R(i, j, 0) \text{ is either } \{ a \in \Sigma \cup \{ e \} : (q_i, a, q_j) \in \Delta \} \text{ if } i \neq j, \text{ or is }
\{ e \} \cup \{ a \in \Sigma \cup \{ e \} : (q_i, a, q_j) \in \Delta \} \text{ if } i = j.
\]

Solution

**Step 1** \( L(M) = R(1, 2, 2) \)

**Step 2** \( R(1, 2, 2) = R(1, 2, 1) \cup R(1, 2, 1)R(2, 2, 1)^* R(2, 2, 1) \)

**Step 3** \( R(1, 2, 1) = R(1, 2, 0) \cup R(1, 1, 0)R(1, 1, 0)^* R(1, 2, 0) \)

**Step 4** \( R(1, 2, 0) = \{ b \}, \ R(1, 1, 0) = \{ e \} \cup \{ a \} \)

\[
R(1, 2, 1) = \{ e \} \cup \{ a \} \cup (\{ e \} \cup \{ a \})(\{ e \} \cup \{ a \})^* \{ b \}
\]

**Question 3** Evaluate \( r \), such that \( L(r) = L(M) \)

using the Generalized Automata Construction, as described in example 2.3.2 page 80.

\[
M = (\{ q_1, q_2 \}, \{ a, b \}, \ s = q_1, \ \Delta = \{(q_1, a, q_1), (q_1, b, q_2), (q_2, a, q_2), (q_2, b, q_1)\}, \ F = \{ q_2 \})
\]

**Step 1:** Construct a generalized \( GM \) that extends \( M \), i.e. such that \( L(M) = L(GM) \)

Solution

\[
GM = (\{ q_1, q_2, q_3, q_4 \}, \{ a, b \}, \ s = q_3, \ F = \{ q_4 \})
\]

\[
\Delta = \{(q_1, a, q_1), (q_1, b, q_2), (q_2, a, q_2), (q_2, b, q_1), (q_3, e, q_1), (q_2, e, q_4)\}
\]

**Step 2:** Construct \( GM_1 \simeq GM \simeq M \) by elimination of \( q_1 \).

Solution

\[
GM_1 = (\{ q_2, q_3, q_4 \}, \{ a, b \}, \ s = q_3, \ F = \{ q_4 \})
\]

\[
\Delta = \{(q_3, a^* b, q_2), (q_2, a, q_2), (q_2, ba^* b, q_2), (q_2, e, q_4)\}
\]

**Step 3:** Construct \( GM_2 \simeq GM_1 \simeq GM \simeq M \) by elimination of \( q_2 \).
Solution

\[ GM2 = (\{q_3, q_4\}, \{a, b\}, s = q_3, F = \{q_4\} \]
\[ \Delta = \{(q_3, a^*b(ba^*b \cup a)^*, q_4)\} \]

Answer : the language is

\[ L(M) = a^*b(ba^*b \cup a)^* \]

QUESTION 4 Show that the class of regular languages is not closed with respect to subset relation.

Solution Consider

\[ L_1 = \{ a^n b^n : n \in N \}, \quad L_2 = a^*b^* \]

\[ L_1 \subseteq L_2 \] and \( L_1 \) is a non-regular subset of a regular \( L_2 \).

QUESTION 5

1. If \( L_1, L_2 \) are regular languages, is \( L_1 \cap L_2 \) also regular? Explain.

Solution YES, class of regular languages is closed under \( \cap \).

2. If \( L_1 \cap L_2 \) is a regular language, are \( L_1 \) and \( L_2 \) also regular? Explain.

Solution NO. Take

\[ L_1 = \{ a^n b^n : n \in N \}, \quad L_2 = \{ a^n : n \in Prime \} \]

\[ L_1 \cap L_2 = \emptyset \] is a regular language and \( L_1, L_2 \) are not regular

QUESTION 6 Show that the language

\[ L = \{ xyx^R : x, y \in \Sigma^* \} \]

is regular for any \( \Sigma \).

Solution Take \( x = e \in \Sigma^* \). The language

\[ L_1 = \{ eye^R : e, y \in \Sigma^* \} \subseteq L \]

and \( L_1 = \Sigma^* \). We get \( \Sigma^* \subseteq L \subseteq \Sigma^* \) and hence \( L = \Sigma^* \) is regular.

QUESTION 7 Show that if \( L \) is regular, so is the language

\[ L_1 = \{ xy : x \in L, y \notin L \}. \]

Solution Observe that \( L_1 = L(\Sigma^* - L) \) and \( L \) regular, hence \( \Sigma^* - L \) is regular (closure under complement), so is \( L_1 \) by closure under concatenation.