1 YES/NO questions

1. For any binary relation $R \subseteq A \times A$, $R^*$ exists.
   
   \textbf{Justify:} definition

2. For any binary relation $R \subseteq A \times A$, $R^{-1}$ exists.
   
   \textbf{Justify:} The set $R^{-1} = \{(b,a) : (a,b) \in R\}$ always exists.

3. For any function $f$ from $A \neq \emptyset$ onto $A$, $f$ has property $f(a) \neq a$ for certain $a \in A$.
   
   \textbf{Justify:} $f(x) = x$ is always "onto".

4. All infinite sets have the same cardinality.
   
   \textbf{Justify:} $|N| < |2^N|$ by Cantor Theorem and $N, 2^N$ are infinite

5. Set $A$ is uncountable iff $R \subseteq A$ ($R$ is the set of real numbers).
   
   \textbf{Justify:} $R, 2^R$ are both uncountable and $R$ is not a subset of $2^R$ ($R \nsupseteq 2^R$) but $R \in 2^R$.

6. Let $A \neq \emptyset$ such that there are exactly 25 partitions of $A$. It is possible to define 20 equivalence relations on $A$.
   
   \textbf{Justify:} one can define up to 25 (as many as partitions) of equivalence classes

7. There is a relation that is equivalence and order at the same time.
   
   \textbf{Justify:} equality relation

8. Let $A = \{n \in N : n^2 + 1 \leq 15\}$. It is possible to define 8 alphabets $\Sigma \subseteq A$.
   
   \textbf{Justify:} $A$ has 4 elements, so we have $2^4 > 8$ subsets

9. There is exactly as many languages over alphabet $\Sigma = \{a\}$ as real numbers.
   
   \textbf{Justify:} $|\Sigma^*| = \aleph_0, |2^{\Sigma^*}| = |R| = C$.

10. Let $\Sigma = \{a, b\}$. There are more than 20 words of length 4 over $\Sigma$.
    
    \textbf{Justify:} There are exactly $2^4 = 16$ words of length 4 over $\Sigma$ and $16 < 20$.

11. $L^* = \{w_1...w_n : w_i \in L, i = 1, 2, ...n, n \geq 1\}$.
    
    \textbf{Justify:} $n \geq 0$.
    
    $L^+ = LL^*$.
    
    \textbf{Justify:} the problem is only with cases $e \in L$ or $e \notin L$. When $e \in L$, then $e \in L^+$, and always $e \in L^*$, hence $e \in LL^*$.
    
    When $e \notin L$, then $e \notin L^+$, and always $e \in L^*$, hence $e \in LL^*$ and $L^+ \neq LL^*$

12. $L^+ = L^* - \{e\}$.
    
    \textbf{Justify:} only when $e \notin L$. When $e \in L$ we get that $e \in L^+$ and $e \notin L^* - \{e\}$.
13. If \( L = \{ w \in \{0, 1\}^* : w \text{ has an unequal number of 0's and 1's} \} \), then \( L^* = \{0, 1\}^* \).

**Justify:** \( 1 \in L, 0 \in L \) so \( \{0, 1\} \subseteq L \subseteq \Sigma^* \), hence \( \{0, 1\}^* \subseteq L^* \subseteq (\Sigma^*)^* = \Sigma^* = \{0, 1\}^* \) and \( L^* = \{0, 1\}^* \). 

14. For any languages \( L_1, L_2 \), \( (L_1 \cup L_2) \cap L_1 = L_1 \).

**Justify:** languages are sets and \( (A \cup B) \cap A = A \). 

15. For any languages \( L_1, L_2 \),
\[
L_1^* = L_2^* \text{ i f f } L_1 = L_2
\]

**Justify:** Consider \( L_1 = \{a, e\}, L_2 = \{a\} \). Obviously, \( L_1 \neq L_2 \) and \( L_1^* = L_2^* \).

16. For any languages \( L_1, L_2 \), \( (L_1 \cup L_2)^* = L_1^* \).

**Justify:** languages are sets so it is true only when \( L_1 \subseteq L_2 \).

17. \( (\emptyset^* \cap a) \cup b^* \cap \emptyset^* \) describes a language with only one element.

**Justify:** \( \emptyset \cup b^* = b^*, b^* \cap \{e\} = \{e\} \)

18. \( (\emptyset^* \cap a) \cup b^* \cap a^* \) is a finite regular language.

**Justify:** \( b^* \cap a^* = \{e\} = 0^* \)

19. \( (\{a\} \cup \{e\}) \cap \{ab\}^* \) is a finite regular language.

**Justify:** \( (\{a\} \cup \{e\}) \cap \{ab\}^* = \{a, e\} \cap \{ab\}^* = \{e\} \cap \emptyset^* \)

20. Any regular language has a finite description.

**Justify:** by definition \( L = L(r) \) and \( r \) is a finite string.

21. Any finite language is regular.

**Justify:** \( L = \{w_1\} \cup \ldots \cup \{w_1\} \) and \( \{w_1\} \) has a finite description \( w_i \)

22. Every deterministic automata is also non-deterministic.

**Justify:** any function is a relation

The set of all configurations of any non-deterministic state automata is always non-empty.

