Radial Restraint: A Semantically Clean Approach to Bounded Rationality for Logic Programs

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Abstract

Declarative logic programs (LP) based on the well-founded semantics (WFS) are widely used for knowledge representation (KR). Logical functions are desirable expressively in KR, but when present make LP inferencing become undecidable. In this paper, we present *radial restraint*: a novel approach to bounded rationality in LP. Radial restraint is parameterized by a norm that measures the syntactic complexity of a term, along with an abstraction function based on that norm. When a term exceeds a bound for the norm, the term is assigned the WFS's third truth-value of *undefined*. If the norm is finitary, radial restraint guarantees finiteness of models and decidability of inferencing, even when logical functions are present. It further guarantees soundness, even when non-monotonicity is present. We give a fixed-point semantics for *radially restrained well-founded models* which soundly approximate well-founded models. We also show how to perform correct inferencing relative to such models, via SLG_{ABS}, an extension of tabled SLG resolution that uses norm-based abstraction functions. Finally we discuss how SLG_{ABS} is implemented in the engine of XSB Prolog, and scales to knowledge bases with more than 10^8 rules and facts.

Introduction

Declarative logic programs (LP) based on the well-founded semantics (WFS) are widely used for knowledge representation (KR), e.g., in databases, business rules, and semantic web. They represent logical non-monotonicity, and offer much better scalability than answer-set programs (ASP) or first-order logic (FOL). Logical functions are desirable expressively in KR overall, and in particular in Rulelog, the logical extension of LP that has been developed and employed in the SILK project and system (SILK 2013). Functions are needed there to support three expressive features: hilog (Chen, Kifer, and Warren 1993); defeasibility via argumentation theories (Wan et al. 2009); and existentials in FOL-like *omniform* rules. These three features are in turn used for reasoning about causal processes/actions, qualitative reasoning, and text-based knowledge acquisition (KA) — in SILK's pilot application domain of cell biology at the first year college level. However, functions pose a fundamental computational complexity challenge in LP and thus Rulelog/SILK, as functions do also in FOL and ASP. LP inferencing is tractable (worst-case quadratic) in the size of the ground-instantiated rulebase. But when functions are present, the Herbrand universe is infinite, and LP inferencing is undecidable. The LP model might be infinite, and a single LP query might have an infinite set of answers. Today's commercially dominant systems for databases, business rules, and semantic web (e.g., SQL, SPARQL, production rules, and event-condition-action rules) avoid use of functions. We are thus motivated to seek a way to allow functions in LP yet to mitigate their impact on scalability — including, as a first step, to ensure finiteness of models and decidability of inferencing. Our approach builds on the idea of bounded rationality.

While AI bounded rationality research has largely focused on decision-theoretic optimization (e.g., (Russell and Subramanian 1995; Anderson and Oates 2007)), a strand has focused on limiting reasoning via deduction principles that derive some beliefs explicitly but leave others implicit (Konolige 1983; Levesque 1984; Fisher and Ghidini 1999; Grant, Kraus, and Perlis 2000; Fisher et al. 2007). To date, however, this strand has lacked much practical impact. For resource-limiting logic programming (i.e., LP), the main approach that has emerged in practice is to set (manually or heuristically) an inferencing engine parameter — for instance, a timeout or a term-depth bound in Prolog — and to treat as false any atom that is not inferred before the parameter bound is exceeded. However, incompleteness about an atom A can lead to unsoundness if another atom A' depends negatively on A. In addition, the results of such inferencing depend on the implementation code or session. Radial restraint overcomes both these shortcomings. It introduces (to our knowledge) the use of the truth value *undefined* to represent implicit deductions that have not been made explicit.

Our starting point is recent work on termination properties of logic programs with negation (default negation, a.k.a. negation-as-failure, a.k.a. weak negation). When tabled evaluation is extended with *subgoal abstraction*, first introduced in (Tamaki and Sato 1986), tabling can ensure termination to queries to safe normal programs that are *strongly bounded term size (SBTS)* (Riguzzi and Swift 2013b; 2013a). Such programs have well-founded models (van Gelder, Ross, and Schlipf 1991) that are representable via finite sets of true and undefined ground atoms, and have

been shown to properly include the finitely ground programs of (Calimeri et al. 2008), a class motivated by the needs of ASP grounders (cf. (Alviano, Faber, and Leone 2010)). Although these results are powerful, they do have drawbacks. It is not decidable whether a program is SBTS. In addition, while SBTS-programs are Turing-complete (shown for finitely ground programs in (Calimeri et al. 2008)), some natural programs are not SBTS: such as those that contain a predicate to determine list membership. From a theoretical level, this weakness can be addressed by defining program classes that terminate for various types of queries (cf. e.g., (Bonatti 2004)). However, membership in such classes is again not decidable.

The tabled evaluation method of (Riguzzi and Swift 2013a) is complete for SBTS-programs and queries. Building on this evaluation method, and making use of the *undefined* truth value as discussed above, we ensure that both the evaluations and the (sub-)models they produce are finite. We introduce aspects of the approach through the following example. to motivate the formalism that follows.

Example 1 *Consider the program P_{inf}:*

p(s(X)) ← p(X).
p(0).
q(0).

P_{inf} is not SBTS as the sets both of true atoms of its well-founded model, $true(WFM^{P_{inf}})$, and the false atoms, $false(WFM^{P_{inf}})$, are infinite. However, an approximation of the answers to p(X) *can be made by restraining inferencing. For instance, if a depth norm were used with a level of 4, then the answers* p(0)*,* p(s(0))*, and* p(s(s(0))) *would be derived. However the answer* p(s(s(s(0)))) *would be* abstracted *to* p(s(s(s(X)))) *and all ground atoms unifying with this answer would be assigned the truth value of* undefined*. This infinite set of answers is represented by the sentence* ∀X. p(s(s(s(X)))). *By allowing the set of atoms in $WFM^{P_{inf}}$ whose truth value is undefined (*undef*($WFM^{P_{inf}}$)) to be represented by such sentences in addition to atoms, a finite representation of the* radially restrained model *is constructed. In addition, because the default negation of any undefined atom is itself undefined in the well-founded semantics, the answer abstraction preserves the soundness of negation. For instance the literal* not p(s(s(s(s(0))))) *is also assigned undefined.*

Intuitively, atoms that are true or false in the well-founded model of a program remain true or false in the radially restrained model as long as they don't exceed a bound defined by a norm on atoms. Those atoms that exceed the bound are abstracted, and have the truth value *undefined*. In this way, radially restrained models are sound approximations to the well-founded model. Further, because variables in the abstraction are regarded as universally quantified, the resulting model can be represented using finite sets of true and undefined atoms.

This paper thus explores radially restrained models along with their efficient query evaluation. Specifically,

- We define the radially restrained well-founded model of a normal program P as parameterized by an abstraction function $abs(\cdot)$. We show that such a model soundly approximates the well-founded model of P, and that if $abs(\cdot)$ is replaced by a weaker abstraction, the approximation of the radially restrained model becomes tighter.
- By extending SLG resolution with subgoal abstraction (Riguzzi and Swift 2013a) to incorporate an abstraction function for answers, we introduce SLG$_{\mathsf{ABS}}$, which correctly evaluates queries with respect to radially restrained models. Given a finitary abstraction function, SLG$_{\mathsf{ABS}}$ terminates with an asymptotic complexity that is equal to the best complexity that is known.
- Finally, based on the the SLG-WAM of XSB (Swift and Warren 2012), we describe an implementation of SLG$_{\mathsf{ABS}}$ that is declarative, efficient and scalable.

Background

We assume a general knowledge of logic programming terminology, including tabled resolution and the well-founded semantics. In addition we make use of the following terminology and assumptions.

Throughout this paper we restrict our attention to normal programs, and to queries and subgoals that are atoms. We also assume a fixed strategy for selecting literals in a clause: without loss of generality we assume the selection strategy is left-to-right. In accordance with this strategy, a normal rule has the form

$$r = A_0 \leftarrow A_1, \ldots, A_m, not\ A_{m+1}, \ldots, not\ A_n$$

where $A_0, \ldots, A_n$ are atoms. A program P is *safe* if each rule r in P is such that every variable in r occurs in a positive literal in the body of r. Our attention is also restricted to three-valued (partial) interpretations and models, such as the well-founded model. Each such interpretation is represented as a pair of true and false atoms: $\langle true(\mathcal{I}); false(\mathcal{I})\rangle$. For two interpretations, $\mathcal{I}$ and $\mathcal{J}$, $\mathcal{I} \subseteq \mathcal{J}$ iff $true(\mathcal{I}) \subseteq true(\mathcal{J})$ and $false(\mathcal{I}) \subseteq false(\mathcal{J})$. Alternatively, a three-value interpretation can be represented as a set of literals.

Symbols within a term may be represented through *positions* which are members of the set Π. A *position* in a term is either the empty string Λ that reaches the root of the term, or the string $\pi.i$ that reaches the ith child of the term reached by π, where π is a position and i an integer. For a term t we denote the symbol at position π in t by t_π. For example, $p(a, f(X))_{2.1} = X$. We assume that a program P is defined over a language $\mathcal{L}$, containing a finite set $\mathcal{FN}$ of predicate and function symbols, and a countable set of variables from the set $\mathcal{V} \cup \hat{\mathcal{V}}$. Elements of the set $\mathcal{V}$ are referred to as *program variables*. Elements of the set $\hat{\mathcal{V}}$, called *position variables*, are of the form X_π, where π is a position. These variables are used when it is convenient to mark certain positions of interest in a term. The Herbrand Universe of $\mathcal{L}$ is denoted $\mathcal{H}_\mathcal{L}$, or as $\mathcal{H}_P$ if $\mathcal{L}$ consists of the predicate and function symbols in P; similarly the Herbrand Base is denoted as $\mathcal{B}_\mathcal{L}$ or as $\mathcal{B}_P$. Throughout the paper variant terms are considered to be equal.

Dynamic Stratification One of the most important formulations of stratification is that of *dynamic* stratification. (Przymusinski 1989) shows that a program has a 2-valued

well-founded model iff it is dynamically stratified, so that it is the weakest notion of stratification that is consistent with the well-founded semantics. As presented in (Przymusinski 1989), dynamic stratification computes strata via operators on interpretations of the form $\langle Tr; Fa\rangle$, where Tr and Fa are subsets of $\mathcal{H}_P$.

