CSE 373 Analysis of Algorithms
Fall 2016
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LEC20: SATISFIABILITY

Lecture slide courtesy of Prof. Steven Skiena

THE MAIN IDEA

Suppose I gave you the following algorithm to solve the *bandersnatch* problem:

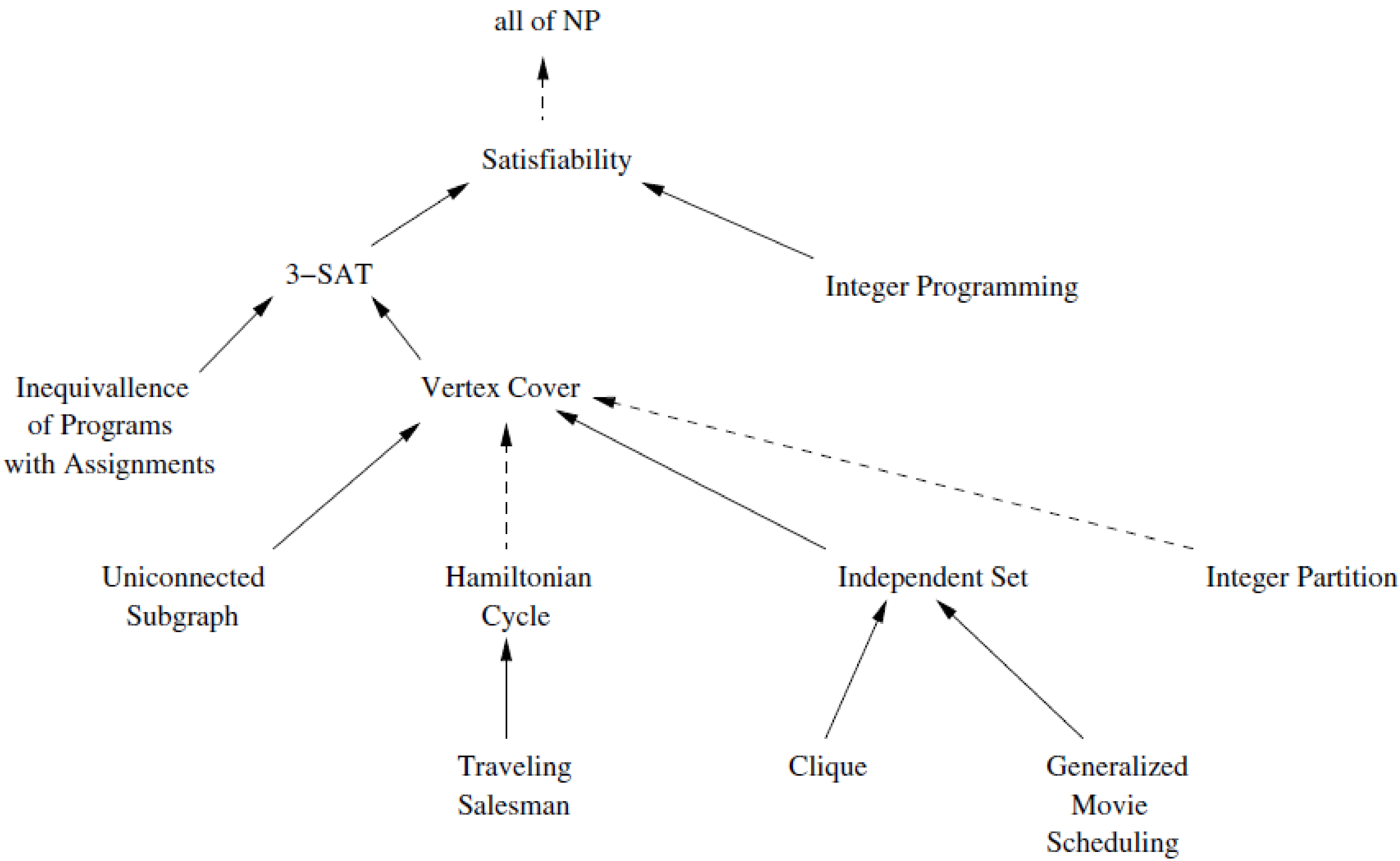
- Bandersnatch(G)
  - Convert G to an instance of the Bo-billy problem Y .
  - Call the subroutine Bo-billy on Y to solve this instance.
  - Return the answer of Bo-billy(Y) as the answer to G.

Such a translation from instances of one type of problem to instances of another type such that answers are preserved is called a *reduction*.