**Justify:** \( K \neq \emptyset \), because \( s \in K \). If \( \Sigma = \emptyset \) the set of all configuration of non-deterministic automata (book definition) is a subset of \( K \times \emptyset \cup \{e\} \neq \emptyset \) as it always contains \( (s, e) \). For the lecture definition, the set of all configuration is a subset of \( K \times \Sigma^* \) and always \( e \in \Sigma^* \) hence always \( (s, e) \in K \times \Sigma^* \).

23. Let \( M \) be a finite state automaton, \( L(M) = \{w \in \Sigma^* : (q, w) \xrightarrow{\Sigma, M} (s, e)\} \).

**Justify:** \( L(M) = \{w \in \Sigma^* : \exists q \in F((s, w) \xrightarrow{\Sigma, M} (q, e))\} \)

24. For any automata \( M \), \( L(M) \neq \emptyset \).

**Justify:** if \( \Sigma = \emptyset \) or \( F = \emptyset \), \( L(M) = \emptyset \)

25. \( L(M_1) = L(M_2) \) iff \( M_1, M_2 \) are deterministic.

**Justify:** Let \( M_1 \) be an automata over \( \{a, b\} \) with with \( \Delta = \{(q_0, ab, q_0)\} \), \( F = \{q_0\}, s = q_0 \) and let \( M_2 \) be an automata over \( \{a, b\} \) with with \( \Delta = \{(q_0, ab, q_0), (q_0, e, q_1)\} \), \( F = \{q_1\}, s = q_0 \).

\( L(M_1) = L(M_2) = (ab)^* \) and both are non-deterministic
26. DFA and NDFA compute the same class of languages.
   **Justify:** basic theorem
   
27. Let $M_1$ be a deterministic, $M_2$ be a nondeterministic FA, $L_1 = L(M_1)$ and $L_2 = L(M_2)$ then there is a deterministic automaton $M$ such that $L(M) = (L^* \cup (L_1 - L_2)^*) L_1$
   **Justify:** the class of finite automata is closed under $\ast, \cup, -, \cap$

TWO DEFINITIONS OF NON DETERMINISTIC AUTOMATA

**BOOK DEFINITION:** $M = (K, \Sigma, \Delta, s, F)$ is non-deterministic when
\[ \Delta \subseteq K \times (\Sigma \cup \{e\}) \times K \]

OBSERVE that $\Delta$ is always finite because $K, \Sigma$ are finite sets.

**LECTURE DEFINITION:** $M = (K, \Sigma, \Delta, s, F)$ is non-deterministic when $\Delta$ is finite and
\[ \Delta \subseteq K \times \Sigma^* \times K \]

OBSERVE that we have to say in this case that $\Delta$ is finite because $\Sigma^*$ is infinite.

SOLVING PROBLEMS you can use any of these definitions.

2 Problems

**Problem 1** Let $L$ be a language defines as follows
\[ L = \{ a^m b^n : \text{every } a \text{ is either immediately proceeded or followed by } b \} \]

1. Describe a regular expression $r$ such that $L(r) = L$ (Meaning of $r$ is $L$).
   **Solution** $L = (b \cup ab \cup ba \cup bab)^*$

2. Construct a finite state automata $M$, such that $L(M) = L$.
   **Solution**
   Components of $M$ are:
   \[ K = \{ s \}, \{ a, b \}, s, F = \{ s \}, \]
   \[ \Delta = \{(s, b, s), (s, ab, s), (s, ba, s), (s, bab, s)\} \]
   Some elements of $L(M)$ are: $b, bb, baab, abab, abbba, bbabbbabbabb$

**Problem 2**

1. Let $M = (K, \Sigma, \delta, s, F)$ be a deterministic finite automaton. Under exactly what conditions $e \in L(M)$?
   **Solution**
   \[ e \in L(M) \text{ iff } s \in F. \]

2. Let $M = (K, \Sigma, \Delta, s, F)$ be a non-deterministic finite automaton. Under exactly what conditions $e \in L(M)$?


Solution Now we have two possibilities: \( s \in F \) (computation of length 0) or there is a computation of length \( > 0 \) from \((s, e)\) to \((q, e)\) for \( q \in F \) when \( s \notin F \).