Definition 1 *For a normal program P, sets Tr and Fa of ground atoms and a 3-valued interpretation I (sometimes called a pre-interpretation):*

$True_I^P(Tr) = \{A | A$ *is not true in I; and there is a clause $B \leftarrow L_1, ..., L_n$ in P, a grounding substitution θ such that $A = B\theta$ and for every $1 \leq i \leq n$ either $L_i\theta$ is true in I, or $L_i\theta \in Tr\}$;*

$False_I^P(Fa) = \{A | A$ *is not false in I; and for every clause $B \leftarrow L_1, ..., L_n$ in P and grounding substitution θ such that $A = B\theta$ there is some i $(1 \leq i \leq n)$ such that $L_i\theta$ is false in I or $L_i\theta \in Fa\}$.*

(Przymusinski 1989) shows that $True_I^P$ and $False_I^P$ are both monotonic, and defines $\mathcal{TR}_I^P$ as the least fixed point of $True_I^P(\emptyset)$ and $\mathcal{FA}_I^P$ as the greatest fixed point of $False_I^P(\mathcal{H}_P)$. In words, the operator $\mathcal{TR}_I^P$ extends the interpretation I to add the new atomic facts that can be derived from P knowing I; $\mathcal{FA}_I^P$ adds the new negations of atomic facts that can be shown false in P by knowing I (via the uncovering of unfounded sets). An iterated fixed point operator builds up dynamic strata by constructing successive partial interpretations as follows.

Definition 2 (Iterated Fixed Point and Dynamic Strata) *For a normal program P let*

$$\begin{array}{rcl} WFM_0 & = & \langle\emptyset;\emptyset\rangle; \\ WFM_{\alpha+1} & = & WFM_\alpha \cup \langle \mathcal{TR}^P_{WFM_\alpha}; \mathcal{FA}^P_{WFM_\alpha}\rangle; \\ WFM_\alpha & = & \bigcup_{\beta<\alpha} WFM_\beta, \text{ for limit ordinal } \alpha. \end{array}$$

$WFM(P)$ denotes the fixed point interpretation WFM_δ, where δ is the smallest (countable) ordinal such that both sets $\mathcal{TR}^P_{WFM_\delta}$ and $\mathcal{FA}^P_{WFM_\delta}$ are empty. The stratum *of atom A, is the least ordinal β such that $A \in WFM_\beta$.*

(Przymusinski 1989) shows that $WFM(P)$ is in fact the well-founded model and that any undefined atoms of the well-founded model do not belong to any stratum – i.e. they are not added to WFM_δ for any ordinal δ. Thus, a program is *dynamically stratified* if every atom belongs to a stratum.

Radially Restrained Models

Norms and Abstractions

Abstraction functions may be understood with respect to norms, which can specify families of abstraction functions. Typically, if the norm of an atom A is greater than a given integer bound, A is abstracted.

A *norm* $N(\cdot)$ is a function from terms to non-negative integers such that

1. $N(t) = 0$ iff $t = \Lambda$ (the empty term)
2. t subsumes t' implies $N(t) \leq N(t')$

A norm is *finitary* iff for any finite non-negative integer k, the cardinality of the set $\{t | t \in \mathcal{H}_\mathcal{L} \wedge N(t) < k\}$ is finite.

An *abstraction* of a term t, denoted $abs(t)$, may replace subterms of t by position variables: formally, $abs(t)$ is a term such that if $abs(t)|_\pi \in (\mathcal{FN} \cup \mathcal{V})$, then $abs(t)|_\pi = t|_\pi$. For instance $p(f(g(X_{1.1.1}), X_{1.2}), X_2)$ is an abstraction of $p(f(g(a), X), X)$. It is easy to see that $abs(t)$ subsumes t, so for any norm $N(\cdot)$, $N(abs(t)) \leq N(t)$. An abstraction $abs(\cdot)$ is finitary if the cardinality of $\{abs(t) | t \in \mathcal{H}_\mathcal{L}\}$ is finite. Given two abstractions, $abs_1(\cdot) \leq abs_2(\cdot)$ if for all terms t, $abs_1(t)$ subsumes $abs_2(t)$. Note that if $abs_1(\cdot) \leq abs_2(\cdot)$, then $\{abs(t) | t \in \mathcal{H}_\mathcal{L}\} \subseteq \{abs(t) | t \in \mathcal{H}_\mathcal{L}\}$. Norms and abstractions are applied to atoms by taking those atoms as terms, and to rules by applying the operation to each atom underlying a literal in the rule.

Example 2 *A* depth norm*, $depth(\cdot)$, maps a term t to the maximal* depth *of any position in t, where the depth of the outermost function symbol of t is 1 and the depth of a position $\pi.i$ is the depth of π plus 1 if $t|_{\pi.i}$ is a not a position variable, and is the depth of π otherwise. For a positive integer k, a* depth-k abstraction *is an abstraction that maps t to itself if $depth(t)$ is less than or equal to k; and otherwise to the abstraction of t with depth k that is maximal with respect to subsumption. It is easy to see that such a maximal depth-k abstraction of t must be unique. Within the atom $A = p(a, f(b, g(c)))$ the depth of c is 4. The depth 3 abstraction of A is $p(a, f(b, g(X_{2.2.1})))$, and the depth 2 abstraction of A is $p(a, f(X_{2.1}, X_{2.2}))$. Both the depth norm and the family of depth-k abstractions (for positive integer k) are finitary.*

Depth-k abstractions are simple to understand and to implement. However the number of terms whose depth is less than k may grow exponentially. Thus, other abstractions, based on the size of a term, or that weigh the occurrence of certain types of function symbols over others (e.g., lists) can be practically useful. Finally, note that the identity function on terms, $Id(\cdot)$, is an abstraction function, but is not finitary for languages that contain non-constant function symbols. In fact, $Id(\cdot)$ is the maximal abstraction function.

Radially Restrained Models

The operators for Radially Restrained Models are based on abstraction functions rather than on norms, both to provide a basis for Theorem 2, and to highlight the correspondence with the resolution method described in the next section.

Definition 3 *For a normal program P, abstraction function $abs(\cdot)$, sets Tr and Fa of ground atoms, and a 3-valued interpretation I (sometimes called a pre-interpretation):*

$True_I^P(abs, Tr) = \{A |$ *there is a clause $B \leftarrow L_1, ..., L_n$ in P, a grounding substitution θ such that $A = B\theta = abs(B\theta)$, and for every $1 \leq i \leq n$ either $L_i\theta$ is true in I, or $L_i\theta \in Tr\}$;*

$False_I^P(abs, Fa) = \{A |$ *for every clause $B \leftarrow L_1, ..., L_n$ in P and grounding substitution θ such that $A = B\theta = abs(B\theta)$ and there is some i $(1 \leq i \leq n)$ such that $L_i\theta$ is false in I or $L_i\theta \in Fa\}$.*

Unlike Definition 1, Definition 3 requires that $abs(B\theta) = B\theta$ in order for an atom to be considered either true or false. Clearly both $True_I^P$ and $False_I^P$ are monotonic in their second arguments; and as with the well-founded model, we define $\mathcal{TR}_I^P(abs)$ as the least fixed point of $True_I^P(abs, \emptyset)$ and $\mathcal{FA}_I^P(abs)$ as the greatest fixed point of $False_I^P(abs, \mathcal{H}_P)$.

Definition 4 (Radially Restrained Model) *For a normal program P and abstraction function $abs(\cdot)$*

$$\begin{array}{rcl} WFM_0 & = & \langle\emptyset;\emptyset\rangle; \\ WFM_{\alpha+1} & = & WFM_\alpha \cup \\ & & \langle \mathcal{TR}^P_{WFM_\alpha}(abs); \mathcal{FA}^P_{WFM_\alpha}(abs)\rangle; \\ WFM_\alpha & = & \bigcup_{\beta<\alpha} WFM_\beta(abs), \textit{ for limit ordinal } \alpha. \end{array}$$

The radially restrained model $WFM(abs, P)$ denotes the fixed point interpretation WFM_δ, where δ is the smallest ordinal such that both sets $\mathcal{TR}^P_{WFM_\delta}(abs)$ and $\mathcal{FA}^P_{WFM_\delta}(abs)$ are empty.

The following statement follows directly from Definition 3. Since the language of P has a finite number of function symbols and predicates, and since $abs(\cdot)$ is finitary, $True_I^P(abs, Tr)$ can only produce a finite number of grounded rules, even if I or Tr were infinite [1].

Proposition 1 *For a program P and finitary abstraction function $abs(\cdot)$ let*

$$WFM(abs, P) = \langle TrueAtoms; FalseAtoms\rangle.$$

The cardinality of $TrueAtoms$ is finite.

Because $\mathcal{TR}(abs)$ is monotonic, due to Proposition 1 it must reach fixed point for some finite ordinal. Accordingly, if $abs(\cdot)$ is finitary, $WFM(abs, P)$ will also reach fixed point at some finite ordinal.

Theorem 1 *Given a program P and finitary abstraction function $abs(\cdot)$, then $WFM(abs, P) = WFM(abs, P)_\delta$ for some finite ordinal δ.*

The main theorem about radially restrained well-founded models is as follows.

Theorem 2 *Let $abs_1(\cdot), abs_2(\cdot)$ be abstraction functions such that $abs_1(\cdot) \leq abs_2(\cdot)$. Then for any program P, $WFM(abs_1, P) \subseteq WFM(abs_2, P)$.*

Since the identity function, $Id(\cdot)$ is the maximal abstraction function, and since $WFM(Id, P) = WFM(P)$, Theorem 2 implies:

Corollary 1 *For a program P and abstraction function $abs(\cdot)$, $WFM(abs, P) \subseteq WFM(P)$.*

For any program P, Theorem 2 also implies that a chain of abstraction functions $abs_1(\cdot), abs_2(\cdot), \ldots$ such that for $i \leq j$, $abs_i(\cdot) \leq abs_j(\cdot)$ is associated with a chain of models: $WFM(abs_1, P), WFM(abs_2, P), \ldots, WFM(abs_j, P) \ldots$ such that for $i \leq j$, $WFM(abs_i, P) \subseteq WFM(abs_j, P)$. Thus, families of finitary abstraction functions, based on depth, size or other measures, provide successively more powerful finite approximations of the well-founded model.