WHAT DOES THIS IMPLY?

- Now suppose my reduction translates G to Y in $O(P(n))$:
  - 1. If my Bo-billy subroutine ran in $O(P'(n))$ I can solve the Bandersnatch problem in $O(P(n) + P'(n'))$
  - 2. If I know that $\Omega(P'(n))$ is a lower-bound to compute Bandersnatch, then $\Omega(P'(n) - P(n'))$ must be a lowerbound to compute Bo-billy.
  - Why? If I could solve Bo-billy any faster, then I could violate my lower bound by solving Bandersnatch using the above reduction. This implies that there can be no way to solve Bo-billy any faster than claimed.!

A PORTION OF THE REDUCTION TREE FOR NP-COMPLETE PROBLEMS.



USING REDUCTION TO SHOW NP-COMPLETENESS

- A decision problem C is NP-complete if:
  - 1. C is in NP, and
  - 2. Every problem in NP is reducible to C in polynomial time.

- C can be shown to be in NP by demonstrating that a c andidate solution to C can be verified in polynomial ti me.
- Note that a problem satisfying condition 2 is said to be NP-hard, whether or not it satisfies condition 1.

SATISFIABILITY

We must start with a single problem that is absolutely, certifiably, undeniably hard: *satisfiability problem*

- *Problem:* Satisfiability
- *Input:* A set of Boolean variables *V* and a set of clauses *C* over *V* .
- *Output:* Does there exist a satisfying truth assignment for *C*—i.e. , a way to set the variables $v_1, \ldots, v_n$ true or false so that each clause contains at least one true literal?

SATISFIABILITY

- Example 1: $V = v_1, v_2$ and $C = \{\{v_1, \overline{v_2}\}, \{\overline{v_1}, v_2\}\}$
- A clause is satisfied when at least one literal in it is TRUE. C is satisfied when $v_1 = v_2 =$TRUE.

- Example 2: $V = v_1, v_2$

$$C = \{\{v_1, v_2\}, \{v_1, \overline{v_2}\}, \{\overline{v_1}\}\}$$

- Although you try, and you try, and you try and you try, you can get no satisfaction.
- There is no satisfying assignment since v_1 must be FALSE (third clause), so v_2 must be FALSE (second clause), but then the first clause is not satisfiable!

SATISFIABILITY IS HARD

- Satisfiability is known/assumed to be a hard problem in the worst case.
- Every top-notch algorithm expert in the world has tried and failed to come up with a fast algorithm to test whether a given set of clauses is satisfiable.
- Further, many strange and impossible-to-believe things have been shown to be true if someone in fact did find a fast satisfiability algorithm.

3-SATISFIABILITY

- *Problem:* 3-Satisfiability (3-SAT)
- *Input:* A collection of clauses *C* where each clause contains exactly 3 literals, over a set of Boolean variables *V* .
- *Output:* Is there a truth assignment to *V* such that each clause is satisfied?

- Note that this is a more restricted problem than SAT.
- If 3-SAT is NP-complete, it implies SAT is NP-complete but not visa-versa, perhaps long clauses are what makes SAT difficult?!
- After all, 1-SAT is trivial!

PROVING 3-SAT IS COMPLETE

- To prove it is complete, we give a reduction from *SAT* ∝ *3-SAT*.
- We will transform each clause independently based on its *length*.
- Suppose the clause C_i contains k literals.
  - If $k = 1$, meaning $C_i = \{z_1\}$, create two new variables v_1; v_2 and four new 3-literal clauses:

$$\{v_1, v_2, z_1\}, \{v_1, \overline{v}_2, z_1\}, \{\overline{v}_1, v_2, z_1\}, \{\overline{v}_1, \overline{v}_2, z_1\}$$

  - Note that the only way all four of these can be satisfied is if z is TRUE.

- If k = 2, meaning $\{z_1; z_2\}$, create one new variable v_1 and two new clauses: $\{v_1; z_1; z_2\}$, $\{\overline{v}_1; z_1; z_2\}$
- If k = 3, meaning $\{z_1; z_2; z_3\}$, copy into the 3-SAT instance as it is.
- If k > 3, meaning $C_i = \{z_1; z_2; \ldots; z_n\}$, create n - 3 new variables and n - 2 new clauses in a chain: where for $2 \le j \le n-3$, $C_{i,j} = \{v_{i,j-1}, z_{j+1}, \overline{v}_{i,j}\}$, $C_{i,1} = \{z_1, z_2, \overline{v}_{i,1}\}$, and $C_{i,n-2} = \{v_{i,n-3}, z_{n-1}, z_n\}$.

