Recurrence relation of SELECT with PARTITION

\[ T(n) = \begin{cases} \theta_1 & \text{when } n < 50 \\ T(n) = T(n/5) + T(3n/4) + cn & \text{when } n \geq 50 \end{cases} \]

Proof by induction on \( n \) that \( T(n) \leq an \):

Randomized Algorithm (RA)

1. **Algorithm** is randomized if its behavior is determined by:
   - **Input**
   - Values produced by random-number generator into Algorithm

*Analysis* of RA takes expected run-time over the distribution of values returned by the random-number generator.

*Probabilistic Analysis*

Choosing PE. \( m \) at random. (Assume equal prob of choosing \( m = \frac{j}{n} \))

**Base**: if \( |S| = 1 \)

then return that element

(Case assume elements are distinct)

Choose index \( i \) from any of \( 1, 2, \ldots, n \)

\[ \frac{1}{n} \]

Let the rank of PE. \( m \) is \( i \)

\[ S_1 \leq S_2 \leq \ldots \leq S_k \]

\[ i > k \]
S1 ① Input n
    \[ S_1 P \]
    \[ S_2 P \]
    \[ S_3 P \]
    \[ S_4 P \]
    \[ S_5 P \]
    \[ S_6 P \]
    \[ S_7 P \]
    \[ S_8 P \]
    \[ i_0 > k \]

S2 ② if \( \frac{n}{2} = k \) then \( \lfloor \frac{S_1}{m} \rfloor \) and \( \lfloor \frac{S_1 + S_2}{m} \rfloor \) return \( m \) as the constant.

S3 ③ if \( \frac{n}{2} < k \) then \( \lfloor \frac{S_1}{m} \rfloor \neq \lfloor \frac{S_3}{m} \rfloor = n - i \)

S1 ③ if \( i_0 > k \) need to recurse on \( S_1 \)

\( S_1 (i = \frac{n}{2}) \)

\[ = \frac{1}{n} \left[ \frac{k!}{k-1} T(n-k) + \sum_{i=k+1}^{n} T(i) \right] \]

\( \approx \frac{1}{n} \left[ \sum_{i=n-k+1}^{n-1} T(i) \right] + \sum_{i=k}^{n-1} T(i) \]

\( T(n) \leq \max_{1 \leq k \leq n} \left[ \frac{k!}{k-1} T(n-k) + \sum_{i=k+1}^{n} T(i) \right] \)

* Recurrence relationship of Randomized SELECT

\( \Delta T(n) \leq C \)

\( n = 1 \)

\( T(n) \leq Cn + \max_{1 \leq k \leq n} \left[ \frac{h-1}{h} T(n-k) + \sum_{i=k+1}^{n-1} T(i) \right] \)

Claim: \( T(n) \leq \alpha n \) where \( \alpha = 9C \)

Proof: by induction on \( n \).
proof: by induction $n$.

Basis ($n=1$)

$T(1) \leq C \leq a \cdot 1$

Induction step ($n > 1$)

Assume as IH that claim holds for all $n' < n$

$T(n) \leq Cn + \max_{1 \leq k \leq n} \left[ \frac{a}{n} \left[ \sum_{i=1}^{n-1} i + \sum_{i=k+1}^{n} \frac{n^2}{i} \right] \right]$

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$= Cn + \max_{1 \leq k \leq n} \left[ \frac{a}{n} \left( \frac{n(n+1)}{2} - \frac{k(k+1)}{2} \right) \right]$

$\frac{a}{2n} \left[ -2k^2 + 2nk + n^2 - n \right]$

$\frac{d}{dk} = -4k + 2n = 0 \implies k = \frac{n}{2}$

$\left( Cn + \frac{a}{4n} \left[ 3n^2 - 2n \right] \right)_{k=\frac{n}{2}} \leq \left( C + \frac{3}{4} a \right)n \leq (C + \frac{3}{4} a)n = Cn$

$C + \frac{3}{4} a = a$

$c = \frac{1}{2} a$

$a = 4c$