CSE 373 Analysis of Algorithms
Fall 2016
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LEC18: APPLICATIONS OF DYNAMIC PROGRAMMING

Lecture slide courtesy of Prof. Steven Skiena

THREE STEPS OF DP

- There are three steps involved in solving a problem by dynamic programming:
  - 1. Formulate the answer as a recurrence relation or recursive algorithm.
  - 2. Show that the number of different parameter values taken on by your recurrence is bounded by a (hopefully small) polynomial.
  - 3. Specify an order of evaluation for the recurrence so the partial results you need are always available when you need them.

APP1: LONGEST INCREASING SEQUENCE

- Develop an algorithm to find the longest monotonically increasing subsequence within a sequence of *n* numbers.
- Difference between increasing sequence and *run*
  - *run :* elements must be physical neighbors of each other.
    - EX> Given S = *{2, 4, 3, 5, 1, 7, 6, 9, 8}*,
    - There are four longest increasing runs of length *2: (2, 4), (3, 5), (1, 7), and (6, 9).*
    - Finding the longest increasing run in a numerical sequence is straightforward
  - longest increasing subsequence (LIS) of S does not require neighborhood.
    - S has eight longest increasing subsequence in S of length 5, including *{2,3,5,6,8}.*
    - Finding the longest increasing subsequence is considerably trickier.

CONSTRUCT A RECURRENCE

- To find the right recurrence, ask what information about the first $n - 1$ elements of S would help you to find the answer for the entire sequence.
  - The length of the LIS in $s_1, s_2, \ldots, s_{n-1}$ seems a useful thing to know.
  - In addition, we need to know the length of the longest sequence that s_n will extend.
- Define l_i to be the length of the longest sequence ending with s_i

- The LIS containing the *n*th number will be formed by appending it to the longest increasing sequence to the left of *n* that ends on a number smaller than s_n.
- The following recurrence computes l_i:

$$l_i = \max_{0<j<i} l_j + 1 \text{ , where } (s_j < s_i)$$

$$l_0 = 0$$

- Goal Cell: The length of the LIS of the entire permutation is given by $\max_{0 \le i \le n} l_i$, since the winning sequence will have to end somewhere.

- Table associated with our previous example:

$$S = \{2, 4, 3, 5, 1, 7, 6, 9, 8\}$$

Sequence s_i	2	4	3	5	1	7	6	9	8
Length l_i	1	2	2	3	1	4	4	5	5
Predecessor p_i	–	1	1	2	–	4	4	6	6

auxiliary information: index *pi* of the element that appears immediately before *si* in the longest increasing sequence ending at *si*.

- Reconstruction: Start from the last value of the longest sequence and follow the pointers to the other items in the sequence

TIME COMPLEXITY

- Each one of the n values of l_i is computed by comparing s_i against (up to) $i-1 \leq n$ values to the left of it,
- so this analysis gives a total of $O(n^2)$ time.

APP2: THE PARTITION PROBLEM

- *Problem:* Integer Partition without Rearrangement
- *Input:* An arrangement S of nonnegative numbers $\{s_1, \ldots, s_n\}$ and an integer k.
- *Output:* Partition S into k or fewer ranges, to minimize the maximum sum over all the ranges, without reordering any of the numbers.
- Example: three workers are given the task of scanning through a shelf of books in search of a given piece of information. What is the fairest way to divide the workload (i.e. Sum # of pages in the partitions are even) : 100 200 300 400 500 | 600 700 | 800 900

RECURSIVE, EXHAUSTIVE SEARCH APPROACH

- Notice that the kth partition starts right after we placed the $(k-1)$st divider.
- Where can we place this last divider? Between the ith and $(i + 1)$st elements for some i, where $1 \leq i \leq n$.
- Let $M[n, k]$ be the minimum possible cost over all partitionings of $\{s_1, \ldots, s_n\}$ into k ranges, where the cost of a partition is the largest sum of elements in one of its parts.

RECURRENCE RELATION

- What is the cost of this? The total cost will be the larger of two quantities—
  - (1) the cost of the last partition $\sum_{j=i+1}^{n} s_j$, and
  - (2) the cost of the largest partition formed to the left of *i*.
    - See the recursion?

$$M[n,k] = \min_{1 \le i \le n} \max(m[i,k-1], \sum_{j=i+1}^{n} s_j)$$

BOUNDARY CONDITIONS

The smallest reasonable value of the

- first argument is $n = 1$ (first partition consists of a single element)

$$M[1, k] = s_1, \text{for all } k > 0 \text{ and,}$$

- second argument is $k = 1$ (we do not partition S at all).

$$M[n, 1] = \sum_{i=1}^{n} s_i$$

TIME ANALYSIS

- When we store the partial results, total of $k \cdot n$ cells exist in the table.
- How much time does it take to compute the result $M[n, k]$?
  - find the minimum of n' quantities each of which is the maximum of the table lookup and a sum of at most n elements
  - -> at most n^2 time per box
- Total recurrence can be computed in $O(kn^3)$ time