[1] Proofs of all results, along with a full presentation of SLG_{ABS} (introduced in the next section) are available at http://www.cs.sunysb.edu/~tswift/webpapers/radial.pdf.

Tabled Resolution for Bounded Rationality

SLG_{ABS} is a tabled resolution method that correctly evaluates queries to radially restrained models of programs. SLG_{ABS} strictly extends SLG evaluation (Chen and Warren 1996) which models well-founded computation at an operational level, ensuring goal-directedness, termination and optimal complexity for a normal programs. SLG evaluation, along with numerous extensions of it, are well-described in the literature. Accordingly in this section we present only those extensions used in SLG_{ABS}, after a brief review of the terminology required by the extensions.

Terminology Used

In the forest-of-trees model of SLG (Swift 1999), an evaluation is a possibly transfinite sequence of forests (sets) of trees in which each tree corresponds to a subgoal that has been encountered in an evaluation. When a new tabled subgoal S is encountered, a tree with root $S \leftarrow |S$ is added to the current forest by a NEW SUBGOAL operation, and children of the root are added through PROGRAM CLAUSE RESOLUTION operations. Other positive selected literals are resolved through the POSITIVE RETURN operation; while ground negative selected subgoals are resolved through the NEGATIVE RETURN operation, or their resolution may be delayed through the DELAYING operation. These delayed literals may later be evaluated through SIMPLIFICATION or ANSWER COMPLETION operations. The need to delay some literals arises because modern Prolog engines rely on a fixed order for selecting literals in a rule. However, well-founded computations cannot be performed using a fixed-order literal selection function. When it is determined that no more resolution may be performed for non-delayed literals in nodes of trees for a mutually dependent set of subgoals, the trees are marked as *complete* using the COMPLETION operation. If a subgoal S has been marked as *complete* and S has no answers, literals of the form $not\, S$ can be resolved away by the NEGATIVE RETURN operation.

More specifically, the nodes in each tree have the form

$$Ans \leftarrow Delays|Goals \qquad \text{or} \qquad \textit{fail}.$$

In the first form, Ans is an atom while $Delays$ and $Goals$ are sequences of literals. The second form is called a *failure node*. *Goals* represents the sequence of literals left to be examined, while *Delays* represents those literals that have been examined, but their resolution delayed. A node N is an *answer* when it is a leaf node for which $Goals$ is empty. If the $Delays$ of an answer is empty, it is termed an *unconditional answer*, otherwise, it is a *conditional answer*.

SLG resolution is used to resolve an answer A against a node N.

Definition 5 (SLG Resolution) *Let N be a node $A \leftarrow D|L_1, ..., L_n$, where $n > 0$. Let $Ans = A' \leftarrow D'|$ be an answer whose variables are disjoint from N. If $\exists i$, $1 \leq i \leq n$,*

such that L_i and A' are unifiable with mgu θ, then the resolvent *of N and Ans on L_i has the form:*

$$(A \leftarrow D|L_1, ..., L_{i-1}, L_{i+1}, ..., L_n)\theta$$

if D' is empty; otherwise the resolvent has the form:

$$(A \leftarrow D, L_i|L_1, ..., L_{i-1}, L_{i+1}, ..., L_n)\theta.$$

Example 3 below further illustrates the foregoing concepts within an SLG_{ABS} evaluation.

Definition 6 relates SLG forests to interpretations, and is used for the statement of correctness in Theorem 4.

Definition 6 *Let $\mathcal{F}$ be an SLG forest. The* interpretation induced by *$\mathcal{F}$, $\mathcal{I}_\mathcal{F}$, is the smallest set such that:*

- *A (ground) atom $A \in true(\mathcal{I}_\mathcal{F})$ iff A is in the ground instantiation of an unconditional answer $Ans \leftarrow |$ in $\mathcal{F}$.*
- *A (ground) atom $A \in false(\mathcal{I}_\mathcal{F})$ iff A is in the ground instantiation of a subgoal whose tree in $\mathcal{F}$ is marked as* complete*, and A is not in the ground instantiation of any answer in a tree in $\mathcal{F}$.*

An atom S is successful *(resp.* failed*) in $\mathcal{F}$ if S' is in $true(\mathcal{I}_\mathcal{F})$ ($false(\mathcal{I}_\mathcal{F})$) for every S' in the ground instantiation of S. A non-ground subgoal $not\ S$ succeeds (fails) if S fails (succeeds). Given an interpretation $\mathcal{J}$ and forest $\mathcal{F}$, the* restriction of *$\mathcal{J}$ to $\mathcal{F}$, $\mathcal{J}|_\mathcal{F}$ is the interpretation such that $true(\mathcal{J}|_\mathcal{F})$ ($false(\mathcal{J}|_\mathcal{F})$) consists of those atoms in $true(\mathcal{J})$ ($false(\mathcal{J})$) that are in the ground instantiation of some subgoal whose tree is in $\mathcal{F}$*

SLG_{ABS}

SLG_{ABS} extends SLG to use abstraction both when creating a tree for a new subgoal, and when deriving an answer.

Definition 7 (Subgoal Abstraction (Riguzzi and Swift 2013a)) NEW SUBGOAL*: Let $abs(\cdot)$ be an abstraction function, and let a forest $\mathcal{F}_n$ contain a tree with non-root node*

$$N = Ans \leftarrow Delays|G, Goals$$

where S is the underlying subgoal of the literal G. Assume $\mathcal{F}_n$ contains no tree with root $abs(S)$. Then add the tree $abs(S) \leftarrow |abs(S)$ to $\mathcal{F}_n$.

Abstaction is also used when an answer Ans is derived; If the abstraction is non-trivial, i.e., if $Ans \neq abs(Ans)$, then a special atom $undefined_{abs}$ is added to the $Delays$ of Ans.

Definition 8 (Answer Abstraction) POSITIVE RETURN*: Let $abs(\cdot)$ be an abstraction function, and let a forest $\mathcal{F}_n$ contain a tree with non-root node N whose selected literal S is positive. Let Ans be an answer for S in $\mathcal{F}_n$ and $N' = A \leftarrow Delays|Goals$ be the SLG resolvent of N and Ans on S.*

- *If $Goals$ is non-empty, then $N_{child} = N'$;*
- *Otherwise, if $abs(N') = N'$, then $N_{child} = N'$;*
- *Otherwise, if $abs(N') \neq N'$, $N_{child} = abs(A \leftarrow Delays, undefined_{abs})|$.*

If N does not have a child N_{child} in $\mathcal{F}_n$, then add N_{child} as a child of N.

For SLG_{ABS} to be correct with respect to radially restrained models of normal programs, negation must be extended to handle the lack of safety that is introduced by abstraction. The following example shows how this can occur, and illustrates the SLG and SLG_{ABS} terminology used so far.

Example 3 *Figure 1 shows the SLG_{ABS} evaluation of the query* r(X) *against the safe program $P_{abs\text{-}unsafe}$:*

p(s(X)) ← p(X). p(0).
r(X) ← p(X),not q(X). q(0).

where a depth-2 abstraction function is used (local scheduling is assumed for this evaluation, cf. (Swift and Warren 2012)). The evaluation begins in a manner identical to SLG evaluation. The initial forest consists simply of node 0. Children of root nodes are created by PROGRAM CLAUSE RESOLUTION*, which creates node 1. The selected (leftmost) literal of node 1 is* p(X)*, which is new at this point of the evaluation. A* NEW SUBGOAL *operation creates node 2, (although an abstraction is applied, it is trivial), and* PROGRAM CLAUSE RESOLUTION *creates node 3, an unconditional answer. Reapplication of* PROGRAM CLAUSE RESOLUTION *also creates node 4, whose selected literal is not new to the evaluation. There is already an answer for* p(X) *so that* POSITIVE RETURN *is applicable to this node; repeated applications of* POSITIVE RETURN *produce nodes 5 and 6. Although abstraction is performed for all answers, it is trivial except when producing node 6. Once node 6 is produced, the tree for* p(X) *is completely evaluated, and a* COMPLETION *operation marks it* complete*. Another* POSITIVE RETURN *operation produces node 7 which has a selected negative literal. Evaluation of the subgoal* q(0) *shows that* q(0) *is successful, and a* NEGATIVE RETURN *operation creates a failure node as child 10. The evaluation proceeds until finally the conditional answer, node 6 is resolved against the selected literal of node 14. Because the answer was conditional, the selected literal* p(s(s(X)) *is moved to the $Delays$ after resolution (Definition 10). Because of the abstraction used to produce node 6, the next selected literal* not q(s(s(X))) *is non-ground. Nonetheless, the atom* q(s(s(X))) *becomes failed (Definition 6), once its tree is completed with no answers (step 15a). Because* q(s(s(X))) *is failed, a* NEGATIVE RETURN *operation resolves the selected literal away, leading to the conditional answer node 16.*

Thus SLG_{ABS} has the following extensions over SLG:

1. Abstraction is used both when creating new trees (in the NEW SUBGOAL operation), and when adding an answer (in the POSITIVE RETURN operation);
2. A special atom $undefined_{abs}$ is added to the $Delays$ of each non-trivially abstracted answer A (in the POSITIVE RETURN operation). The truth value of $undefined_{abs}$ is always *undefined*, so it can never be removed from the $Delays$ of A, forcing A to have a truth value of *undefined* as well; and
3. NEGATIVE RETURN is defined so that literal $not\ A$ can be resolved away in a forest $\mathcal{F}$ if A is failed in $\mathcal{F}$, regardless of whether A is ground.

Of course, NEW SUBGOAL and POSITIVE RETURN in SLG_{ABS} can be reduced to the classical definitions of SLG by setting the $abs(\cdot)$ to the identity function.

If a finitary abstraction function is used in SLG_{ABS}, then any forest has a finite number of trees and answers. This fact together with other tabling properties ensures the following.

Theorem 3 *Let Q be a query to a normal program P, and let $abs(\cdot)$ be a finitary abstraction function. Then any SLG_{ABS} evaluation $\mathcal{E}$ of Q against P reaches a final forest $\mathcal{F}_{fin}$ after a finite number of steps.*

Regardless of whether $abs(\cdot)$ is finitary, a SLG_{ABS} evaluation is complete with respect to a model that is restrained by the same abstraction function. If P is unsafe, SLG_{ABS} may derive truth values that are not in $WFM(abs, P)$, but that are in $WFM(P)$. This occurs if SLG_{ABS} derives a non-ground answer A for which $A = abs(A)$, and for some atom A' in the ground instantiation of A, $A' \neq abs(A')$. In this case A' is undefined in $WFM(abs, P)$, although A is true in the interpretation induced by the final forest of the SLG_{ABS} evaluation ($\mathcal{I}_{\mathcal{F}_{fin}}$).