Problem 3 Let 
\[
M = (K, \Sigma, s, \Delta, F)
\]
for \( K = \{q_0, q_1, q_2, q_3\} \), \( s = q_0 \)
\( \Sigma = \{a, b\} \), \( F = \{q_1, q_2, q_3\} \) and
\[
\Delta = \{(q_0, a, q_1), (q_0, b, q_3), (q_1, a, q_2), (q_1, b, q_1), (q_3, a, q_3), (q_3, b, q_2)\}
\]

1. List some elements of \( L(M) \).

Solution \( a, b, aa, bb, aba, abbba \)

2. Write a regular expression for the language accepted by \( M \). Simplify the solution.

Solution 
\[
L(M) = ab^* \cup ab^*a \cup ba^*b = ab^*(e \cup a) \cup ba^*(e \cup b).
\]

3. Define a deterministic \( M' \) such that \( M \equiv M' \), i.e. \( L(M) = L(M') \).

Solution We complete \( M \) do a deterministic \( M' \) by adding a TRAP state \( q_4 \) and put
\[
\Delta' = \delta = \Delta \cup \{(q_2, a, q_4), (q_2, b, q_4), (q_4, a, q_4), (q_4, b, q_4)\}
\]

Justify why \( M \approx M' \).

Solution \( q_4 \) is a trap state, it does not influence \( L(M) \).

Problem 4 Let \( M \) be defined as follows 
\[
M = (K, \Sigma, s, \Delta, F)
\]
for \( K = \{q_0, q_1, q_2, q_3\} \), \( s = q_0 \)
\( \Sigma = \{a, b, c\} \), \( F = \{q_0, q_2, q_3\} \) and
\[
\Delta = \{(q_0, abc, q_0), (q_0, a, q_1), (q_0, c, q_3), (q_1, bc, q_1), (q_1, b, q_2), (q_2, a, q_2), (q_2, b, q_3), (q_3, a, q_3)\}.
\]

Find the regular expression describing the \( L(M) \). Simplify it as much as you can. Explain your steps. Does \( e \in L(M) \)?

Solution 
\[
L = (abc)^*a(bc)^* \cup e \cup a^*ba^* \cup (abd)^*
\]

This is not the only solution.

Observe that \( e \in L \) as \( q_0 \in F \) and also \((q_0, e, q_3) \in \Delta \) and \( q_3 \in F \).

This is not the only solution.

Write down (you can draw the diagram) an automata \( M' \) such that \( M' \equiv M \) and \( M' \) is defined by the BOOK definition.

Solution
Solution We apply the "stretching" technique to \( M \) and the new \( M' \) is as follows.

\[
M' = (K \cup \{p_1, p_2, p_3\}, \Sigma, s = q_0, \Delta', F' = F)
\]

for \( K = \{q_0, q_1, q_2\}, s = q_0 \)
\( \Sigma = \{a, b\}, F = \{q_0, q_2, q_3\} \) and
\( \Delta' = \{(q_0, a, q_1), (q_0, e, q_3), (q_1, b, q_2), (q_2, a, q_2), (q_2, b, q_3), (q_3, a, q_3)\} \cup \{(q_0, a, p_1), (p_1, b, p_2), (p_2, c, q_0), (q_1, b, p_3), (p_3, b, q_1)\}

Problem 5 For \( M \) defined as follows

\[
M = (K, \Sigma, s, \Delta, F)
\]

for \( K = \{q_0, q_1, q_2, q_3\}, s = q_0 \)
\( \Sigma = \{a, b\}, F = \{q_2\} \) and
\( \Delta = \{(q_0, a, q_3), (q_0, e, q_3), (q_0, b, q_1), q_0, e, q_1), (q_1, a, q_2), (q_2, b, q_3), (q_2, e, q_3)\}\)

Write 2 steps of the general method of transformation the NDFA \( M \) defined above into an equivalent DFA \( M' \).

Step 1: Evaluate \( \delta(E(q_0), a) \) and \( \delta(E(q_0), b) \).

Step 2: Evaluate \( \delta \) on all states that result from step 1.

Reminder: \( E(q) = \{p \in K : (q, e) \stackrel{\star\rightarrow}{M} (p, e)\} \) and
\[
\delta(Q, \sigma) = \bigcup \{E(p) : \exists q \in Q, (q, \sigma, p) \in \Delta\}.
\]

Solution Step 1: First we need to evaluate \( E(q) \), for all \( q \in K \).
\[
E(q_0) = \{q_0, q_1, q_3\} = S, E(q_1) = \{q_1\}, E(q_2) = \{q_2, q_3\} \in F, E(q_3) = \{q_3\}
\]
\[
\delta(E(q_0), a) = \delta(\{q_0, q_1, q_3\}, a) = E(q_3) \cup E(q_2) \cup \emptyset = \{q_2, q_3\} \in F
\]
\[
\delta(E(q_0), b) = \delta(\{q_0, q_1, q_3\}, b) = E(q_1) \cup \emptyset \cup \emptyset = \{q_1\}
\]

Solution Step 2:
\[
\delta(\{q_2, q_3\}, a) = \emptyset \cup \emptyset = \emptyset
\]
\[
\delta(\{q_2, q_3\}, b) = E(q_3) \cup \emptyset = \{q_3\}
\]
\[
\delta(\{q_1\}, a) = E(q_2) = \{q_2, q_3\} \in F
\]
\[
\delta(\{q_1\}, b) = \emptyset
\]