Lecture 18:
Applications of Dynamic Programming

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Problem of the Day

We wish to compute the laziest way to dial a given n-digit number on a standard push button phone using two fingers. We assume the two fingers start on the * and # keys, and the effort to move a finger from one key to another is proportional to the Euclidean distance between them.

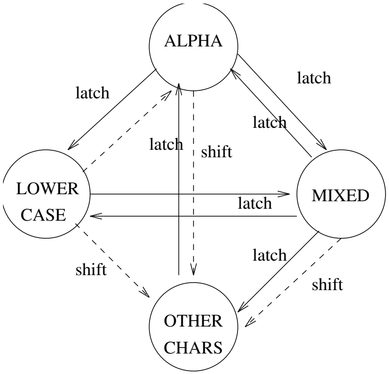
Design an algorithm to that computes the method of dialing that involves moving your figures the smallest amount of total distance, where k is the number of distinct keys on the keypad ($k = 16$ for standard telephones).

Try to use $O(nk^3)$ time.

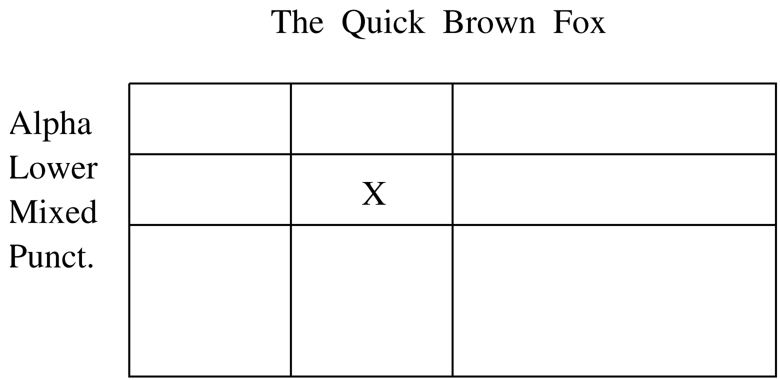
Dynamic Programming and High Density Bar Codes

Symbol Technology has developed a new design for bar codes, PDF-417 that has a capacity of several hundred bytes. What is the best way to encode text for this design?

They developed a complicated mode-switching data compression scheme.



Latch commands permanently put you in a different mode. Shift commands temporarily put you in a different mode. Symbol used a greedy algorithm to encode a string, making local decisions only. But we realized that for any prefix, you want an optimal encoding which might leave you in every possible mode.



$M[i, j] = \min(M[i-1, k]+$ the cost of encoding the ith character and ending up in node j).

Our simple dynamic programming algorithm improved the capacity of PDF-417 by an average of 8%!

Dividing the Work

Suppose the job scanning through a shelf of books is to be split between k workers. To avoid the need to rearrange the books or separate them into piles, we can divide the shelf into k regions and assign each region to one worker.

What is the fairest way to divide the shelf up?

If each book is the same length, partition the books into equal-sized regions,

$$100\ 100\ 100 \mid 100\ 100\ 100 \mid 100\ 100\ 100$$

But what if the books are not the same length? This partition would yield

$$100\ 200\ 300 \mid 400\ 500\ 600 \mid 700\ 800\ 900$$

Which part of the job would you volunteer to do?
How can we find the fairest possible partition, i.e.

$$100\ 200\ 300\ 400\ 500 \mid 600\ 700 \mid 800\ 900$$

The Linear Partition Problem

Input: A given arrangement S of nonnegative numbers $\{s_1, \ldots, s_n\}$ and an integer k.

Problem: Partition S into k ranges, so as to minimize the maximum sum over all the ranges.

- Does a single fixed partition work for all instances of size (n, k)?
- Does taking the average value of each part $(\Sigma_{i=1}^{n} s_i/k)$ from the left always yield the optimal partition?

Recursive Idea

Consider a recursive, exhaustive search approach. Notice that the kth partition starts right after we placed the $(k-1)$st divider.

Where can we place this last divider? Between the ith and $(i+1)$st elements for some i, where $1 \leq i \leq n$.

What is the cost of this? The total cost will be the larger of two quantities, (1) the cost of the last partition $\Sigma_{j=i+1}^{n} s_j$ and (2) the cost of the largest partition cost formed to the left of i.

What is the size of this left partition? To partition the elements $\{s_1, \ldots, s_i\}$ as equally as possible. *But this is a smaller instance of the same problem!*

Dynamic Programming Recurrence

Define $M[n, k]$ to be the minimum possible cost over all partitionings of $\{s_1, \ldots, s_n\}$ into k ranges, where the cost of a partition is the largest sum of elements in one of its parts. Thus defined, this function can be evaluated:

$$M[n, k] = \min_{i=1}^{n} \max(M[i, k-1], \sum_{j=i+1}^{n} s_j)$$

with the natural basis cases of

$$M[1, k] = s_1, \text{ for all } k > 0 \text{ and,}$$

$$M[n, 1] = \sum_{i=1}^{n} s_i$$

Running Time

What is the running time?
It is the number of cells times the running time per cell.
A total of $k \cdot n$ cells exist in the table.
Each cell depends on n others, and can be computed in linear time, for a total of $O(kn^2)$.

When can you use Dynamic Programming?

Dynamic programming computes recurrences efficiently by storing partial results. Thus dynamic programming is efficient when there are few partial results to compute!

- There are $n!$ permutations of an n-element set, so we cannot hope to to store the best solution for each subpermutation *polynomially*.
- There are 2^n subsets of an n-element set, so we cannot hope to store the best solution for each *polynomially*.
- But there are *only* $n(n-1)/2$ continguous substrings of a string, so we can use it for string problems.

- But there are only n possible subtrees of a rooted tree (cut edge to the root) so we can use it optimization problems on rooted trees.

Dynamic programming works best on objects which are linearly ordered and cannot be rearranged – characters in a string, matrices in a chain, points around the boundary of a polygon, the left-to-right order of leaves in a search tree. Whenever your objects are ordered in a left-to-right way, you should smell dynamic programming!

The Principle of Optimality

To use dynamic programming, the problem must observe the *principle of optimality*, that whatever the initial state is, remaining decisions must be optimal with regard the state following from the first decision.

- Dijkstra's algorithm works because we care about the length of the shortest path to x, not how we got there.
- Edit distance works because we care about the cheapest way edit given prefixes, not how we got here.

This would not be true if we charged more for a deletion if there were other deletions nearby.

Example: The Traveling Salesman Problem

Combinatorial problems may observe this property but still use too much memory/time to be efficient.

Let $T(i; j_1, j_2, \ldots, j_k)$ be the cost of the optimal tour from 1 to i that goes thru each of the other cities once

$$T(i; i_1, j_2, \ldots, j_i) = Min_{1 \leq m \leq k} C[i, j_m] + T(j_m; j_1, j_2, \ldots, j_k)$$

$$T(i, j) = C(i, j) + C(j, 1)$$

Here there can be any subset of $j_1, j_2, \ldots, j_k$ instead of any subinterval - hence exponential.

But it is $O(n2^n)$ instead of $n!$.