YES/NO questions Circle the correct answer. Write SHORT justification.

1. For any finite language $L$ there is a deterministic automata $M$, such that $L = L(M)$.
   **Justify:** Any finite language is regular

2. Any regular language is finite.
   **Justify:** $L = a^*$ is infinite

3. Any finite language is regular.
   **Justify:** $L = \bigcup \{L_w : w \in L\}$, each $L_w$ is regular and regular languages are closed under finite union.

4. Given $L_1, L_2$ regular languages over $\Sigma$, then $(L_1 \cup (\Sigma^* - L_1))L_2$ is regular.
   **Justify:** closure of regular languages over union and complement

5. For any $M$, $L(M) = \bigcup \{R(1, j, n) : q_j \in F\}$, where $R(1, j, n)$ is the set of all strings in $\Sigma^*$ that may drive $M$ from state initial state to state $q_j$ without passing through any intermediate state numbered $n + 1$ or greater, where $n$ is the number of states of $M$.
   **Justify:** only when $M$ is a finite automaton

6. The Generalized Finite Automaton accepts regular expressions.
   **Justify:** accepts regular expressions

7. There is an algorithm that for any finite automata $M$ computes a regular expression $r$, such that $L(M) = r$ (short hand notation).
   **Justify:** defined in the proof of Main Theorem

8. Pumping Lemma says that we can always prove that a language is regular.
   **Justify:** it gives certain characterization of infinite regular languages

9. Pumping Lemma proves that a language is not regular.
   **Justify:** PL is usually used to prove that an infinite language is not regular

10. $L = \{a^n : n \geq 0\}$ is not regular.
    **Justify:** $L = a^*$

11. $L = \{b^na^n : n \geq 0\}$ is not regular.
    **Justify:** proved using Pumping Lemma

12. $L = \{a^{2n} : n \geq 0\}$ is regular.
    **Justify:** $L = (aa)^*$
13. Let \( L \) be a regular language, and \( L_1 \subseteq L \), then \( L_1 \) is regular.

**Justify:** \( L_1 = \{ b^n a^n : n \geq 0 \} \subseteq L = b^* a^* \) and \( L \) is regular, and \( L_1 \) is not regular.

14. Let \( L \) be a regular language. The language \( L^R = \{ w^R : w \in L \} \) is regular.

**Justify:** \( L^R \) is accepted by a finite automata \( M^R = (K \cup s', \Sigma, \Delta', s', F = \{ s \}) \), where \( K \) is the set of states of \( M \) accepting \( L \), \( s' \notin K \), \( s \) the initial state of \( M \), \( F \) is the set of final states of \( M \) and

\[
\Delta' = \{(r, \sigma, p) : (p, \sigma, r) \in \Delta \} \cup \{(s', e, q) : q \in F \},
\]

where \( \Delta \) is the set of transitions of \( M \).

**QUESTION 1** Give a direct construction for the closure under intersection of the languages accepted by finite automata. Which of the two constructions, the one given in the textbook or one suggested in this problem, is more efficient when the two languages are given in terms of nondeterministic automata?

**Solution**

**Case 1: deterministic** Let

\[ M_1 = (K_1, \Sigma, \delta_1, s_1, F_1), \ M_2 = (K_2, \Sigma, \delta_2, s_2, F_2) \]

be two deterministic automata. We construct

\[ M = (K, \Sigma, \delta, s, F), \]

such that

\[ L(M) = L(M_1) \cap L(M_2) \]

as follows.

\[
K = K_1 \times K_2, \ s = (s_1, s_2), \ F = F_1 \times F_2, \quad \delta((q_1, q_2), \sigma) = (\delta_1(q_1, \sigma), \delta_2(q_2, \sigma)) = (p_1, p_2).
\]

Obviously \( w \in L(M) \) iff \( ((s_1, s_2), w) \xrightarrow{\ast} ((f_1, f_2), e) \) and \( f_1 \in F_1, f_2 \in F_2 \) iff \( (s_1, w) \xrightarrow{\ast} (f_1, e) \) for \( f_1 \in F_1 \) and \( (s_2, w) \xrightarrow{\ast} (f_2, e) \) for \( f_2 \in F_2 \) iff \( w \in L(M_1) \) and \( w \in L(M_2) \) iff \( w \in L(M_1) \cap L(M_2) \).