Theorem 4 *Let $\mathcal{E}$ be an SLG_{ABS} evaluation of a query Q to a normal program P using abstraction function $abs(\cdot)$, such that $\mathcal{E}$ has a final forest $\mathcal{F}_{fin}$. Then*

$$WFM(abs, P)|_{\mathcal{F}_{fin}} \subseteq \mathcal{I}_{\mathcal{F}_{fin}} \subseteq WFM(P)|_{\mathcal{F}_{fin}}.$$

Complexity of SLG_{ABS}

The best currently known bound on worst case complexity for computing the well-founded semantics of a program P is $size(P) \times |atoms(P)|$ (van Gelder, Ross, and Schlipf 1991). In order to relate the complexity of SLG_{ABS} to this result, we extend the cost model of (Riguzzi and Swift 2013b).

The first aspect of our cost model, $\mathcal{C}_{SLG_{\text{ABS}}}$, addresses the fact that evaluations may terminate on ground programs that are not finite. Let P be a ground (normal) program, and Q an atomic query to P (not necessarily ground). Then the *atomic search space* of Q, P_Q, consists of the union of all ground instantiations of Q in $\mathcal{B}_P$ together with all atoms reachable in the atom dependency graph of P from any ground instantiation of Q. By Theorem 3 a SLG_{ABS} evaluation $\mathcal{E}$ of Q against P that uses a finitary abstraction function will produce a final forest $\mathcal{F}_{fin}$ after a finite number of steps, and $\mathcal{F}_{fin}$ will itself be finite. It is evident that the set of subgoals corresponding to trees in $\mathcal{F}_{fin}$ ($subgoals(\mathcal{F}_{fin})$) is finite. Because $\mathcal{F}_{fin}$ may contain non-ground subgoals, it is not the case that $subgoals(\mathcal{F}_{fin}) \subseteq P_Q$; however if depth-$k$ abstraction is used, it can be shown that $|subgoals(\mathcal{F}_{fin})| \leq 2 \times |atoms(P_Q)|$.

Next, given the finite sequence $\mathcal{E}$, we can construct the set of (ground) rules that were used in some PROGRAM CLAUSE RESOLUTION operation and denote this set as $P_Q(\mathcal{E})$. It is evident that $P_Q(\mathcal{E}) \subseteq P_Q \subseteq P$, and that $P_Q(\mathcal{E})$ must always be finite. Define for a rule r, $size(r)$ as one plus the number of body literals in r. Extending this, $size(P_Q(\mathcal{E}))$ is defined as the sum of sizes of rules in $P_Q(\mathcal{E})$. $\mathcal{C}_{SLG_{\text{ABS}}}$ thus does not consider the size of terms within an atom or literal.

Finally, $\mathcal{C}_{SLG_{\text{ABS}}}$ determines the cost of each SLG_{ABS} operation. Note, since the scope of an abstraction function is an atom, the cost of applying an abstraction function is constant in $\mathcal{C}_{SLG_{\text{ABS}}}$ [2]. Accordingly under $\mathcal{C}_{SLG_{\text{ABS}}}$ the NEW SUBGOAL, PROGRAM CLAUSE RESOLUTION, POSITIVE RETURN, NEGATIVE RETURN, DELAYING, and SIMPLIFICATION operations each affect one goal or delay literal and are considered constant time. The COMPLETION operation, however, applies to a set of subgoals $\mathcal{S}$ in a forest $\mathcal{F}$ and its cost is proportional to the size of $\mathcal{S}$: in the worst case this is $|subgoals(\mathcal{F})|$. Similarly, the ANSWER COMPLETION operation must determine an unsupported set of answers and its worst case is $size(P_Q(\mathcal{E}))$.

The cost model $\mathcal{C}_{SLG_{\text{ABS}}}$ thus consists of

1. The definition of $subgoals(\mathcal{F})$ which is finite, and is $\mathcal{O}(atoms(P))$ if $atoms(P)$ is finite;
2. The definition of $size(P_Q(\mathcal{E}))$ which finite and is $\mathcal{O}(size(P))$ if $size(P)$ is finite; and
3. Costs for each individual SLG_{ABS} operation.

Theorem 5 *Let P be a ground normal program, Q a ground query, and $\mathcal{E}$ a terminating SLG_{ABS} evaluation of Q against P that uses depth-k abstraction, and with final forest $\mathcal{F}_{fin}$. Then under the cost mode $\mathcal{C}_{SLG_{\text{ABS}}}$, the cost of $\mathcal{E}$ is $\mathcal{O}(|subgoals(\mathcal{F}_{fin})| \times size(P_Q(\mathcal{E})))$.*

Implementation, Performance and Scalability

SLG_{ABS} is implemented using depth-k abstraction in version 3.3.7 (publicly available) of XSB (XSB 2013), based in part on a prior implementation of subgoal abstraction. From the programmer's perspective, depth-k abstraction is not used by default, but can be invoked using different values of k on a predicate basis. Answer abstraction is performed in the tabling engine of XSB, the SLG-WAM, during the check/insert step which checks whether an answer exists in a given table, and inserts the answer into the table if not. A counter maintains the current depth of the answer $Ans \leftarrow Delays|$ being traversed; if the depth of Ans is greater than k then the current subterm is replaced by a free (position) variable. In addition, the atom $undefined_{abs}$, a reserved atom in XSB, is added to $Delays$ if it is not already included, indicating that Ans is *undefined*. The overhead of answer abstraction is thus the cost of maintaining the depth-counter, along with that of copying $undefined_{abs}$ into $Delays$ if the depth bound is exceeded.

If no answer abstraction function is specified (so that answers will not be abstracted) the overhead consists solely of the cost of maintaining the the depth counter within the answer check/insert operation. For various forms of linear recursion, we measured this overhead at $0 - 4\%$ based on the ratio of answers to subgoals in a given benchmark.

A series of independent studies have shown XSB to be highly scalable (OpenRuleBench 2011). In addition, recent work with trace-based analysis in XSB has performed sophisticated analysis on trace logs with 10^7 to 10^8 and more

[2] Of course a practical implementation of an abstraction function should have a low cost as a function of the actual size of an atom to which it is applied.

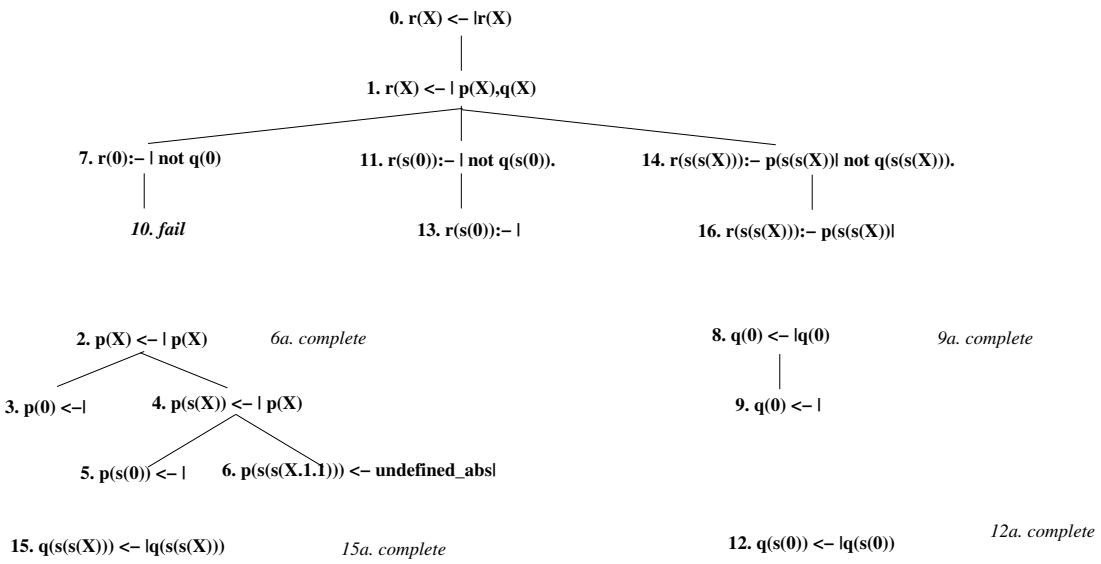

Figure 1: Final forest for the query $r(X)$ to $P_{abs\text{-}unsafe}$.

events, where each event corresponds to a Prolog fact that is dynamically loaded for the analysis. This scalability has not been affected by the extension to SLG_{ABS} in the SLG-WAM.

Discussion and Current Work

This paper has shown how radially restrained well-founded models of a program approximate the well-founded model in a clear manner (Theorem 2). Queries to these models terminate correctly (Theorems 3 and 4) with low abstract complexity (Theorem 5). Tabled resolution for restrained models can be implemented with low overhead on performance, without impacting the scalability of query evaluation.

Current work has been studying how this kind of bounded rationality is best exploited for practical KR. To date, this study has been primarily in the SILK project, which has sought to provide a framework for scalable logic-based KR. SILK's logic, Rulelog, derives its scalability in part from its reliance on the well-founded semantics, which offers a low computational complexity and supports top-down query evaluation. SILK has been used to evaluate sets of high-level rules about cell biology at the first-year college level. These rules are constructed by a team of knowledge engineers who, as an experiment, constructed rules directly from textual knowledge. While the heavy use of tabling makes evaluation of such rule sets possible, they are often not well-behaved, so that knowledge engineers may be faced with "run-away" computations. Radial restraint bounds these computations so that they will terminate. Then the results of queries may be analyzed — via SILK's justification-based debugging and other introspection routines — and used to modify problematic rules.

Besides such application piloting, current work is also been addressing several areas of mechanisms and theory around radial restraint. One area is been formulating results on conditions that ensure computational tractability of LP and Rulelog. A second area is developing methods to coordinate radial restraint with temporal bounds (i.e., time-outs). A third area is developing analysis, such as estimated maximum number of answers, that is specific to various abstraction norms, e.g., term size, term depth, and list vs. non-list functions. A fourth area is developing justification mechanisms to inform users whether an atom was undefined because of restraint, versus because negation lacked stratifiability. A fifth area is developing methods for introspection during long-running queries (within which SILK permits interrupting and resuming computation), so that a knowledge engineer can obtain information about what *subqueries* in a current paused state of a computation have a truth value that is undefined due to radial restraint.

