1 YES/NO questions

1. All infinite sets have the same cardinality.
   \textbf{Justify}: \(|N| \neq |R|\) and \(N\) (natural numbers) and \(R\) (real numbers) are infinite sets.

2. Regular language is a regular expression.
   \textbf{Justify}: Regular language is a language defined by a regular expression.

3. \(L^+ = \{w_1...w_n : w_i \in L, i = 1, 2, ..n, n \geq 1\}\).
   \textbf{Justify}: definition

4. \(L^+ = L^* - \{e\}\).
   \textbf{Justify}: only when \(e \notin L\). When \(e \in L\) we get that \(e \in L^+\) and \(e \notin L^* - \{e\}\).

5. \((\emptyset^* \cap b^*) \cup \emptyset^*\) describes a language with only one element.
   \textbf{Justify}: \(\{e\} \cap \{b\}^* \cup \{e\} = \{e\} \cup \{e\} = \{e\}\)

6. Let \(M\) be a finite state automaton, \(L(M) = \{\omega \in \Sigma^* : (s, \omega) \overset{*}{\rightarrow}_M (q, e)\}\).
   \textbf{Justify}: only when \(q \in F\)

7. DFA and NDFA recognize the same class of languages.
   \textbf{Justify}: theorem proved in class

2 Two definitions of a non-deterministic automaton

\textbf{BOOK DEFINITION}: \(M = (K, \Sigma, \Delta, s, F)\) is non-deterministic when
\[
\Delta \subseteq K \times (\Sigma \cup \{e\}) \times K
\]

\textbf{OBSERVE} that \(\Delta\) is always finite because \(K, \Sigma\) are finite sets.

\textbf{LECTURE DEFINITION}: \(M = (K, \Sigma, \Delta, s, F)\) is non-deterministic when \(\Delta\) is finite and
\[
\Delta \subseteq K \times \Sigma^* \times K
\]

\textbf{OBSERVE} that we have to say in this case that \(\Delta\) is finite because \(\Sigma^*\) is infinite.

\textbf{SOLVING PROBLEMS} you can use any of these definitions.
3 Very short questions

For the QUESTIONS below do the following.

1. Determine whether it defines a finite state automaton.
2. Determine whether it is a deterministic / non-deterministic automaton.
3. Write full definition of $M$ by listing all its components.
4. Describe the language by writing a regular expression or a property that defines it.

Q1 Solution: $M = (K, \Sigma, s, \Delta, F)$ for $K = \{q_0\} = F$, $s = q_0$, $\Sigma = \emptyset$, $\Delta = \emptyset$. $M$ is deterministic and

$L(M) = \{\epsilon\} \neq \emptyset$

Q2 Solution: $M = (K, \Sigma, s, \Delta, F)$ for $\Sigma = \{a, b\}$, $K = \{q_0, q_1\}$, $s = q_0$, $F = \{q_0\}$, $\Delta = \{(q_0, a, q_1), (q_0, b, q_1), (q_1, a, q_1), (q_1, e, q_2), (q_2, ab, q_2)\}$. $M$ is non deterministic; $\Delta$ is not a function on $K \times \Sigma$.

$L(M) = (ab)^*$

Q3 Solution: $M = (K, \Sigma, s, \Delta, F)$ for $\Sigma = \{a, b\}$, $K = \{q_0, q_1, q_2\}$, $F = \{q_1\}$, $\Delta = \{(q_0, a, q_1), (q_0, b, q_1), (q_1, a, q_1), (q_1, e, q_2), (q_2, ab, q_2)\}$. It is NOT an automaton. It has no initial state.

Q4 $M = (K, \Sigma, s, \Delta, F)$ for $\Sigma = \{a, b\}$, $K = \{q_0, q_1, q_2, q_3\}$, $s = q_0$, $F = \emptyset$, $\Delta = \{(q_0, a, q_1), (q_0, b, q_2), (q_2, a, q_1), (q_0, e, q_3), (q_2, a, q_3)\}$. $M$ is non deterministic; $\Delta \subseteq K \times \Sigma \cup \{\epsilon\} \times K$.

$L(M) = \emptyset$

4 Problems

PROBLEM 1 Let $L$ be a language defines as follows

$L = \{w \in \{a, b\}^*: \text{between any two a's in } w \text{ there is an even number of consecutive b's.}\}$.