We denote

\[ M = M_1 \cap M_2. \]
Case 2: nondeterministic Let

\[ M_1 = (K_1, \Sigma, \Delta_1, s_1, F_1), \quad M_2 = (K_2, \Sigma, \Delta_2, s_2, F_2) \]

be two nondeterministic automata. We construct

\[ M = (K, \Sigma, \delta, s, F), \]

such that

\[ L(M) = L(M_1) \cap L(M_2), \]

and denoted by \( M = M_1 \cap M_2 \) as follows.

\[
K = K_1 \times K_2, \quad s = (s_1, s_2), \quad F = F_1 \times F_2,
\]

\[
\Delta = \{(q_1, q_2, \sigma, (p_1, p_2)) : (q_1, \sigma, p_1) \in \Delta_1 \text{ and } (q_2, \sigma, p_2) \in \Delta_2 \text{ and } \sigma \in \Sigma \}
\]

or \( (q_1, e, p_1) \in \Delta_1 \text{ and } q_2 = p_2 \), i.e. \( (q_2, \sigma, q_2) \in \Delta_2 \)

or \( (q_2, e, p_2) \in \Delta_2 \text{ and } q_1 = p_1 \), i.e. \( (q_1, \sigma, q_1) \in \Delta_1 \}

This is called a DIRECT construction of \( M = M_1 \cap M_2 \), as opposed to

the in the proof of the theorem of closure of regular languages under set

intersection.

Observe that if \( M_1, M_2 \) have each at most \( n \) states, our direct construc-

tion of produces \( M = M_1 \cap M_2 \) with at most \( n^2 \) states. The construction

from the proof of the theorem might generate \( M \) with up to \( 2^{n^2+1} \) states.

**QUESTION 2** Using the construction in the proof of theorem

*A language is regular iff it is accepted by a finite automata*

construct a a finite automata \( M \) accepting

\[ L = L = a^*(ab \cup ba \cup \emptyset^*)b^* \]

You can just draw a diagrams.

1. Define \( M_1, M_2, M_3 \) such that \( L(M_1) = ab, L(M_2) = ba, L(M_3) = \emptyset^* \)

**Solution**

**M1** components:

\[ K_1 = \{q_1, q_2\}, \quad s = q_1, \quad F_1 = \{q_2\}, \quad \Delta_1 = \{(q_1, ab, q_2)\} \]
M2 components:
\[ K_2 = \{q_3, q_4\}, s = q_3, F_2 = \{q_4\}, \Delta_2 = \{(q_3, ba, q_4)\} \]

M3 components:
\[ K_3 = \{q_5, q_6\}, s = q_5, F_3 = \{q_6\}, \Delta_3 = \{(q_5, e, q_6)\} \]

2. Define \( M_4 \) such that \( L(M_4) = L(M_1) \cup L(M_2) \cup L(M_3) \)

Solution

M4 components:
\[ K_4 = K_1 \cup K_2 \cup K_3 \cup \{q_7\}, s = q_7, F_7 = F_1 \cup F_2 \cup F_3, \]
\[ \Delta_3 = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \{(q_7, e, q_1), (q_7, e, q_3), (q_7, e, q_5)\} \]

3. Define \( M_5, M_6 \) such that \( L(M_5) = a^*, L(M_6) = b^* \).

Solution

M5 components:
\[ K_5 = \{q_8\}, s = q_8, F_5 = \{q_8\}, \Delta_5 = \{(q_8, a, q_8)\} \]

M6 components:
\[ K_6 = \{q_9\}, s = q, F_6 = \{q_9\}, \Delta_6 = \{(q_9, b, q_9)\} \]

4. Use \( M_1 - M_6 \) to define \( M \) such that \( L = L(M) \).

Solution

M components:
\[ K = K_1 \cup K_4 \cup K_5 \cup K_6 = \{q_1, \ldots, q_0\}, s = q_8, F = \{q_9\}, \]
\[ \Delta = \bigcup_{i=1}^{6} \Delta_i \cup \{(q_8, e, q_7), (q_2, e, q_3), (q_4, e, q_3), (q_6, e, q_9)\} \]