A final area of current work is developing techniques and theory for *non-radial* kinds of restraint. Restraint appears to be potentially a rich realm for work in the field of KR overall.

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Appendix

For the convenience of the reviewers, we include the proofs of the technical results in this appendix.

Proof of Theorem 2

Theorem 2

Let $abs_1(\cdot), abs_2(\cdot)$ be abstraction functions such that $abs_1(\cdot) \leq abs_2(\cdot)$. Then for any program P, $WFM(abs_1, P) \subseteq WFM(abs_2, P)$.

Proof Proof is by double induction, first on the number of iterations of $WFM(abs_1, P)$ and $WFM(abs_2, P)$, and within that by the number of iterations of $True_I^P(abs)$ and $False_I^P$.

- *Outer Induction Base Case:* First, we show $WFM(abs_1, P) \subseteq WFM(abs_2, P)$, by showing

  $$\mathcal{TR}_\emptyset^P(abs_1) \subseteq \mathcal{TR}_\emptyset^P(abs_2)$$

  For the base step, note that since $abs_1(\cdot) \leq abs_2(\cdot)$, then for any term t, $abs_1(t)$ subsumes $abs_2(t)$: thus if $abs_1(t)$ is ground, then $abs_2(t)$ must also be ground, and

  $$True_\emptyset^P(abs_1, \emptyset) \subseteq True_\emptyset^P(abs_2, \emptyset)$$

  For the induction step, assume that $Tr \subseteq Tr'$ to show that

  $$True_\emptyset^P(abs_1, Tr) \subseteq True_\emptyset^P(abs_2, Tr')$$

  Note that for a rule $B \leftarrow L_1, ..., L_n$ in P and any grounding substitution θ

  $$L_1, ..., L_n\theta \in Tr \Rightarrow L_1, ..., L_n\theta \in Tr'$$

  Furthermore if $abs(B)\theta$ is ground, then $abs(B')\theta$ must also be ground. Finally, by definition $\mathcal{FA}_\emptyset^P$ is not affected by the use of an abstraction function, and the Outer Induction Base Case holds.

- *Outer Induction Successor Case:* To distinguish between the various sets, we sometimes denote as $WFM_\delta(abs, P)$ the set WFM_δ produced using abstraction function $abs(\cdot)$ on program P. Assume that $WFM_\delta(abs_1, P) \subseteq WFM_\delta(abs_2, P)$ to show $WFM_{\delta+1}(abs_1, P) \subseteq WFM_{\delta+1}(abs_2, P)$. First, for any abstraction function $a(\cdot)$ if $I \subseteq I'$

  $$True_I^P(a, Tr) \subseteq True_{I'}^P(a, Tr).$$

  In other words, $True_I^P(a, Tr)$ is monotonic on its "pre-interpretation" I as well as on its parameter Tr. Next,

  $$True_I^P(abs_1, Tr) \subseteq True_I^P(abs_2, Tr)$$

  since every grounding substitution of $abs_1(\cdot)$ is a grounding substitution of $abs_2(\cdot)$. Thus,

  $$\mathcal{TR}_{WFM_{\delta+1}}^P(abs_1) \subseteq \mathcal{TR}_{WFM_{\delta+1}}^P(abs_2).$$

  To show that

  $$\mathcal{FA}_{WFM_{\delta+1}(abs_1,P)}^P \subseteq \mathcal{FA}_{WFM_{\delta+1}(abs_2,P)}^P$$

  For the sub-induction, the base case:

  $$False_{WFM_\delta(abs_1,P)}(\emptyset) \subseteq False_{WFM_\delta(abs_2,P)}(\emptyset)$$

  is immediate, as $False_I^P(abs, Fa)$ is monotonic on its subscripted "pre-interpretation', as well the abstraction function in its first argument. For the inductive step, note that for $Fa \subseteq Fa'$,

  $$False_{WFM_\delta}^P(abs_1, Fa) \subseteq False_{WFM_\delta}^P(abs_2, Fa')$$

  as $False_I^P(Fa)$ is monotonic on both of its arguments, as well as on its subscripted "pre-interpretation'. Thus

  $$\mathcal{FA}_{WFM}^P(abs_1) \subseteq \mathcal{FA}_{WFM}^P(abs_2)$$

  and the statement holds for the successor case.

- *Outer Induction Limit Case* For non-finitary abstractions, the limit case must also be considered. The statement $WFM_\lambda(abs_1, P) \subseteq WFM_\lambda(abs_2, P)$ follows for limit ordinal λ immediately from the successor case.

SLG$_{\mathsf{ABS}}$Evaluation

SLG resembles other Prolog-like tabling formalisms in the case of programs that do not use default negation However, for negation it introduces the concept of delaying literals in order to be able to find witnesses of failure anywhere in a rule, along with the concept of simplifying these delayed literals whenever their truth value becomes known.

An SLG evaluation proceeds by constructing a forest according to the set of SLG operations. Such a forest, and the trees and nodes it contains are defined in the following defintion, which formally defines the terminology used in the body of the paper.

Definition 9 *A* node *has the form*

$$AnswerTemplate \leftarrow Delays|Goals \qquad or \qquad \text{fail}.$$

In the first form, AnswerTemplate *is an atom, while* Delays *and* Goals *are sequences of literals. The second form is called a* failure node*. An SLG* tree T *has a root of the form $S \leftarrow |S$ for some atom S: we call S* the root node for T *and T* the tree for S*. An* SLG forest $\mathcal{F}$ *is a set of SLG trees. A node N is an* answer *when it is a leaf node for which* Goals *is empty. If the* Delays *of an answer is empty, it is termed an* unconditional answer*, otherwise, it is a* conditional answer*. A program tree T may be marked with the symbol* complete*.*

The underlying subgoal *of a literal L is L if L is a positive literal; otherwise it is S if $L = not\ S$.*

An SLG evaluation $\mathcal{E}$ of an atomic query Q to a program P is a sequence of forests. $\mathcal{E}$ starts with an initial forest containing the single node $Q \leftarrow |Q$ and creates the n^{th} forest in the sequence by applying an SLG operation if n is a successor ordinal, or by taking the union of forests in previous sequences if n is a limit ordinal. If no further operation is applicable, then the *final forest* for the evaluation of the query has been reached. We introduce SLG operations incrementally, in Definitions 11, 12, and 15. Before we present the first set of operations, we present the definition of answer resolution, which differs in SLG from resolution in Horn rules in order to take account of delay literals in conditional answers.

Definition 10 *Let N be a node $A \leftarrow D|L_1, ..., L_n$, where $n > 0$. Let $Ans = A' \leftarrow D'|$ be an answer whose variables are disjoint from N. N is* SLG resolvable *with Ans if $\exists i$, $1 \leq i \leq n$, such that L_i and A' are unifiable with a most general unifier θ. The* SLG resolvent *of N and Ans on L_i has the form:*

$$(A \leftarrow D|L_1, ..., L_{i-1}, L_{i+1}, ..., L_n)\theta$$

if D' is empty; otherwise the resolvent has the form:

$$(A \leftarrow D, L_i|L_1, ..., L_{i-1}, L_{i+1}, ..., L_n)\theta.$$

This form of resolution delays L_i rather than propagating the answer's delay list D', which means that L_i in the delay list is only simplified once D' in the conditional answer has been simplified. Preventing the propagation of the underlying delay literals in D' is necessary, as shown in (Chen and Warren 1996), to ensure polynomial data complexity.[3]

The previous SLG definitions are not changed in SLG_{ABS}. However in the the next set of operations, both NEW SUBGOAL is changed (as in (**?**)) and POSITIVE RETURN is changed.

Definition 11 (SLG_{ABS} Operations: 1) *Let P be a program and assume that a fixed selection function is used to select a literal from the $Goals$ in a node. Finally, assume that $abs(\cdot)$ is an abstraction function.*

Given a forest $\mathcal{F}_n$ of an SLG evaluation of P $\mathcal{F}_{n+1}$ may be produced by one of the following operations.

1. NEW SUBGOAL*: Let $\mathcal{F}_n$ contain a tree with non-root node*

   $$N = Ans \leftarrow Delays|G, Goals$$

   where S is the underlying subgoal of G. Assume $\mathcal{F}_n$ contains no tree with root $abs(S)$. Then add the tree $abs(S) \leftarrow |abs(S)$ to $\mathcal{F}_n$.
2. PROGRAM CLAUSE RESOLUTION*: Let $\mathcal{F}_n$ contain a tree with root node $N = S \leftarrow |S$ and P contain a rule $A \leftarrow Body$ such that A unifies with S with mgu θ.*
   - *If $Body$ is not empty, then $N_{child} = (S \leftarrow |Body)\theta$;*
   - *Otherwise, if $Body$ is empty and $abs(S\theta) = S\theta$, then $N_{child} = (S\theta \leftarrow |)$;*
   - *Otherwise, if $Body$ is empty and $abs(S\theta) \neq S\theta$, then $N_{child} = abs(S\theta \leftarrow undefined_{abs}|)$.*

   If N does not have a child N_{child} then add N_{child} as a child of N.
3. POSITIVE RETURN*: Let a forest $\mathcal{F}_n$ contain a tree with non-root node N whose selected literal S is positive. Let Ans be an answer for S in $\mathcal{F}_n$ and $N' = A \leftarrow Delays|Goals$ be the SLG resolvent of N and Ans on S.*
   - *If N' has a non-empty $Goals$ sequence, $N_{child} = N'$.*
   - *Otherwise, if $abs(N') = N'$, then $N_{child} = N'$*
   - *Otherwise, if $abs(N') \neq N'$ then*

     $$N_{child} = abs(A \leftarrow Delays, undefined_{abs}|).$$

[3] If delay lists were propagated directly, then delay lists could effectively contain all derivations which could be exponentially many in the worst case.

In SLG, the NEGATIVE RETURN operation is restricted to apply to ground negative literals. However, this is unnecessary. Recall from Definition **??**hat an atom S is *successful* (resp. *failed*) in $\mathcal{F}$ if S' is in $true(\mathcal{I}_{\mathcal{F}})$ ($false(\mathcal{I}_{\mathcal{F}})$) for every S' in the ground instantiation of S. SLG_{ABS} removes this restriction from the NEGATIVE RETURN operation.

