CSE 373 Analysis of Algorithms
Fall 2016
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LEC16: INTRODUCTION TO DYNAMIC PROGRAMMING

Lecture slide courtesy of Prof. Steven Skiena

DYNAMIC PROGRAMMING

- Dynamic programming is a very powerful, general tool for solving optimization problems on left-right-ordered items such as character strings.
- Once understood it is relatively easy to apply, it looks like magic until you have seen enough examples.
- Floyd's all-pairs shortest-path algorithm was an example of dynamic programming.

GREEDY VS. EXHAUSTIVE SEARCH

- ***Greedy* algorithms** focus on making the best local choice at each decision point. In the absence of a correctness proof such greedy algorithms are very likely to fail.
- **Dynamic programming** gives us a way to design custom algorithms which systematically search all possibilities (thus guaranteeing correctness) while storing results to avoid recomputing (thus providing efficiency).

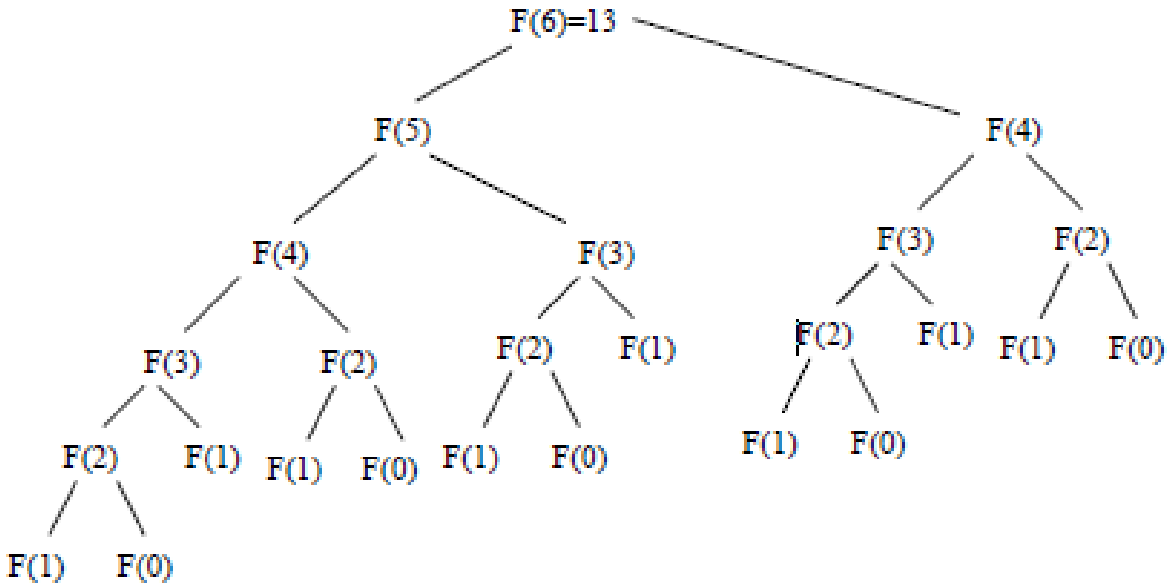
RECURRENCE RELATIONS

- A recurrence relation is an equation which is defined in terms of itself.
- They are useful because many natural functions are easily expressed as recurrences:
  - Polynomials: $a_n = a_{n-1} + 1;\ a_1 = 1 \rightarrow a_n = n$
  - Exponentials: $a_n = 2a_{n-1};\ a_1 = 2 \rightarrow a_n = 2^n$
  - Weird: $a_n = na_{n-1};\ a_1 = 1 \rightarrow a_n = n!$
- Computer programs can easily evaluate the value of a given recurrence even without the existence of a nice closed form.

COMPUTING FIBONACCI NUMBERS

$$F_n = F_{n-1} + F_{n-2}; F_0 = 0; F_1 = 1$$

- Implementing this as a recursive procedure is easy, but slow because we keep calculating the same value over and over.