QUESTION 3 For the automaton \( M \)

\[ M = \{(q_1, q_2, q_3), \{a, b\}, s = q_1, \]
\[ \Delta = \{(q_1, b, q_2), (q_1, a, q_3), (q_2, a, q_1), (q_2, b, q_1), (q_3, a, q_1), (q_3, b, q_1)\}, F = \{q_1\} \]
1. Evaluate 4 steps, in which you must include at least one \( R(i, j, 0) \), in the construction of regular expression that defines \( L(M) \) that uses the formulas:

\[
L(M) = \bigcup\{R(1, j, n) : q_j \in F\}
\]

\[
R(i, j, k) = R(i, j, k - 1) \cup R(i, k, k - 1)R(k, k, k - 1)^*R(k, j, k - 1)
\]

where \( n \) is the number of states of \( M \), \( k = 1, \ldots, n \) and \( R(i, j, 0) \) is either \( \{a \in \Sigma \cup \{e\} : (q_i, a, q_j) \in \Delta\} \) if \( i \neq j \), or is

\[
\{e\} \cup \{a \in \Sigma \cup \{e\} : (q_i, a, q_j) \in \Delta\}
\]

if \( i = j \).

Solution

**Step 1**  \( L(M) = R(1, 1, 3) \).

**Step 2**  \( R(1, 1, 3) = R(1, 1, 2) \cup R(1, 3, 2)R(3, 3, 2)^*R(3, 1, 2) \).

**Step 3**  \( R(1, 1, 2) = R(1, 1, 1) \cup R(1, 2, 1)R(2, 2, 1)^*R(2, 1, 1) \).

**Step 4**  \( R(1, 1, 1) = R(1, 1, 0) \cup R(1, 1, 0)^*R(1, 1, 0) \) and

\[
R(1, 1, 0) = \{e\} \cup \emptyset = \{e\}, \quad R(1, 1, 1) = \{e\} \cup \{e\}^*\{e\} = \{e\}.
\]

2. Evaluate \( r \), such that \( L(r) = L(M) \)

using the Generalized Automata Construction, as described in example 2.3.2 page 80.

Solution

**Step 1:** We extend \( M \) to a generalized \( GM \), such that \( L(M) = L(G(M)) \) as follows:

\[
GM = (\{q_1, q_2, q_3, q_4, q_5\}, \{a, b\}, s = q_4, \Delta = \{(q_1, b, q_2), (q_1, a, q_3), (q_2, a, q_1), (q_2, b, q_1), (q_3, a, q_1), (q_3, b, q_1), (q_4, e, q_1), (q_1, e, q_5)\}, F = \{q_5\})
\]

**Step 2:** Construct \( GM1 \simeq GM \simeq M \) by elimination of \( q_2 \).

\[
GM1 = (\{q_1, q_3, q_4, q_5\}, \{a, b\}, s = q_4, \Delta = \{(q_1, a, q_3), (q_1, bb\cdot ba), q_1), (q_3, a, q_1), (q_3, b, q_1), (q_4, e, q_1), (q_1, e, q_5)\}, F = \{q_5\})
\]
Step 3: Construct $GM_2 \simeq GM_1 \simeq GM \simeq M$ by elimination of $q_3$.

$$GM_2 = \langle \{q_1, q_4, q_5\}, \{a, b\}, s = q_4, \Delta = \{(q_1, (bb \cup ba), q_1), (q_1, (aa \cup ab), q_1), (q_4, e, q_1), (q_1, e, q_5)\}, F = \{q_5\}\rangle$$

Step 4: Construct $GM_3 \simeq GM_2 \simeq GM_1 \simeq GM \simeq M$ by elimination of $q_1$.

$$GM_3 = \langle \{q_4, q_5\}, \{a, b\}, s = q_4, \Delta = \{(q_4, (bb \cup ba \cup aa \cup ab)^*, q_5)\}, F = \{q_5\}\rangle$$

Answer

$$L(M) = L(MG_4) = (bb \cup ba \cup aa \cup ab)^* = ((a \cup b)(a \cup b))^*.$$