5$not\ A$. If an atom A is failed, then we can simplify away $not\ A$.

Definition 12 (SLG_{ABS} Operations: 2) *Let P be program and assume selection and abstraction functions as in Definition 11.*

Given a forest $\mathcal{F}_n$ of an SLG evaluation of P, $\mathcal{F}_{n+1}$ may be further produced by one of the following operations.

4. NEGATIVE RETURN*: Let $\mathcal{F}_n$ contain a tree with a leaf node, whose selected literal is $not\ S$*

   $$N = Ans \leftarrow D|not\ S, Goals.$$

   (a) NEGATION SUCCESS*: If S is failed in $\mathcal{F}_n$ then create a child N_{child} of N where*
   - *If $Goals$ is non-empty, $N_{child} = Ans \leftarrow D|Goals$;*
   - *Otherwise, if $Goals$ is empty and $Ans \leftarrow D = abs(Ans \leftarrow D)$,*

     $$N_{child} = Ans \leftarrow D|;$$

   - *Otherwise, if $Goals$ is empty and $Ans \leftarrow D \neq abs(Ans \leftarrow D)$,*

     $$N_{child} = Ans \leftarrow D, undefined_{abs}|.$$

   (b) NEGATION FAILURE*: If S succeeds in $\mathcal{F}_n$, then create a child for N of the form* fail.
5. DELAYING*: Let $\mathcal{F}_n$ contain a tree with leaf node*

   $$N = Ans \leftarrow Delays|not\ S, Goals$$

   S is neither successful nor failed in $\mathcal{F}_n$. Then create a child for N of the form $Ans \leftarrow Delays, not\ S|Goals$.
6. SIMPLIFICATION*: Let $\mathcal{F}_n$ contain a tree with leaf node*

   $$N = Ans \leftarrow Delays|$$

   and let $L \in Delays$

   (a) *If L is failed in $\mathcal{F}$ then create a child* fail *for N.*

   (b) *If L is successful in $\mathcal{F}$, then create a child $Ans \leftarrow Delays'|$ for N, where $Delays' = Delays - L$.*

The remaining operations and definitions are unchanged between SLG and SLG_{ABS}. SLG also includes an operation that marks a set of trees as complete if the corresponding set of subgoals is completely evaluated.

Definition 13 *A set $\mathcal{S}$ of subgoals in a forest $\mathcal{F}$ is* completely evaluated *if at least one of the conditions holds for each $S \in \mathcal{S}$*

1. *The tree for S contains an answer $S \leftarrow |$; or*
2. *For each node N in the tree for S:*

   (a) *The underlying subgoal of the selected literal of N is marked as complete; or*

(b) The underlying subgoal of the selected literal of N is in $\mathcal{S}$ and there are no applicable NEW SUBGOAL, PROGRAM CLAUSE RESOLUTION, POSITIVE RETURN *(Definition 11),* NEGATIVE RETURN*or* DELAYING *(Definition 12) operations for N.*

Once a set of subgoals is determined to be completely evaluated, a COMPLETION operation marks the trees for each subgoal (Definition 9). If a subgoal S is completed due to condition 1 holding, we say that S is *early completed*. If condition 1 does not hold, condition 2b of the above definition prevents the COMPLETION operation from being applied to a tree from a set if certain other operations are applicable to the trees in the set. This notion of completion is incremental in the sense that once a set $\mathcal{S}$ of mutually dependent subgoals is fully evaluated, the derivation need not be concerned with the trees for $\mathcal{S}$ apart from any answers they contain. In an actual implementation most resources for such trees can be reclaimed.

In certain cases the propagation of conditional answers through SLG resolution (Definition 10) can lead to a set of *unsupported answers* — conditional answers that are false in the well founded model (see, e.g., Example 1 of (**?**)).[4] Intuitively, these answers, which have positive mutual dependencies, correspond to an unfounded set, but their technical definition is based on the form of conditional answers.

Definition 14 *Let $\mathcal{F}$ be an SLG forest, and* Answer *be an atom that occurs in the head of some answer in a tree with root S. Then $Answer$ is supported in $\mathcal{F}$ if and only if:*

1. *S is not completely evaluated; or*
2. *there exists an answer node $Answer' \leftarrow Delays|$ in S such that $Answer'$ subsumes* Answer *and for every positive literal $L \in Delays$, L is supported in $\mathcal{F}$.*

We are now able to characterize the last two SLG operations: one allows the completion of trees, and the other removes unsupported answers.

Definition 15 (SLG Operations: 3) *Let P be a program. Given a forest $\mathcal{F}_n$ of an SLG evaluation of P, $\mathcal{F}_{n+1}$ may also be produced by one of the following operations.*

8. COMPLETION*: Given a completely evaluated set $\mathcal{S}$ of subgoals (Definition 13), mark the trees for all subgoals in $\mathcal{S}$ as complete.*
9. ANSWER COMPLETION*: Given a set of unsupported answers $\mathcal{UA}$, create a failure node as a child for each answer $Ans \in \mathcal{UA}$.*

Each of the operations (1)–(9), in Definitions 11, 12 and 15, can be seen as a function that associates a forest with a new forest by adding a new tree, adding a new node (possibly a failure node) to an existing tree, or marking a set of trees as complete.

[4] As an aside, we note that unsupported answers appear to be uncommon in practical evaluations which minimize the use of delay as does XSB (Swift and Warren 2012).

Proof of Theorem 3 (SLG$_{\mathsf{ABS}}$ Termination)

Theorem 3
Let Q be a query to a program P, and let $abs(\cdot)$ be a finitary abstraction function. Then any SLG$_{\mathsf{ABS}}$ evaluation $\mathcal{E}$ of Q reaches a final forest $\mathcal{F}_{fin}$ after a finite number of steps.

Proof The proof is by induction on the maximal dynamic stratum of any answer in $\mathcal{E}$.

- For the base case, assume the maximal stratum is 1 (Definition 2). Because Q is in stratum 1, the only applicable SLG operations in $\mathcal{E}$ are NEW SUBGOAL, PROGRAM CLAUSE RESOLUTION, POSITIVE RETURN, and COMPLETION. Note that each of these operations is applicable only once to a given node or set of subgoals.
  - NEW SUBGOAL: The use of a finitary abstraction operation means that there may only be a finite number of NEW SUBGOAL operations in $\mathcal{E}$, and hence a finite number of trees in any forest of $\mathcal{E}$.
  - PROGRAM CLAUSE RESOLUTION: Since there are a finite number of trees, and a finite number of program clauses resolvable against the root subgoal of any tree, $\mathcal{E}$ contains only a finite number of PROGRAM CLAUSE RESOLUTION operations, and the root of any tree has only a finite number of immediate children.
  - POSITIVE RETURN: By Proposition 1 since $abs(\cdot)$ is finitary, $true(WFM^P)$ is finite. Also, since the maximal stratum of $\mathcal{E}$ is 1, any answer returned will be unconditional. Accordingly there are a finite number of answers that can be resolved against any selected subgoal. Because interior nodes SLG trees in stratum 1 can only be extended by POSITIVE RETURN operations, any non-root node in any tree may have only a finite number of children. In addition, the depth of any tree in $\mathcal{E}$ is bounded by the maximal number of body literals in any rule in P, which is finite. Thus the non-root nodes of any tree in $\mathcal{E}$ have a finite depth and a finite branching factor, and so are finite. Because all trees are finite and since each POSITIVE RETURN operation creates a new node, there can thus be only a finite number of POSITIVE RETURN operations.
  - COMPLETION: Finally since there are a finite number of trees in any forest, there can be only a finite number of COMPLETION operations.

  Since the number of occurrences of each type of operation in $\mathcal{E}$ is finite, $\mathcal{E}$ itself must be finite.
- For the inductive case, assume that the statement is true for all atoms whose (finite) stratum is less than n in order to prove it true for those atoms whose stratum is n. By Theorem 1 the maximum stratum of the restrained well-founded model of P is finite, so that the inductive case need not consider limit ordinals.
  - POSITIVE RETURN: Because the stratum under consideration is greater than 1, there is a possibility that an answer may be unconditional or conditional. The case for unconditional answers is argued as in the base case above. However, it must be shown that there are a finite number of conditional answers in $\mathcal{F}_{fin}$. Consider first that because the maximal number of literals,

$maxLit$ in any rule of P is finite, each node produced by PROGRAM CLAUSE RESOLUTION will have a finite number of literals in $Goals$, and an empty $Delays$. The only operations that add literals to $Delays$ are DELAYING and POSITIVE RETURN (through SLG resolution). However, Each of these operations must remove a corresponding literal from $Goals$ so that there are at most $maxLit$ literals in the $Delays$ of any conditional answer. When a conditional answer Ans is produced by POSITIVE RETURN, each literal in its $Delays$ is also abstracted. There are at most maxlit literals in the $Delays$ of Ans, and since $abs(\cdot)$ is finitary, each such literal is chosen from a finite set. Accordingly there are only a finite number of conditional answers in $\mathcal{F}_{fin}$.

Having shown that there are at most a finite number of answers (conditional or unconditional) each node that has a positive selected literal may have at most a finite number of children in $\mathcal{F}_{fin}$ the argument that there are a finite number of POSITIVE RETURN operations in $\mathcal{E}$ is essentially the same as in the base case.

– NEGATIVE RETURN: First, consider that a NEGATIVE RETURN operation can be applied at most once to any node N. As a result of this operation, any node N to which a NEGATIVE RETURN operation is applied can have only a single child: either a failure node in the case of NEGATION FAILURE, or a single child with the selected literal removed from the $Goals$ of N in the case of NEGATION SUCCESS. In the case of NEGATION FAILURE this is enough to show that the finiteness of $\mathcal{E}$ is not affected, as a failure node cannot be further expanded. In the case of NEGATION SUCCESS, the fact that the selected literal is removed from $Goals$, means that the child of N will have a smaller $Goals$ sequence. Since $Goals$ is finite, any path from N may may have been produced by only a finite number of NEGATIVE RETURN operations. Since there are only a finite number of such paths, $\mathcal{E}$ may have only a finite number of NEGATIVE RETURN operations.

– DELAYING: Analogous considerations show that DELAYING will not affect the finiteness of $\mathcal{E}$.

