YES/NO questions (10pts)

1. For any finite language $L \subseteq \Sigma^*$, $\Sigma \neq \emptyset$ there is a finite automata $M$, such that $L = L(M)$.
   **Justify:** any finite language is regular

2. Given $L_1, L_2$ languages over $\Sigma$, then $((L_1 \cup (\Sigma^* - L_2)) \cap L_2)$ is regular.
   **Justify:** only when both are regular languages

3. For any deterministic automata $M$, $L(M) = \bigcup \{R(1, j, n) : q_j \in K\}$, where $R(1, j, n)$ is the set of all strings in $\Sigma^*$ that may drive $M$ from state initial state to state $q_j$ without passing through any intermediate state numbered $n + 1$ or greater, where $n$ is the number of states of $M$.
   **Justify:** only when $q_j \in F$

4. If $L_1 \cap L_2$ is a regular language, so are $L_1$ and $L_2$.
   **Justify:** No, $L_1$ and $L_2$ may not be regular. Take $L_1 = \{a^n b^n : n \in \mathbb{N}\}, \ L_2 = \{a^n : n \in \text{Prime}\}$ $L_1 \cap L_2 = \emptyset$ is a regular language and $L_1, L_2$ are not regular.

5. $L = \{a^n a^n : n \geq 0\}$ is not regular.
   **Justify:** $L = a^n a^n = a^{2n} = (aa)^*$ and hence regular

6. If $L$ is regular, so is the language $L_1 = \{xy : x \in L, y \notin L\}$.
   **Justify:** Observe that $L_1 = L(\Sigma^* - L)$ and $L$ regular, hence $(\Sigma^* - L)$ is regular (closure under complement), so is $L_1$ by closure under concatenation.

7. Let $L$ be a regular language, and $L_1 \subseteq L$, then $L_1$ is regular.
   **Justify:** $L_1 = \{a^n b^n : n \geq 0\}$ is a non-regular subset of regular $L = a^* b^*$

8. Let $L$ be a language. The language $L^R = \{w^R : w \in L\}$ is regular.
   **Justify:** $L^R$ is accepted by finite automata $M^R$ constructed from $M$ such that $L(M) = L$

9. Let $L$ be a regular language $\Sigma$. Then the following condition holds.
   $$\exists n \geq 1 \forall \omega \in L(|\omega| \geq n \Rightarrow \forall x, y, z \in \Sigma^*(w = xyz \land y \neq e \land |xy| \leq n \land \forall i \geq 0(xy^iz \in L))$$
   **Justify:** $\exists x, y, z \in \Sigma^*$. 

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10. Let $L$ be a regular language over $\Sigma \neq \emptyset$. Then the following holds:

\[ \exists w \in \Sigma^* \exists x, y, z \in \Sigma^* (w = xyz \land y \neq e \land \forall n \geq 0 (xy^n z \in L)) \]

**Justify:** only when $L$ is infinite.

**PROBLEMS**

**QUESTION 1 (5pts)** Using the construction in the proof of theorem

*A language is regular iff it is accepted by a finite automata*

construct a a finite automata $M$ accepting

$L_1 = \mathcal{L} = ((ba)^* \cup (cb)^*)ab$

You can just draw a diagrams.

**Solution**

1. Diagrams for $M_1, M_2, M_3$ such that $L(M_1) = ab, L(M_2) = bc, L(M_3) = ba$

**Solution**

**M1** components:

$K = \{q_1, q_2\}, \Sigma = \{a, b, c\}, s = q_1, F = \{q_2\}$,

\[ \Delta_{M_1} = \{(q_1, ba, q_2)\} \]

**M2** components:

$K = \{q_3, q_4\}, \Sigma = \{a, b, c\}, s = q_3, F = \{q_4\}$,

\[ \Delta_{M_2} = \{(q_2, cb, q_4)\} \]

**M3** components:

$K = \{q_5, q_6\}, \Sigma = \{a, b, c\}, s = q_5, F = \{q_6\}$,

\[ \Delta_{M_3} = \{(q_5, ab, q_6)\} \]

