1 YES/NO questions

1. All infinite sets have the same cardinality.
   Justify: \(|N| \neq |R|\) and \(N\) (natural numbers) and \(R\) (real numbers) are infinite sets.
   y

2. Regular language is a regular expression.
   Justify: Regular language is a language defined by a regular expression.
   n

3. \(L^+ = \{w_1...w_n : w_i \in L, i = 1, 2, ..n, n \geq 1}\).
   Justify: definition
   y

4. \(L^* - \{e\}\).
   Justify: only when \(e \notin L\). When \(e \in L\) we get that \(e \in L^+\) and \(e \notin L^* - \{e\}\).
   n

5. \((\emptyset^* \cap b^*) \cup \emptyset^*\) describes a language with only one element.
   Justify: \((\{e\} \cap \{b\}^*) \cup \{e\} = \{e\} \cup \{e\} = \{e\}\)
   y

6. Let \(M\) be a finite state automaton, \(L(M) = \{\omega \in \Sigma^* : (s, \omega) \xrightarrow{s,M} (q, e)\}\).
   Justify: only when \(q \in F\)
   n

7. DFA and NDFA recognize the same class of languages.
   Justify: theorem proved in class
   y

2 Two definitions of a non-deterministic automaton

BOOK DEFINITION: \(M = (K, \Sigma, \Delta, s, F)\) is non-deterministic when
\[
\Delta \subseteq K \times (\Sigma \cup \{e\}) \times K
\]

OBSERVE that \(\Delta\) is always finite because \(K, \Sigma\) are finite sets.

LECTURE DEFINITION: \(M = (K, \Sigma, \Delta, s, F)\) is non-deterministic when \(\Delta\) is finite and
\[
\Delta \subseteq K \times \Sigma^* \times K
\]

OBSERVE that we have to say in this case that \(\Delta\) is finite because \(\Sigma^*\) is infinite.

SOLVING PROBLEMS you can use any of these definitions.
3 Very short questions

For the QUESTIONS below do the following.

1. Determine whether it defines a finite state automaton.
2. Determine whether it is a deterministic / non-deterministic automaton.
3. Write full definition of $M$ by listing all its components.
4. Describe the language by writing a regular expression or a property that defines it.

Q1 Solution: $M = (K, \Sigma, s, \Delta, F)$ for $K = \{q_0\} = F$, $s = q_0$, $\Sigma = \emptyset, \Delta = \emptyset$. $M$ is deterministic and $L(M) = \{e\} \neq \emptyset$

Q2 Solution: $M = (K, \Sigma, s, \Delta, F)$ for $\Sigma = \{a,b\}, K = \{q_0, q_1\}, s = q_0, F = \{q_0\}, \Delta = \{(q_0,a,q_1),(q_1,b,q_0)\}$. $M$ is non deterministic; $\Delta$ is not a function on $K \times \Sigma$.

$L(M) = (ab)^*$

Q3 Solution: $M = (K, \Sigma, s, \Delta, F)$ for $\Sigma = \{a,b\}, K = \{q_0, q_1, q_2\}, F = \{q_1\}$,
\[ \Delta = \{(q_0,a,q_1),(q_0,b,q_1),(q_1,a,q_1),(q_1,e,q_3),(q_2,ab,q_2)\} \]
It is NOT an automaton. It has no initial state.

Q4 $M = (K, \Sigma, s, \Delta, F)$ for $\Sigma = \{a,b\}, K = \{q_0, q_1, q_2, q_3\}, s = q_0, F = \emptyset,$
\[ \Delta = \{(q_0,a,q_1),(q_1,b,q_2),(q_2,a,q_1),(q_0,e,q_3),(q_2,a,q_3)\} \]
$M$ is non deterministic; $\Delta \subseteq K \times \Sigma \cup \{e\} \times K$.