– ANSWER COMPLETION: Next, note that ANSWER COMPLETION will produce a single failure node as a child of each answer node to which it is applied, and a failure node cannot be further expanded. So ANSWER COMPLETION does not affect the finiteness of any forest, no matter how many times it is applied.

– In the case of SIMPLIFICATION, an application of the SIMPLIFICATION opertion that produces a failure node does not affect the finiteness of any forest (Definition 12, 6a) as a failure node cannot be further expanded.. On the other hand, if an application of a SIMPLIFICATION operation to a node N produces a non-failure child (Definition 12, 6b), note that similar to the case of NEGATION SUCCESS, the child of N will have a smaller $Delays$. Since $Delays$ is finite, any path from N to its descendents may have only a finite number of SIMPLIFICATION operations.

Since each operation can be applied only a finite number of times, $\mathcal{E}$ must be finite.

Example 4 *p(f(X)):- p(X). p(1). p(f(f(f(f(2))))).*

Proof of Theorem 4 (SLG$_{\mathsf{ABS}}$ Correctness)

In the proof below, in order to appeal to previous correctness proofs, we distinguish between

- SLG (Chen and Warren 1996; Swift 1999), which uses no abstraction functions and in which the NEGATIVE RETURN and DELAYING operations (Definition 12) require a selected negative literal to be ground;
- SLG$_{\mathsf{SA}}$ (**?**) which uses an abstraction function only in the NEW SUBGOAL operation (Definition 11); and in which the NEGATIVE RETURN and DELAYING operations require a selected negative literal to be ground;
- SLG$_{\mathsf{ABS}}$ introduced in this paper, which uses the same NEW SUBGOAL operation as SLG$_{\mathsf{SA}}$, but uses an abstraction function in the POSITIVE RETURN operation (Definition 11), and generalizes the NEGATIVE RETURN and DELAYING operations so that the they apply to any negative literal, ground or not

Theorem 4
Let $\mathcal{E}$ be an SLG$_{\mathsf{ABS}}$ evaluation of a query Q to a ground program P using abstraction function $abs(\cdot)$, such that $\mathcal{E}$ has a final forest $\mathcal{F}_{fin}$. Then

$$WFM(abs, P)|_{\mathcal{F}_{fin}} \subseteq \mathcal{I}_{\mathcal{F}_{fin}} \subseteq WFM(P)|_{\mathcal{F}_{fin}}.$$

Proof (**?**) provided a correctness theorem for SLG$_{\mathsf{SA}}$ which stated that if P was safe, then $\mathcal{I}_{\mathcal{F}_{fin}} = WFM(P)|_{\mathcal{F}_{fin}}$ for any abstraction function used in the NEW SUBGOAL operation. Clearly this theorem holds for SLG$_{\mathsf{ABS}}$ using the identity function as an abstraction function. Accordingly here we focus on the use of general abstraction functions within the POSITIVE RETURN operation, as well as the removal of the ground subgoal restriction for DELAYING and NEGATIVE RETURN. With this in mind, we first show soundness and then completeness.

Soundness is shown by a double induction, first on the maximal stratum of any answer pr subgoal in $\mathcal{E}$, and within each outer induction step, by induction on the sequence $\mathcal{F}_0, \mathcal{F}_1, \ldots$ of forests in $\mathcal{E}$, and for each such forest $\mathcal{F}_i$ we show that $\mathcal{I}_{\mathcal{F}_i} \subseteq WFM(P)|_{\mathcal{F}_i}$.

1. *Outer Induction Base Case* For the base case, assume the maximal stratum of any answer or subgoal in $\mathcal{E}$ is 1.
   (a) *Inner Induction Base Case* For the base case, $\mathcal{F}_0$ is the initial forest to which only a PROGRAM CLAUSE RESOLUTION operation may be applicable. The PROGRAM CLAUSE RESOLUTION operation of SLG$_{\mathsf{ABS}}$ differs from that of SLG$_{\mathsf{SA}}$ only if it creates an answer $abs(\mathcal{A} \leftarrow)$ where $abs(\mathcal{A} \leftarrow) \neq \mathcal{A} \leftarrow$. In this case, S would be neither true nor false in $I_{\mathcal{F}_{fin}}$, and so is sound However, the case must be considered where the PROGRAM CLAUSE RESOLUTION operation produces an answer $(A \leftarrow)$ which equals $abs(A \leftarrow)$, but A is non-ground, and some atom A' in the instantiation of A is

such that $A' \neq abs(A')$. In this case, A' will not be in $WFM(abs, P)$, but it will true in $cI_{\mathcal{F}_1}$. However, but the correctness of SLG$_{\mathsf{SA}}$, A' will still be true in $WFM(P)$.

(b) *Inner Induction Successor Case* For the induction case, assume that $\mathcal{I}_{\mathcal{F}_i} \subseteq WFM(P)|_{\mathcal{F}_i}$ to show that $\mathcal{I}_{\mathcal{F}_{i+1}} \subseteq WFM(P)|_{\mathcal{F}_{i+1}}$. Given that the maximal stratum of any subgoal or answer in $\mathcal{E}$ is 1, the only SLG$_{\mathsf{ABS}}$ operations to consider are NEW SUBGOAL PROGRAM CLAUSE RESOLUTION, POSITIVE RETURN and COMPLETION.

i. NEW SUBGOAL The NEW SUBGOAL operation is defined in the same way SLG$_{\mathsf{ABS}}$ as in SLG$_{\mathsf{SA}}$ and its soundness follows from the proof of correctness of SLG$_{\mathsf{SA}}$.

ii. PROGRAM CLAUSE RESOLUTION The correctness of the PROGRAM CLAUSE RESOLUTION operation is the same as for the Inner Induction Base Case.

iii. POSITIVE RETURN The POSITIVE RETURN operation of SLG$_{\mathsf{ABS}}$ differs from that of SLG$_{\mathsf{SA}}$ only if it creates an answer $abs(\mathcal{A} \leftarrow)$ where $abs(\mathcal{A} \leftarrow) \neq \mathcal{A} \leftarrow$. In this case, S would be neither true nor false in $I_{\mathcal{F}_{fin}}$ and so is sound.

iv. COMPLETION The COMPLETION operation is defined in the same way SLG$_{\mathsf{ABS}}$ as in SLG$_{\mathsf{SA}}$ (and SLG) and its soundness follows from of correctness of SLG$_{\mathsf{SA}}$.

(c) *Inner Induction Limit Case* Note that since $abs(\cdot)$ is finitary, then by Theorem 3 $\mathcal{E}$ will be a finite sequence of forests, so that the limit case does not need to be considered.

2. *Outer Induction Successor Case* Assume that the statement holds for all evaluations with stratum n or less to show that the statement holds for evaluations with stratum n.

(a) *Inner Induction Base Case* For the base case, $\mathcal{F}_0$ is the initial forest to which only a PROGRAM CLAUSE RESOLUTION operation may be applicable. Soundness follows exactly as with case 1(a).

(b) *Inner Induction Successor Case* For the induction case, assume that $\mathcal{I}_{\mathcal{F}_i} \subseteq WFM(P)|_{\mathcal{F}_i}$ to show that $\mathcal{I}_{\mathcal{F}_{i+1}} \subseteq WFM(P)|_{\mathcal{F}_{i+1}}$. We consider all possible cases of SLG$_{\mathsf{ABS}}$ operations that may produce $\mathcal{F}_{i+1}$.

i. NEW SUBGOAL, PROGRAM CLAUSE RESOLUTION, POSITIVE RETURN, COMPLETION. These cases are argued as in case 1(b).

ii. NEGATIVE RETURN The NEGATIVE RETURN operation of SLG$_{\mathsf{ABS}}$ differs from that of SLG$_{\mathsf{SA}}$ in one of two ways.

A. First, if NEGATIVE RETURN is applied to a node

$$N = Ans \leftarrow D|not\, S, Goals.$$

such that S is non-ground, S must be successful or failed in $\mathcal{I}_{\mathcal{F}_i}$. By Definition 6 if S is successful, every literal in the ground instantiation of S must be true, i.e. there must be an answer $S \leftarrow$ in $\mathcal{F}_i$, so that by the induction hypothesis, every literal in the ground instantiation of S is true in $WFM(P)$. Under such conditions application of the NEGATION FAILURE case is sound. By Definition 6 if S is successful, every literal in the ground instantiation of S must be false, i.e. there must be no answers to the completed subgoal S in $\mathcal{F}_i$. Under such conditions the NEGATION SUCCESS case is sound.

B. In addition, the NEGATION SUCCESS case differs from that of SLG$_{\mathsf{SA}}$ only if it creates an answer $abs(\mathcal{A} \leftarrow)$ where $abs(\mathcal{A} \leftarrow) \neq \mathcal{A} \leftarrow$. In this case, S would be neither true nor false in $I_{\mathcal{F}_{fin}}$ and so is sound.

iii. DELAYING For a given node, the DELAYING operation simply moves the selected literal of $Goals$ to $Delays$, and so will not affect the soundness of SLG$_{\mathsf{ABS}}$.

iv. SIMPLIFICATION The SIMPLIFICATION operation is the same as in SLG$_{\mathsf{SA}}$, however in SLG$_{\mathsf{ABS}}$ it can be the case that a delayed literal is non-ground, which could not occur in SLG$_{\mathsf{SA}}$. In this case, the soundness of the SIMPLIFICATION operation is analogous to cases of NEGATION SUCCESS and NEGATION FAILURE in 2(b).ii.A above.

v. ANSWER COMPLETION ANSWER COMPLETION the same as in SLG$_{\mathsf{SA}}$ (and SLG) and its soundness follows from of correctness of SLG$_{\mathsf{SA}}$.

(c) *Inner Induction Limit Case* As with 1(c), since $abs(\cdot)$ is finitary, then by Theorem 3 $\mathcal{E}$ will be a finite sequence of forests, so that the limit case does not need to be considered.

Completeness The statement

$$WFM(abs, P)|_{\mathcal{F}_{fin}} \subseteq \mathcal{I}_{\mathcal{F}_{fin}}$$

is shown by a double induction, first on the stratum of a given atom in $WFM(abs, P)$, and within each outer induction step, by induction on the sequence of applications of $True_I^P(abs)$, along with a separate argument for $False_I^P(abs)$

1. *Outer Induction Base Case* For the base case, assume the stratum of an answer in $WFM(abs, P)$ is 1.