HOW SLOW?

$$\frac{F_{n+1}}{F_n} \approx \phi = \frac{1+\sqrt{5}}{2} \approx 1.61803$$

Thus $F_n \approx 1.6^n$.

Since our recursion tree has 0 and 1 as leaves, computing F_n requires $\approx 1.6^n$ calls!

WHAT ABOUT DYNAMIC PROGRAMMING?

We can calculate F_n in linear time by storing small values:

- $F_0 = 0$
- $F_1 = 1$
- for $i = 1$ to n
  - $F_i = F_{i-1} + F_{i-2}$

Moral: we traded space for time.

BENEFITS OF DYNAMIC PROGRAMMING

- Dynamic programming is a technique for efficiently computing recurrences by storing partial results.
- Once you understand dynamic programming, it is usually easier to reinvent certain algorithms than try to look them up!
- Dynamic programming to be one of the most useful algorithmic techniques in practice:
  - Morphing in computer graphics.
  - Data compression for high density bar codes.
  - Designing genes to avoid or contain specified patterns.

AVOIDING RECOMPUTATION BY STORING PARTIAL RESULTS

- The trick to dynamic program is to see that the naive recursive algorithm repeatedly computes the same subproblems over and over and over again.
- If so, storing the answers to them in a table instead of recomputing can lead to an efficient algorithm.
- Thus we must first hunt for a correct recursive algorithm – later we can worry about speeding it up by using a results matrix.

BINOMIAL COEFFICIENTS

- The most important class of counting numbers are the *binomial coefficients*, where $\binom{n}{k}$ counts the number of ways to choose k things out of n possibilities.
  - *Committees* – How many ways are there to form a k-member committee from n people? By definition, $\binom{n}{k}$.
  - *Paths Across a Grid* – How many ways are there to travel from the upper-left corner of an n m grid to the lower-right corner by walking only down and to the right? Every path must consist of n + m steps, n downward and m to the right, so there are $\binom{n+m}{n}$ such sets/paths.

COMPUTING BINOMIAL COEFFICIENTS

- Since $\binom{n}{k}$ = *n!/((n - k)! k!)*, in principle you can compute them straight from factorials.
- However, intermediate calculations can *easily* cause arithmetic overflow even when the final coefficient fits comfortably within an integer.

PASCAL'S TRIANGLE

- In Pascal's Triangle, each number is the sum of the two numbers directly above it:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

PASCAL'S RECURRENCE

- A more stable way to compute binomial coefficients is using the recurrence relation implicit in the construction of Pascal's triangle, namely, that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

- It works because the nth element either appears or does not appear in one of the $\binom{n}{k}$ subsets of k elements.

BASIS CASE

- No recurrence is complete without basis cases.
- How many ways are there to choose 0 things from a set?
- Exactly one, the empty set.
- The right term of the sum drives us up to $\binom{k}{k}$. How many ways are there to choose k things from a k-element set?
- Exactly one, the complete set.

BINOMIAL COEFFICIENTS IMPLEMENTATION

```
long binomial coefficient(n,m)
int n,m;                    /* compute n choose m */
{
    int i,j;                /* counters */
    long bc[MAXN][MAXN]; /* table of binomial coefficients */

    for (i=0; i<=n; i++) bc[i][0] = 1;

    for (j=0; j<=n; j++) bc[j][j] = 1;

    for (i=1; i<=n; i++)
    for (j=1; j<i; j++)
        bc[i][j] = bc[i-1][j-1] + bc[i-1][j];

    return( bc[n][m] );
}
```

THREE STEPS TO DYNAMIC PROGRAMMING

1. Formulate the answer as a recurrence relation or recursive algorithm.
2. Show that the number of different instances of your recurrence is bounded by a polynomial.
3. Specify an order of evaluation for the recurrence so you always have what you need.