$L(M) = \emptyset$

4 Problems

PROBLEM 1 Let $L$ be a language defines as follows

$L = \{w \in \{a,b\}^* : \text{between any two a's in w there is an even number of consecutive b's.}\}.$

1. Describe a regular expression $r$ such that $L(r) = L$.

Solution Remark that 0 is an even number, hence $a^* \in L$, 

$r = b^* \cup b^*a^*b^* \cup b^*(a(bb)^*a)^*b^* = b^*a^*b^* \cup b^*(a(bb)^*a)^*b^*$

2. Construct a finite state automata $M$, such that $L(M) = L$.

Solution 1

Components of $M$ are:

$\Sigma = \{a, b\}, K = \{q_0, q_1, q_2, q_3\}, s = q_0, F = \{q_0, q_2, q_3\},$

$\Delta = \{(q_0,b,q_0),(q_0,a,q_1),(q_0,e,q_3),(q_3,a,q_3),(q_3,b,q_3),(q_1,bb,q_1),(q_1,a,q_2),(q_2,e,q_0)\}$

Some elements of $L(M)$ as defined by the state diagram are:

$a, aabbb, bbb, aaaaaab, abba, abababab, aabbbbab, ....$
PROBLEM 2  Let  
\[ M = (K, \Sigma, s, \Delta, F) \]
for  \( K = \{q_0\} \), \( s = q_0 \), \( \Sigma = \{a, b\} \), \( F = \{q_0\} \) and 
\[ \Delta = \{(q_0, aba, q_0), (q_0, ab, q_0)\} \]

1. List some elements of \( L(M) \).
Solution

\[ e, ab, abab, ababa, ababaaba, \ldots \]

2. Write a regular expression for the language accepted by \( M \).
Solution

\[ L = (ab \cup aba)^* \]

PROBLEM 3

We know that for any deterministic finite automaton \( M = (K, \Sigma, s, \delta, F) \) the following is true:
\[ e \in L(M) \iff s \in F. \]

Show that the above is not true for all non-deterministic automata.

Solution  Let \( M = (K, \Sigma, s, \Delta, F) \) for \( K = \{q_0, q_1\} \), \( s = q_0 \), \( \Sigma = \emptyset \), \( F = \{q_1\} \), and \( \Delta = \{(q_0, e, q_1)\} \).
\[ L(M) = \{e\} \ and \ s \notin F. \]

PROBLEM 4  For \( M \) defined as follows
\[ M = (K, \Sigma, s, \Delta, F) \]
for  \( K = \{q_0, q_1, q_2, q_3\} \), \( s = q_0 \) 
\( \Sigma = \{a, b\} \), \( F = \{q_2, q_3\} \) and 
\[ \Delta = \{(q_0, a, q_1), (q_0, e, q_1), (q_0, b, q_2), (q_1, b, q_3), (q_1, e, q_3), (q_2, b, q_2), (q_2, e, q_3), (q_3, a, q_3)\} \]

Write a regular expression describing \( L(M) \).
Solution

\[ aa^* \cup a^* \cup aba^* \cup ba^* \cup bb^* \cup bb^a^* \]

Write 4 steps  of the general method of transformation the NDFA \( M \), into an equivalent deterministic \( M' \).

Reminder:  \( E(q) = \{p \in K : (q, e) \xrightarrow{*M} (p, e)\} \) and
\[ \delta(Q, \sigma) = \bigcup \{E(p) : \exists q \in Q, (q, \sigma, p) \in \Delta\}. \]

Solution Step 1:
\[ E(q_0) = \{q_0, q_1, q_3\}, \ E(q_1) = \{q_1, q_3\}, \ E(q_2) = \{q_2, q_3\}, \ E(q_3) = \{q_3\}. \]
Solution Step 2:

\[ \delta(E(q_0), a) = \delta(\{q_0, q_1, q_3\}, a) = E(q_1) \cup E(q_3) = \{q_1, q_3\} \in F, \]

\[ \delta(E(q_0), b) = \delta(\{q_0, q_1, q_3\}, b) = E(q_2) \cup E(q_3) \cup \emptyset = \{q_2, q_3\} \in F, \]

Solution Step 3:

\[ \delta(\{q_1, q_3\}, a) = \emptyset \cup E(q_3) = \{q_3\} \in F, \]

\[ \delta(\{q_1, q_3\}, b) = E(q_3) \cup \emptyset = \{q_3\} \in F, \]

\[ \delta(\{q_2, q_3\}, a) = \emptyset \cup E(q_3) = \{q_3\} \in F, \]

\[ \delta(\{q_2, q_3\}, b) = E(\{q_2\}) \cup \emptyset = \{q_2, q_3\} \in F \]

Solution Step 4:

\[ \delta(\{q_3\}, a) = E(q_3) = \{q_3\} \in F, \quad \delta(\{q_3\}, b) = \emptyset, \]

\[ \delta(\emptyset, a) = \emptyset, \quad \delta(\emptyset, b) = \emptyset \]

End of the construction.