- This transform takes $O(m+n)$ time if there were n clauses and m total literals in the SAT instance.
- Since any SAT solution also satisfies the 3-SAT instance and any 3-SAT solution describes how to set the variables giving a SAT solution, the transformed problem is equivalent to the original.

WHY DOES THE CHAIN WORK?

- If none of the original variables in a clause are TRUE, there is no way to satisfy all of them using the additional variable:

  (F; F; T); (F; F; T); ... ; (F; F; F)

- But if any literal is TRUE, we have n - 3 free variables and n-3 remaining 3-clauses, so we can satisfy each of them.

  (F; F; T); (F; F; T);...; (F;T;F);... ; (T; F; F); (T; F; F)

- Any SAT solution will also satisfy the 3-SAT instance and any 3-SAT solution sets variables giving a SAT solution, so the problems are equivalent.

4-SAT AND 2-SAT

- A slight modification to this construction would prove 4-SAT, or 5-SAT,... also NP-complete.
- However, it breaks down when we try to use it for 2-SAT, since there is no way to stuff anything into the chain of clauses.
- Now that we have shown 3-SAT is NP-complete, we may use it for further reductions. Since the set of 3-SAT instances is smaller and more regular than the SAT instances, it will be easier to use 3-SAT for future reductions.
- Remember the direction to reduction!

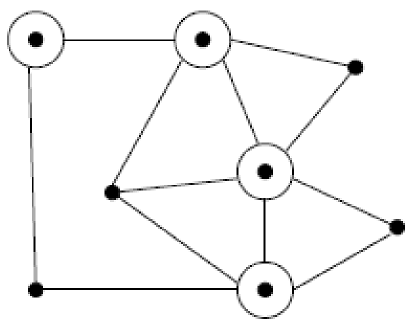
$$SAT \propto 3 - SAT \propto X$$

A PERPETUAL POINT OF CONFUSION

- Note carefully the direction of the reduction.
- We must transform *every* instance of a known NP-complete problem to an instance of the problem we are interested in. If we do the reduction the other way, all we get is a slow way to solve x, by using a subroutine which probably will take exponential time.
- This always is confusing at first - it seems bass-ackwards.
- Make sure you understand the direction of reduction now - and think back to this when you get confused.

VERTEX COVER

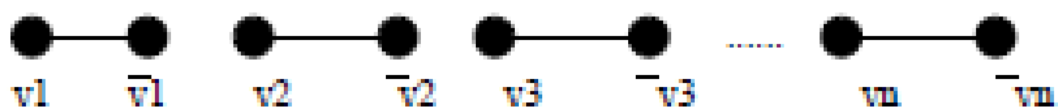
- *Problem:* Vertex Cover
- *Input:* A graph $G = (V,E)$ and integer $k \leq |V|$.
- *Output:* Is there a subset S of at most k vertices such that every $e \in E$ has at least one vertex in S?



- Here, four of the eight vertices suffice to cover.
- It is trivial to find *a* vertex cover of a graph – just take all the vertices. The tricky part is to cover with as small a set as possible.

VERTEX COVER IS NP-COMPLETE

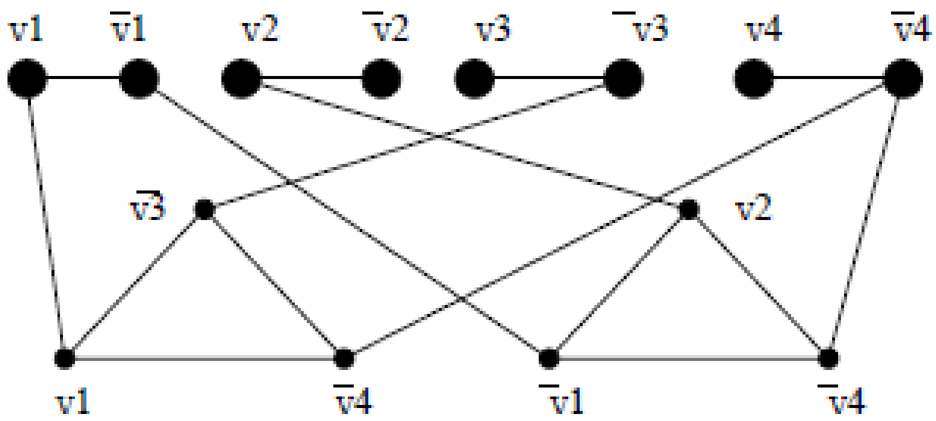
- To prove completeness, we reduce 3-SAT to VC.
- From a 3-SAT instance with n variables and C clauses, we construct a graph with 2N + 3C vertices.
- For each variable, we create two vertices connected by an edge:



- To cover each of these edges, at least n vertices must be in the cover, one for each pair.