CODE

```
partition(int s[], int n, int k)
{
    int m[MAXN+1][MAXK+1];          /* DP table for values */
    int d[MAXN+1][MAXK+1];          /* DP table for dividers */
    int p[MAXN+1];                  /* prefix sums array */
    int cost;                       /* test split cost */
    int i,j,x;                      /* counters */

    p[0] = 0;                       /* construct prefix sums */
    for (i=1; i<=n; i++) p[i]=p[i-1]+s[i];

    for (i=1; i<=n; i++) m[i][1] = p[i]; /* initialize boundaries */
    for (j=1; j<=k; j++) m[1][j] = s[1];

    for (i=2; i<=n; i++) /* evaluate main recurrence */
        for (j=2; j<=k; j++) {
            m[i][j] = MAXINT;
            for (x=1; x<=(i-1); x++) {
                cost = max(m[x][j-1], p[i]-p[x]);
                if (m[i][j] > cost) {
                    m[i][j] = cost;
                    d[i][j] = x;
                }
            }
        }
    }
    reconstruct_partition(s,d,n,k); /* print book partition */
}
```

We keep track of prefix sums $p[i] = \sum_{k=1}^{i} s_k$ for faster run time since $\sum_{k=i}^{j} s_k = p[k] - p[j]$

Since $\sum_{k=i}^{j} s_k = p[k] - p[j]$ Enables us to evaluate the recurrence in linear time per cell, yielding an $O(kn^2)$ algorithm.

RECONSTRUCTING ACTUAL PARTITION

- Final value of *M(n,k)* will be the cost of the largest range in the optimal partition
- Matrix *D* is used to reconstruct the optimal partition by work backward from D[n, k] and add a divider at each specified position.

```
reconstruct_partition(int s[],int d[MAXN+1][MAXK+1], int n, int k)
{
        if (k==1)
                print_books(s,1,n);
        else {
                reconstruct_partition(s,d,d[n][k],k-1);
                print_books(s,d[n][k]+1,n);
        }
}

print_books(int s[], int start, int end)
{
        int i; /* counter */
        for (i=start; i<=end; i++) printf(" %d ",s[i]);
        printf("\n");
}
```

EXAMPLE

M	k				D	k		
n	1	2	3		n	1	2	3
1	1	1	1		1	–	–	–
1	2	1	1		1	–	1	1
1	3	2	1		1	–	1	2
1	4	2	2		1	–	2	2
1	5	3	2		1	–	2	3
1	6	3	2		1	–	3	4
1	7	4	3		1	–	3	4
1	8	4	3		1	–	4	5
1	9	5	3		1	–	4	6

M	k				D	k		
n	1	2	3		n	1	2	3
1	1	1	1		1	–	–	–
2	3	2	2		2	–	1	1
3	6	3	3		3	–	2	2
4	10	6	4		4	–	3	3
5	15	9	6		5	–	3	4
6	21	11	9		6	–	4	5
7	28	15	11		7	–	5	6
8	36	21	15		8	–	5	6
9	45	24	17		9	–	6	7

Partitioning
{1, 1, 1, 1, 1, 1, 1, 1, 1}
into {{1, 1, 1}, {1, 1, 1}, {1, 1, 1}}

Partitioning
{1, 2, 3, 4, 5, 6, 7, 8, 9}
into {{1, 2, 3, 4, 5}, {6, 7}, {8, 9}}

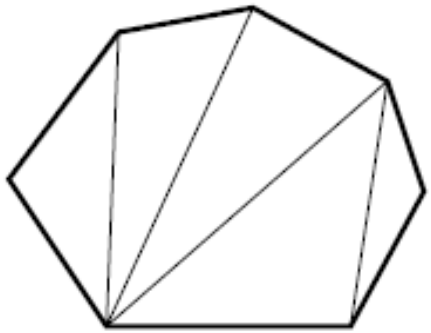
Notice that final value of $M(n, k)$ is the cost of the largest range in the optimal partition.

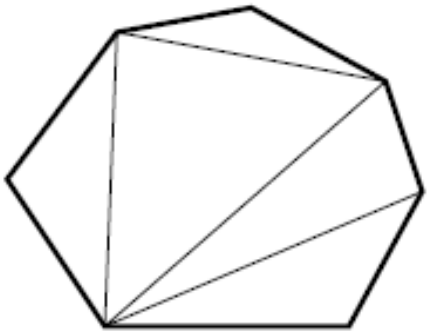
PARSING CONTEXT-FREE GRAMMARS

- Learning it in your compiler class.