2. Diagrams for $M_4, M_5$ such that $L(M_4) = L(M_1)^*, L(M_5) = L(M_2)^*$
Solution

M4 components:

\[ K = \{q_1, q_2, q_7\}, \Sigma = \{a, b, c\}, s = q_7, F = \{q_2, q_7\}, \]
\[ \Delta_{M4} = \{(q_7, e, q_1), (q_1, ba, q_2), (q_2, e, q_1)\} \]

M5 components:

\[ K = \{q_3, q_4, q_8\}, \Sigma = \{a, b, c\}, s = q_8, F = \{q_4, q_8\}, \]
\[ \Delta_{M4} = \{(q_8, e, q_3), (q_3, cb, q_4), (q_4, e, q_3)\} \]

3. Diagram for \( M_6 \) such that \( L(M_5) = L(M_4) \cup L(M_5) \)

Solution

M5 components:

\[ K = \{q_1, q_2, q_3, q_4, q_7, q_8, q_9\}, \Sigma = \{a, b, c\}, s = q_9, F = \{q_2, q_4, q_7, q_8\}, \]
\[ \Delta_{M5} = \{(q_9, e, q_7), (q_3, c, q_3), (q_4, c, q_3), (q_3, cb, q_4), (q_4, e, q_3)\} \]

4. Diagram for \( M = M_5M_3 \), i.e. \( M \) is such that \( L(M) = L(M_5) \cup L(M_3) \).

M components:

\[ K = \{q_1, q_2, q_3, q_4, q_7, q_8, q_9\}, \Sigma = \{a, b, c\}, s = q_9, F = \{q_6\}, \]
\[ \Delta_{M5} = \Delta_{M4} \cup \{(q_7, e, q_5), (q_9, e, q_5), (q_2, e, q_5), (q_4, e, q_5), (q_5, ab, q_6)\} \]
\[ = \{(q_9, e, q_7), (q_9, e, q_8), (q_7, e, q_1), (q_1, ba, q_2), (q_2, e, q_1), (q_8, e, q_3), (q_3, cb, q_4), (q_4, e, q_3), \]
\[ (q_7, e, q_5), (q_9, e, q_5), (q_2, e, q_3), (q_4, e, q_5), (q_5, ab, q_6)\} \]

QUESTION 2 (5pts)

Evaluate \( r \), such that \( L(r) = L(M) \)

using the Generalized Automata Construction, as described in example 2.3.2 page 80.

\[ M = \{q_1, q_2\}, \{a, b\}, s = q_1, \]
\[ \Delta = \{(q_1, a, q_1), (q_1, b, q_2), (q_2, a, q_2), (q_2, b, q_1)\}, F = \{q_2\} \]

Step 1: Construct a generalized \( GM \) that extends \( M \), i.e. such that \( L(M) = L(GM) \)
Solution
\[ GM = \{ q_1, q_2, q_3, q_4 \}, \{ a, b \}, s = q_3, F = \{ q_4 \} \]
\[ \Delta = \{ (q_1, a, q_1), (q_1, a, q_2), (q_2, b, q_2), (q_2, a, q_1) \}, (q_3, e, q_4), (q_2, e, q_4) \} \]

**Step 2:** Construct \( GM_1 \eqsim GM \eqsim M \) by elimination of \( q_1 \).

Solution
\[ GM_1 = \{ q_2, q_3, q_4 \}, \{ a, b \}, s = q_3, F = \{ q_4 \} \]
\[ \Delta = \{ (q_3, b^* a, q_2), (q_2, b, q_2), (q_2, ab^* a, q_2) \}, (q_2, e, q_4) \} \]

**Step 3:** Construct \( GM_2 \eqsim GM_1 \eqsim GM \eqsim M \) by elimination of \( q_2 \).

Solution
\[ GM_2 = \{ q_3, q_4 \}, \{ a, b \}, s = q_3, F = \{ q_4 \} \]
\[ \Delta = \{ (q_3, b^* a(ab^* a \cup b)^*, q_4) \} \]

**Answer:** the language is
\[ L(M) = b^* a(ab^* a \cup b)^* \]

**QUESTION 3** (5pts) Show that the class of regular languages is not closed with respect to subset relation.

**Solution** Consider
\[ L_1 = \{ a^n b^n : n \in N \}, \quad L_2 = a^* b^* \]
\( L_1 \subseteq L_2 \) and \( L_1 \) is a non-regular subset of a regular \( L_2 \).