CLAUSE GADGETS

- For each clause, we create three new vertices, one for each literal in each clause. Connect these in a triangle.
- At least two vertices per triangle must be in the cover to take care of edges in the triangle, for a total of at least *2C* vertices.
- Finally, we will connect each literal in the flat structure to the corresponding vertices in the triangles which share the same literal.



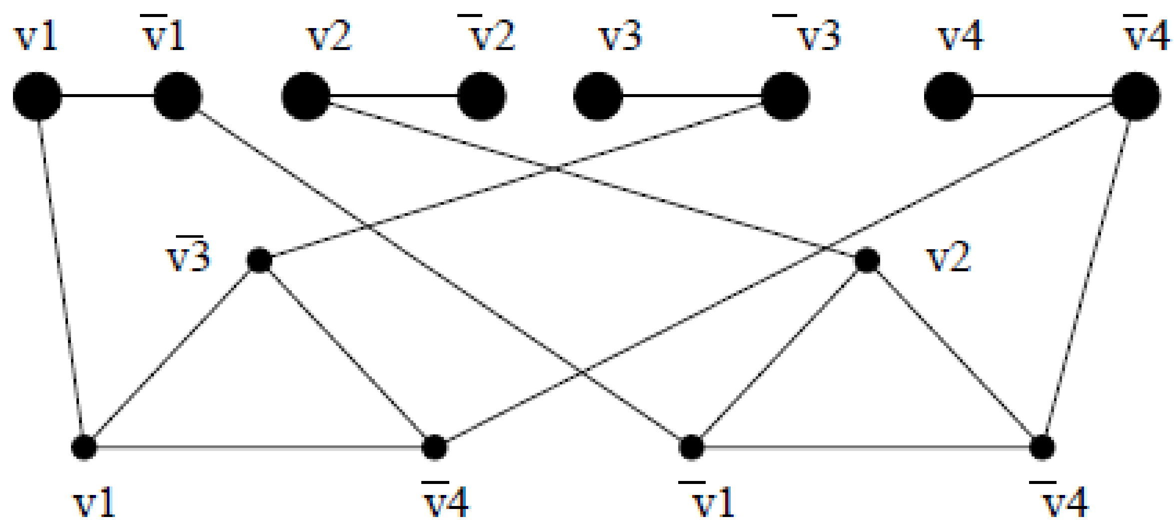
CLAIM: G HAS A VERTEX COVER OF SIZE N + 2C IFF S IS SATISFIABLE

- This graph has been designed to have a vertex cover of size $n + 2c$ if and only if the original expression is satisfiable.
- To show that our reduction is correct, we must show th at:
- 1. *Every satisfying truth assignment gives a cover.*
  - Select the N vertices corresponding to the TRUE literals to be in the cover.
  - Since it is a satisfying truth assignment, at least one of the three cross edges associated with each clause must already be covered – pick the other two vertices to complete the cover.

- 2. *Every vertex cover gives a satisfying truth assignment*.
  - Every vertex cover must contain n first stage vertices and 2C second stage vertices.
  - Let the first stage vertices define the truth assignment.
  - To give the cover, at least one cross-edge must be covered, so the truth assignment satisfies.

EXAMPLE REDUCTION

- *Every SAT defines a cover and Every Cover Truth values for the SAT!*
- Example: *V1 = V2 = True, V3 = V4 = False.*



STARTING FROM THE RIGHT PROBLEM

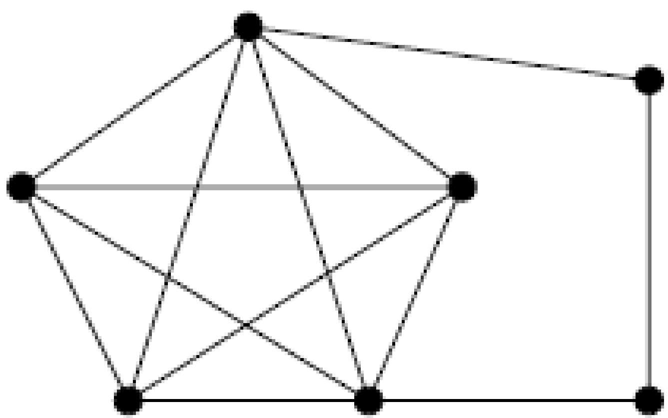
- As you can see, the reductions can be very clever and very complicated.
- While theoretically any NP-complete problem can be reduced to any other one, choosing the correct one makes finding a reduction much easier.

$$3\ SAT \propto VC$$

- As you can see, the reductions can be very clever and
- complicated.
- While theoretically any NP-complete problem will do, choosing the correct one can make it much easier.