MINIMUM WEIGHT TRIANGULATION

- A *triangulation* of a polygon $P = \{v_1, \ldots, v_n, v_1\}$ is a set of nonintersecting diagonals that partitions the polygo n into triangles.
- The *weight* of a triangulation is the sum of the lengths of its diagonals.

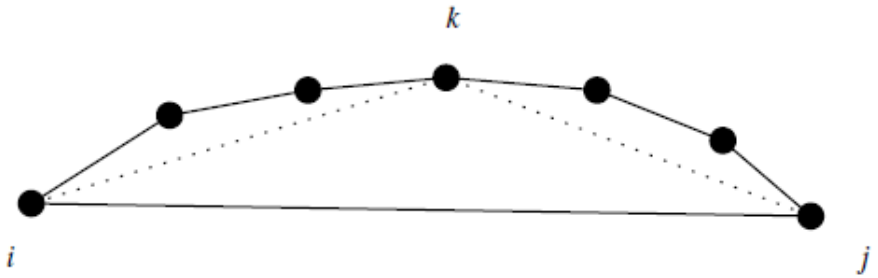




- We seek to find its **minimum weight triangulation** for a given polygon *p*

RECURRENCE

- Observe that every edge of the input polygon must be in volved in exactly one triangle. Turning this edge (i,j) into a triangle means identifying the third vertex,k.



- Let $T[i, j]$ be the cost of triangulating from vertex v_i to v ertex v_j , ignoring the length of the chord d_{ij} from v_i t v_j .

$$T[i,j] = \min_{i+1 \le k \le j-1} (T[i,k] + T[k,j] + d_{ik} + d_{kj})$$

- Basis: when i and j are immediate neighbors, as

$$T[i, i+1] = 0.$$

- Evaluation an proceed in terms of the gap size from *i* to *j*:

```
Minimum-Weight-Triangulation(P)
    for i = 1 to n − 1 do T[i, i + 1] = 0
        for gap = 2 to n − 1
            for i = 1 to n − gap do
                j = i + gap
                T[i, j] = min_{i+1≤k≤j−1} (T[i, k] + T[k, j] + d_ik + d_kj)
    return T[1, n]
```

- There are $\binom{n}{2}$ values of *T*, each of which takes $O(j - i)$ time if we evaluate the sections in order of increasing size.
- Since $j - i = O(n)$, complete evaluation takes $O(n^3)$ time and $O(n^2)$ space.

LIMITATIONS OF DYNAMIC PROGRAMMING: TSP

- Dynamic programming doesn't always work.
- *Working example:*
  - *Problem:* Longest *Simple* **Path**
  - *Input:* A weighted graph *G*, with specified start and end vertices *s* and *t*.
  - *Output:* What is the **most expensive** path from *s* to *t* that does not visit any vertex more than once?

WHEN ARE DP ALGORITHMS CORRECT?

- Suppose we define *LP*[*i*, *j*] as a function denoting the length of the longest simple path from *i* to *j*.
- Note that the longest simple path from *i* to *j* had to visit some vertex *x* right before reaching *j*.
  - Thus, the last edge visited must be of the form (*x*, *j*).
- Recurrence relation: the length of the longest path, where *c*(*x*, *j*) is the cost/weight of edge (*x*, *j*):

$$LP[i,j] = \max_{(x,j)\in E} LP[i,x] + c(x,j)$$

- Can you see the problem?
  - Does not enforce simplicity (we are not allowed to visit any vertex more than once)
  - No evaluation order: It is not clear what the *smaller* subprograms are.

PRINCIPLE OF OPTIMALITY

- Dynamic programming can be applied to any problem that observes the ***principle of optimality***: partial solutions can be optimally extended with regard to the *state* after the partial solution, instead of the specifics of the partial solution itself.
  - Future decisions are made based on the *consequences* of previous decisions, not the actual decisions themselves
  - Problems do not satisfy the principle of optimality when the specifics of the operations matter, as opposed to just the cost of the operations.
- Example: in deciding whether to extend an approximate string matching by a substitution, insertion, or deletion, we did not need to know which sequence of operations had been performed to date.

WHEN ARE DP ALGORITHMS EFFICIENT?

- Running time of DP is a function of following:
  - (1) number of partial solutions we must keep track of, and
  - (2) how long it take to evaluate each partial solution.
- The partial solutions should be completely described by specifying the stopping *places* in the input
  - Once the order is fixed, there are relatively few possible stopping places or states, so we get efficient algorithms.

- When the objects are not firmly ordered, we get an exponential number of possible partial solutions.
- EX> Suppose the state of our partial solution is entire path *P* taken from the start to end vertex.

$$LP[i, j, P + x] = \max_{(x,j) \in E, x,j \notin P} LP[i, x, P] + c(x, j)$$

This is Correct but not efficient:

- The path *P* consists of an ordered sequence of up to $n - 3$ vertices. There can be up to $(n - 3)!$ such paths!