(a) $False_\emptyset^P(abs, \mathcal{B}_P)$ consists of those atoms A for every clause $B \leftarrow L_1, ..., L_n$ in P and grounding substitution θ such that $A = B\theta = abs(B\theta)$ and there is some i $(1 \leq i \leq n)$ such that $L_i\theta \in Fa\}$. For an A that is in the ground instantiation of $subgoals(\mathcal{F}_{fin})$ and for which $A = B\theta = abs(B\theta)$, the statement holds by the correctness of SLG. If $A = B\theta \neq abs(B\theta)$, then A cannot be false in $WFM(abs, P)$.

(b) We show the statement for $True_\emptyset^P(abs)$ by induction on the applications needed to produce the fixed point.

i. *Inner Induction Base Case* $True_\emptyset^P(abs, \emptyset)$ consists of those atoms such that there is a clause $B \leftarrow$ in P and a grounding substitution θ such that $A = B\theta = abs(B\theta)$. Suppose that such an A is in the ground instantiation of some subgoal in $\mathcal{F}_{fin}$. In such a case, a PROGRAM CLAUSE RESOLUTION operation would be applicable to the tree with rood $S \leftarrow |S$ and based

on some program clause in P would derive $abs(S\theta)$. (If $abs(A\theta) = A\theta$ and $A\theta$ is non-ground, then there may be some atoms in the ground instantiation of $abs(A)\theta$ that are in $\mathcal{I}_{\mathcal{F}_{fin}}$ but not in $WFM(abs, P)$; however by soundness, such atoms are in $WFM(P)$.)

ii. *Inner Induction Successor Case* Assume the statement is true up to the i^{th} iteration of $True^P_\emptyset(abs)$ (denoted Tr) to show that it is true for the $i + 1^{st}$. This new iteration consists of atoms such that there is a clause $B \leftarrow L_1, ..., L_n$ in P, a grounding substitution θ such that $A = B\theta = abs(B\theta)$, and for every $1 \leq i \leq n$, $L_i\theta \in Tr$. Thus, assume that A is in the ground instantiation of some subgoal in $\mathcal{F}_{fin}$. In such a case, by the induction hypothesis and the correctness of SLG_{SA}, each L_i will be true in some $\mathcal{F}_j$ in $\mathcal{E}$, and by a PROGRAM CLAUSE RESOLUTION operation plus a sequence of POSITIVE RETURN operations, an answer will be derived of which A is in the ground instantiation.

iii. *Inner Induction Limit Case* This case does not need to be considered, as $True^P_\emptyset$ will reach fixed point after a finite number of iterations.

2. *Outer Induction Successor Case* Assume the statement holds for WFM_i (i.e., $WFM(abs, P)_i$ to show that the statement also holds at WFM_{i+1}.

(a) $False^P_{WFM_i}(\mathcal{B}_P)$. $False^P_{WFM_i}(\mathcal{B}_P)$ consists of those atoms A such that for every clause $B \leftarrow L_1, ..., L_n$ in P and grounding substitution θ such that $A = B\theta$ there is some j $(1 \leq j \leq n)$ such that $L_j\theta$ is false in WFM_i or $L_i\theta \in \mathcal{B}_P$.

i. First we consider only the condition where $L_j\theta$ is false in WFM_i. Accordingly, consider an atom A that is in the ground instantiation of some subgoal in $\mathcal{F}_{fin}$ and a given rule r for A with a leftmost false literal $L_j\theta$ as above. Since $L_j\theta$ is false in WFM_i, then by the induction hypothesis it will be failed in $\mathcal{F}_{fin}$. Since L_j is the leftmost literal in r, it will be the selected literal of some node N in $\mathcal{F}_{fin}$, where N is a descendent of a node N_{anc} created by a PROGRAM CLAUSE RESOLUTION operation that used r.

A. L_j is positive. In this case, one of two things will happen. (1) Since L_j is failed in $\mathcal{F}_{fin}$ it may be that no answers for L_j are resolved against the selected literal of N. In such a case, the derivation path from N_{anc} through N will produce no answers. (2) Within an POSITIVE RETURN operation, SLG resolution (Definition 10) will resolve a conditional answer against N. Since $L_j\theta$ is failed in $\mathcal{F}_{fin}$, either a SIMPLIFICATION or an ANSWER COMPLETION operation will create a failure node as a descendent of N by the correctness of SLG.

B. L_j is negative. In this case, either a NEGATION FAILURE operation will create a failure node as a child of N, or L_j will be delayed, and later simplified away, since L_j is failed in $\mathcal{F}_{fin}$.

ii. The previous subcase, 2(a).i argued that when L_j is the witness of failure for a rule r for an atom A, **and** if L_j is false in WFM_i, then SLG_{ABS} will ensure that there is no answer for A using q rule with literal L_j. However, there is still the case of a rule r in which no literal is false in WFM_i, but for which the witness of failure is simply in the greatest fixed point of $False^P_{WFM_i}$. For this to be the case, A must be an unfounded set in the interpretation WFM_i. If $abs(A) = A$, then the statement holds by the correctness of SLG, while if $abs(A) \neq A$, then A is undefined in $WFM(abs, P)$.

(b) We show the statement for $True^P_{WFM_i}(abs)$ by induction on the applications needed to produce the fixed point.

i. *Inner Induction Base Case* For the base case, $True^P_{WFM_i}(abs, \emptyset)$ consists of those atoms such that there is a clause $B \leftarrow L_1, ..., L_n$ in P, a grounding substitution θ such that $A = B\theta = abs(B\theta)$, and for every $1 \leq i \leq n$ $L_i\theta$ is true in WFM_i. Suppose that such an A is in the ground instantiation of some subgoal in $\mathcal{F}_{fin}$. By the outer induction hypothesis $L_1, ..., L_n$ are true in $\mathcal{F}_{fin}$, and by the correctness of SLG, an answer $abs(B\eta)$ would be derived such that $abs(B\eta)$ subsumes A. Furthermore, since $abs(B\theta)$ equals $B\theta$ (because $B\theta \in True^P_{WFM_i}(abs, \emptyset)$), then $abs(B\eta) = B\eta$ (as $B\eta$ subsumes $B\theta$) so that $undefined_{abs}$ is not added to the $Delays$ of the answer. Thus $B\theta(= A)$ is in $\mathcal{I}_{\mathcal{F}_{fin}}$).

ii. *Inner Induction Successor Case* For the inner induction case we also assume that Tr is the result of the i^{th} iterations of $True^P_{WFM_i}$ and that $Tr|_{\mathcal{F}_{fin}} \subseteq \mathcal{I}_{\mathcal{F}_{fin}}$. The arguments for the successor case are essentially those of the base case 2(b).i

iii. *Inner Induction Limit Case* This case does not need to be considered, as $True^P_{WFM_i}$ will reach fixed point after a finite number of iterations.

3. *Outer Induction Limit Case* By Theorem 1, this case does not need to be considered.

Proof of SLG_{ABS} Complexity

The proof of complexity of SLG_{ABS} is a fairly direct extension of that of SLG_{SA}.

Theorem 6 ((?)) Let P be a ground strongly bounded-term-size program, Q a ground query, and $\mathcal{E}$ a terminating SLG_{SA} evaluation of Q against P that uses depth-k abstraction. Then under the cost mode $\mathcal{C}$, the cost of $\mathcal{E}$ is $\mathcal{O}(|subgoals(\mathcal{F}_{fin})| \times size(P_Q(\mathcal{E})))$.

Theorem 6 is thus the basis of the proof of the complexity bound of SLG_{ABS}.

Theorem 5 *Let P be a ground program, Q a ground query, and $\mathcal{E}$ a terminating SLG_{ABS} evaluation of Q against P that uses depth-k abstraction. Then under the cost mode $\mathcal{C}$, the cost of $\mathcal{E}$ is $\mathcal{O}(subgoals(\mathcal{F}_{fin}) \times P_Q(\mathcal{E}))$.*

Proof We consider first the case where P is a strongly bounded-term-size program. Recall that SLG_{ABS} differs from SLG_{SA} in its use of abstraction in the POSITIVE RETURN operation, and in allowing the NEGATIVE RETURN

and the DELAYING operations to apply to selected negative literals in cases where they are non-ground. We consider each case in turn.

- (POSITIVE RETURN) In $\mathcal{C}$, all atoms are assumed to have a maximal size: since P is strongly bounded-term-size this maximal size as defined by a depth function must exist. Thus, in $\mathcal{C}$ the cost of an POSITIVE RETURN operation is taken to be costant, except if answer abstraction is used when the cost is 1 plus the number of delay literals in the answer. To show that answer abstraction does not affect the complexity of $\text{SLG}_{\mathsf{ABS}}$, consider an arbitrary tree T in $\mathcal{F}_{fin}$. Each child N_r of the root of T is created by a ground rule r. We show that each descendent of N_r has at most one child. This follows directly from the definitions of $\text{SLG}_{\mathsf{ABS}}$for all operations except for POSITIVE RETURN. However note that if an POSITIVE RETURN operation is applied to a ground selected atom A of a node N, N may have only one child; also note that if answer abstraction was applied to A (so that A is non-ground), there may still be only one child for N. Thus, answer abstraction may be applied a maximum of once per rule so that the total cost of all POSITIVE RETURN operations in $\mathcal{E}$ is at most $size(r)$, so that answer abstraction does not affect the worst-case complexity of $\text{SLG}_{\mathsf{ABS}}$ evaluation.
- (NEGATIVE RETURN and DELAYING) Note that the change to allow the NEGATIVE RETURN and the DELAYING operations to apply to selected negative literals in cases where they are non-ground will not affect the complexity of the NEGATIVE RETURN and DELAYING operations under the cost model $\mathcal{C}$.

Thus, if P is strongly bounded-term-size, the statement holds.

The restriction to strongly bounded-term-size programs is necessary in Theorem 6 as SLG_{SA} is not guaranteed to terminate if a program is not strongly bounded-term-size. By Theorem 3, $\text{SLG}_{\mathsf{ABS}}$ will always terminate if a finitary abstraction function is used, and depth-k abstraction functions are finitary. There is thus no difference in complexity between a strongly bounded-term-size program for which abstraction is used in $\mathcal{E}$, and an arbitrary normal program for which abstraction is used in $\mathcal{E}$. The statement thus holds for any normal program P.