MAXIMUM CLIQUE

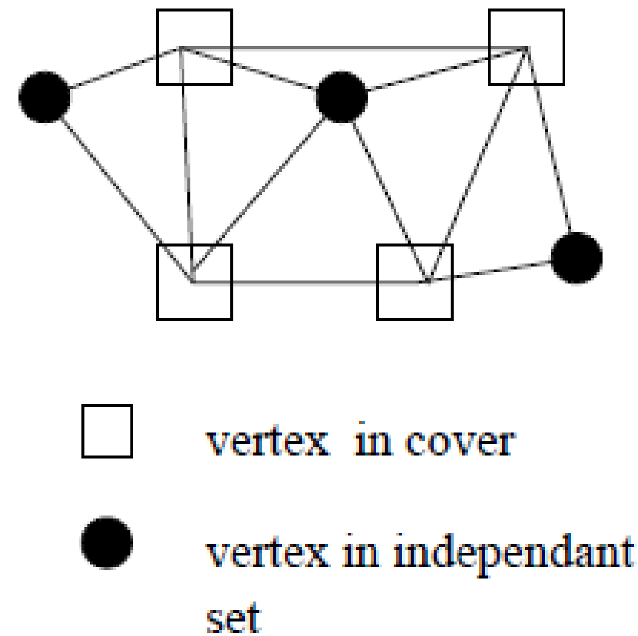
- Instance: A graph G = (V;E) and integer j ≤ v.
- Question: Does the graph contain a clique of j vertices, ie. Is there a subset of v of size j such that every pair of vertices in the subset defines an edge of G?
- Example: this graph contains a clique of size 5.



MAXIMUM CLIQUE IS *NP*-COMPLETE

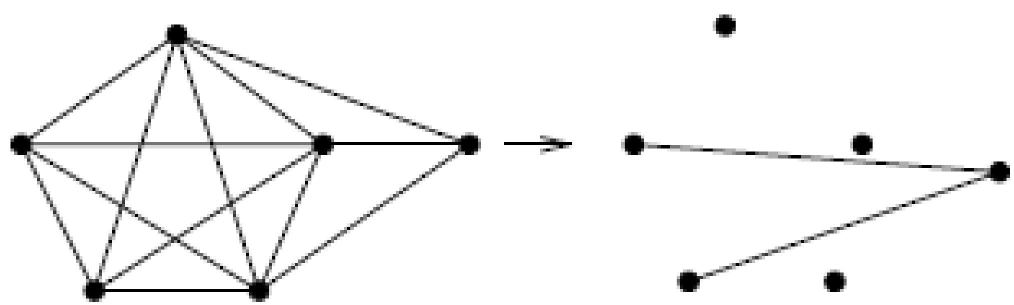
- When talking about graph problems, it is most natural to work from a graph problem - the only *NP*-complete one we have is vertex cover!
- If you take a graph and find its **vertex cover**, the remaining vertices form an **independent set**, meaning <u>there are no edges between any two vertices in the independent set</u>, for if there were such an edge the rest of the vertices could not be a vertex cover.

- Clearly the smallest vertex cover gives the biggest independent set, and so the problems are equivalent
- Delete the subset of vertices in one from the total set of vertices to get the order!
- Thus finding the maximum independent set must be NP - complete!



FROM INDEPENDENT SET

- In an independent set, there are no edges between two vertices. In a clique, there are always between two vertices.
- Thus if we **complement a graph** (have an edge iff there was no edge in the original graph), a clique becomes an independent set and an independent set becomes a Clique!



Max Clique = 5
Max IS = 2

Max Clique = 2
Max IS = 5

PUNCH LINE

- Thus finding the largest clique is NP-complete:
- If VC is a vertex cover in G, then V - VC is a clique in G'.
- If C is a clique in G, V - C is a vertex cover in G'.