1. Describe a regular expression $r$ such that $L(r) = L$.

Solution Remark that 0 is an even number, hence $a^* \in L$,

$r = b^* \cup a^*a^*b^* \cup b^*(a(bb)^*a)^*b^* = b^*a^*b^* \cup b^*(a(bb)^*a)^*b^*$

2. Construct a finite state automata $M$, such that $L(M) = L$.

Solution

Components of $M$ are:

$\Sigma = \{a, b\}$, $K = \{q_0, q_1, q_2, q_3\}$, $s = q_0$, $F = \{q_0, q_2, q_3\}$,

$\Delta = \{(q_0, b, q_0), (q_0, a, q_1), (q_0, e, q_3), (q_3, a, q_3), (q_3, b, q_3), (q_1, bb, q_1), (q_1, a, q_2), (q_2, e, q_0)\}$

Some elements of $L(M)$ as defined by the state diagram are:

$a, aaa, bbaabbb, bbb, aaaaabbb, bbbaaa, abba, abbabbbba, abbbbbbabba, ....$
PROBLEM 2 Let

\[ M = (K, \Sigma, s, \Delta, F) \]

for \( K = \{q_0\}, s = q_0, \Sigma = \{a, b\}, F = \{q_0\} \) and

\[ \Delta = \{(q_0, aba, q_0), (q_0, ab, q_0)\} \]

Solution
1. List some elements of \( L(M) \).

\[ e, ab, abab, ababa, ababaaba, \ldots \]

2. Write a regular expression for the language accepted by \( M \).

\[ L = (ab \cup aba)^* \]

PROBLEM 3 We know that for any deterministic finite automaton \( M = (K, \Sigma, s, \delta, F) \) the following is true:

\[ e \in L(M) \iff s \in F. \]

Show that the above is not true for all non-deterministic automata.

Solution Let \( M = (K, \Sigma, s, \Delta, F) \) for \( K = \{q_0, q_1\}, s = q_0, \Sigma = \emptyset, F = \{q_1\}, \) and \( \Delta = \{(q_0, e, q_1)\} \).

\[ L(M) = \{e\} \text{ and } s \notin F \]

PROBLEM 4 For \( M \) defined as follows

\[ M = (K, \Sigma, s, \Delta, F) \]

for \( K = \{q_0, q_1, q_2, q_3\}, s = q_0 \)
\( \Sigma = \{a, b\}, F = \{q_2, q_3\} \) and

\[ \Delta = \{(q_0, a, q_1), (q_0, e, q_1), (q_0, b, q_2), (q_1, b, q_3), (q_1, e, q_3), (q_2, b, q_2), (q_2, e, q_3), (q_1, a, q_3)\} \]

Write a regular expression describing \( L(M) \).

Solution

\[ aa^* \cup a^* \cup aba^* \cup ba^* \cup bb^* \cup bb^*a^* \]

Write 4 steps of the general method of transformation the NDFA \( M \), into an equivalent deterministic \( M' \).

Reminder: \( E(q) = \{p \in K : (q, e) \xrightarrow{M} (p, e)\} \) and

\[ \delta(Q, \sigma) = \bigcup \{E(p) : \exists q \in Q, (q, \sigma, p) \in \Delta\}. \]

Solution Step 1:

\[ E(q_0) = \{q_0, q_1, q_3\}, E(q_1) = \{q_1, q_3\}, E(q_2) = \{q_2, q_3\}, E(q_3) = \{q_3\}. \]
Solution Step 2:

\[ \delta(E(q_0), a) = \delta(q_0, q_1, q_3, a) = E(q_1) \cup E(q_3) = \{q_1, q_3\} \in F, \]

\[ \delta(E(q_0), b) = \delta(q_0, q_1, q_3, b) = E(q_2) \cup E(q_3) \cup \emptyset = \{q_2, q_3\} \in F, \]

Solution Step 3:

\[ \delta(q_1, q_3, a) = \emptyset \cup E(q_3) = \{q_3\} \in F, \]

\[ \delta(q_1, q_3, b) = E(q_3) \cup \emptyset = \{q_3\} \in F, \]

\[ \delta(q_2, q_3, a) = \emptyset \cup E(q_3) = \{q_3\} \in F, \]

\[ \delta(q_2, q_3, b) = E(q_2) \cup \emptyset = \{q_2, q_3\} \in F \]

Solution Step 4:

\[ \delta(q_3, a) = E(q_3) = \{q_3\} \in F, \quad \delta(q_3, b) = \emptyset, \]

\[ \delta(\emptyset, a) = \emptyset, \quad \delta(\emptyset, b) = \emptyset \]

End of the construction.