PART 1: Yes/No Questions

1. \{\emptyset\} \subseteq \{a, b, c\}
   \textbf{Justify}: \emptyset \not\in \{a, b, c\}

2. Set \(A\) is countable iff \(\mathbb{N} \subseteq A\) (\(\mathbb{N}\) is the set of natural numbers).
   \textbf{Justify}: \(A = \{\emptyset\}\) is countable (finite), but \(\mathbb{N}\) is not a subset of \(\{\emptyset\}\), i.e. \(\mathbb{N} \not\subseteq \emptyset\).
   In fact \(A\) can be ANY finite set, or any infinite set that does not include \(\mathbb{N}\), for example \(A = \{\{n\} : n \in \mathbb{N}\}\).
   In this case \(|A| = |\mathbb{N}|\), but \(\mathbb{N}\) is not a subset of \(A\), i.e. \(\mathbb{N} \not\subseteq A\).

3. \(2^\mathbb{N}\) is infinitely countable.
   \textbf{Justify}: \(|2^\mathbb{N}| = |\mathbb{R}|\) and \(\mathbb{R}\) are uncountable.

4. Let \(A = \{\{n\} : n^2 + 1 \leq 15\}\). \(A\) is infinite.
   \textbf{Justify}: \(\{n \in \mathbb{N} : n^2 + 1 \leq 15\} = \{0, 1, 2, 3\}\),
   hence \(A = \{\{0\}, \{1\}, \{2\}, \{3\}\}\) is a finite set.

5. Let \(\Sigma = \{a\}\). There are countably many languages over \(\Sigma\).
   \textbf{Justify}: There is any many languages as subsets of \(\Sigma^*\), i.e. uncountably many and exactly as many as real numbers.

6. For any \(L\), \(L^+ = L^* - \{e\}\).
   \textbf{Justify}: Only when \(e \not\in L\).

7. \(L^* = \{w_1...w_n : w_i \in L, i = 1, 2, ..n, n \geq 1\}\).
   \textbf{Justify}: \(n \geq 0\).

8. For any language \(L\) over an alphabet \(\Sigma\), \(L^+ = L \cup L^*\).
   \textbf{Justify}: Take \(L\) such that \(e \not\in L\). We get that \(e \in L \cup L^*\) as \(e \in L^*\)
   and \(e \not\in L^+\).
PART 2: PROBLEMS

QUESTION 1
Let $\Sigma$ be any alphabet, $L_1, L_2$ two languages over $\Sigma$ such that $e \in L_1$ and $e \in L_2$. Show that

$$(L_1 \Sigma^* L_2)^* = \Sigma^*$$

Solution: By definition, $L_1 \subseteq \Sigma^*$, $L_2 \subseteq \Sigma^*$ and $\Sigma^* \subseteq \Sigma^*$. Hence

$$(L_1 \Sigma^* L_2) \subseteq \Sigma^*.$$  

Now we use the following property:

Property  For any languages $L_1, L_2$,  

if $L_1 \subseteq L_2$, then $L_1^* \subseteq L_2^*$  

and obtain that

$$(L_1 \Sigma^* L_2)^* \subseteq \Sigma^{**} = \Sigma^*.$$  

We have to show that also

$$\Sigma^* \subseteq (L_1 \Sigma^* L_2)^*.$$  

Let $w \in \Sigma^*$ we have that also $w \in (L_1 \Sigma^* L_2)^*$ because $w = ewe$ and $e \in L_1$ and $e \in L_2$.

QUESTION 2
Let $\mathcal{L}$ be a function that associates with any regular expression $\alpha$ the regular language $\mathcal{L}(\alpha)$.

1. Evaluate $\mathcal{L}(((a \cup b)^* a))$.

Solution: $\mathcal{L}(((a \cup b)^* a)) = \mathcal{L}((a \cup b)^*) \mathcal{L}(a) = (\mathcal{L}(a \cup b)^*) \{a\} = (\mathcal{L}(a) \cup \mathcal{L}(b))^* \{a\} = (\{a\} \cup \{b\})^* \{a\} = \{a, b\}^* \{a\}$.

2. Describe a property that defines the language $\mathcal{L}(((a \cup b)^* a))$.

Solution  $\{a, b\}^* \{a\} = \Sigma^* \{a\} = \{w \in \{a, b\}^* : w \text{ ends with } a \}$.

QUESTION 3
Let $\Sigma = \{a, b\}$. Let $L_1 \subseteq \Sigma^*$ be defined as follows:

$$L_1 = \{w \in \Sigma^* : \text{number of } b \text{ in } w \text{ is divisible three} \}$$

Write a regular expression $\alpha$, such that $\mathcal{L}(\alpha) = L_1$. You can use shorthand notation. Explain shortly your answer.
Solution:

\[ \alpha = a^* (a^* ba^* ba^*)^* = a^* (ba^* ba^*)^*. \]

**Explanation:** the part \( a^* ba^* ba^* \) says that there must be 3 occurrences of \( b \) in \( L_1 \). The part \( (a^* ba^* ba^*)^* \) says that we the number of \( b \)'s is \( 3n \) for \( n \geq 1 \).

Observe that 0 is divisible by 3, so we need to add the case of 0 number of \( b \)'s \( (n = 0) \), i.e. words \( e, a, aa,aaa, \ldots \). We do so by concatenating \( (a^* ba^* ba^*)^* \) with \( a^* \).