CSE 373 Analysis of Algorithms
Fall 2016
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LEC11: BREADTH-FIRST SEARCH

Lecture slide courtesy of Prof. Steven Skiena

TRAVERSING A GRAPH

- One of the most fundamental graph problems is to traverse every edge and vertex in a graph.
  - E.g. printing or copying graphs, and converting between alternate representations
- For *efficiency*, we must make sure we don't visit each edge repeatedly.
- For *correctness*, we must do the traversal in a systematic way so that we don't miss anything.
- Since a maze is just a graph, such an algorithm must be powerful enough to enable us to get out of an arbitrary maze.

MARKING VERTICES

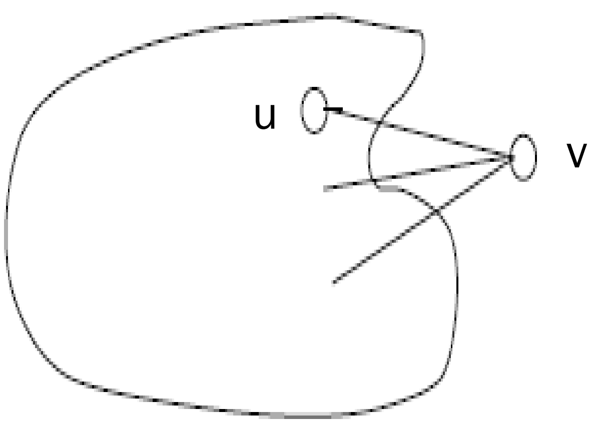
- The key idea is that we must **mark** each vertex when we first visit it, and **keep track** of what have not yet completely explored.
- Each vertex will always be in one of the following three states:
  - *Undiscovered* – the vertex in its initial, virgin state.
  - *Discovered* – the vertex after we have encountered it, but before we have checked out all its incident edges.
  - *Processed* – the vertex after we have visited all its incident edges.
- Obviously, a vertex cannot be *processed* before we discover it, so over the course of the traversal the state of each vertex progresses from *undiscovered -> discovered -> processed*.

TO DO LIST

- We must also maintain a structure containing all the vertices we have discovered but not yet completely processed.
- Initially, only a single start vertex is considered to be discovered.
- To completely process a vertex, we look at each edge going out of it.
- For each edge which goes to an undiscovered vertex, we mark it *discovered* and add it to the list of work to do. (do nothing to vertices already processed & vertices *discovered* but not *processed*.)

CORRECTNESS OF GRAPH TRAVERSAL

- Each undirected edge will be considered exactly twice,
  - once when each of its endpoints is explored.
- Directed edges will be considered only once,
  - when exploring the source vertex.
- Every edge and vertex in the connected component must eventually be visited.
  - Suppose that there exists a vertex u that remains unvisited, whose neighbor v was visited.
  - This neighbor v will eventually be explored, after which we will certainly visit u.
  - Thus, we must find everything that is there to be found.



BREADTH-FIRST TRAVERSAL

- The basic operation in most graph algorithms is completely and systematically traversing the graph.
- We want to visit every vertex and every edge exactly **once** in some well-defined order.
- **Breadth-first search** is appropriate if we are interested in shortest paths on unweighted graphs.
  - In a breadth-first search of an undirected graph, we assign a direction to each edge, from the discoverer *u* to the discovered *v*. We thus denote *u* to be the parent of *v*.

BFS(G, s)

for each vertex $u \in V[G] - \{s\}$ do

    state[u] = “undiscovered”

    $p[u]$ = *nil*, i.e. no parent is in the BFS tree

state[s] = “discovered”

$p[s]$ = *NULL*

$Q = \{s\}$

while $Q = \emptyset$ do

    u = dequeue[Q]

    process vertex u as desired

    for each $v \in Adj[u]$ do

        process edge (u, v) as desired

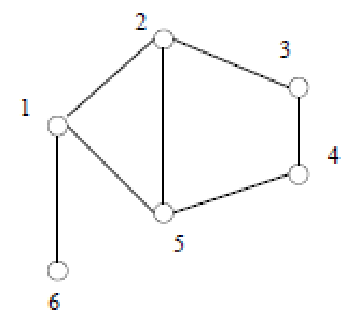
        if *state*[v] = “undiscovered” then

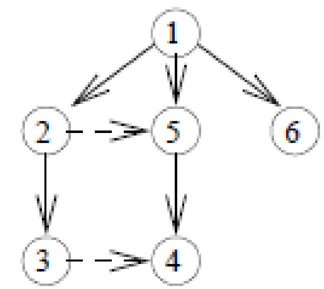
            state[v] = “discovered”

            $p[v] = u$

            enqueue[Q, v]

    state[u] = “processed”





DATA STRUCTURES FOR BFS

- We use <u>two Boolean arrays</u> to maintain our knowledge about each vertex in the graph.
  - A vertex is **discovered** the first time we visit it.
  - A vertex is considered **processed** after we have traversed all outgoing edges from it.
- Once a vertex is discovered, it is placed on a **<u>FIFO queue</u>**.
  - Thus the oldest vertices / closest to the root are expanded first.

BFS IMPLEMENTATION - INITIALIZING BFS

Global Variables:

- bool processed[MAXV+1];
- bool discovered[MAXV+1];
- int parent[MAXV+1];

Each vertex is initialized as undiscovered:

```
initialize search(graph *g)
{
        int i;
        for (i=1; i<=g->nvertices; i++) {
                processed[i] = discovered[i] = FALSE;
                parent[i] = -1;
        }
}
```

BFS IMPLEMENTATION

Once a vertex is discovered, it is placed on a queue

```
bfs(graph *g, int start)
{
    queue q; /* queue of nodes to visit */
    int v;       /* current vertex */
    int y;       /* successor vertex */
    edgenode *p;    /* temporary pointer */

    init queue(&q);
    enqueue(&q,start);
    discovered[start] = TRUE;

    while (empty_queue(&q) == FALSE) {
        v = dequeue(&q);
        process_vertex_early(v);
        processed[v] = TRUE;
        p = g->edges[v];
        while (p ! = NULL) {
            y = p->y;
            if ((processed[y] == FALSE||g-->directed)
                process_edge(v,y);
            if (discovered[y] == FALSE) {
                enqueue(&q,y);
                discovered[y] = TRUE;
                parent[y] = v;
            }
            p = p->next;
        }
        process_vertex_late(v);
    }
}
```

EXPLOITING TRAVERSAL

- The exact behavior of bfs depends upon the functions process vertex early(), process vertex late(), and process edge().By setting the functions to
- By setting the active functions to

```
process_vertex(int v) {
        printf("processed vertex %d\n",v);
}
process_edge(int x, int y) {
        printf(”processed edge (%d,%d) ”,x,y);
}
```

- we print each vertex and edge exactly once.

FINDING PATHS

- The **parent** array set within **bfs()** is very useful for finding interesting paths through a graph.
- The vertex which discovered vertex *i* is defined as **parent[i].**
- The parent relation defines a tree of discovery with the initial search node as the root of the tree.

SHORTEST PATHS AND BFS

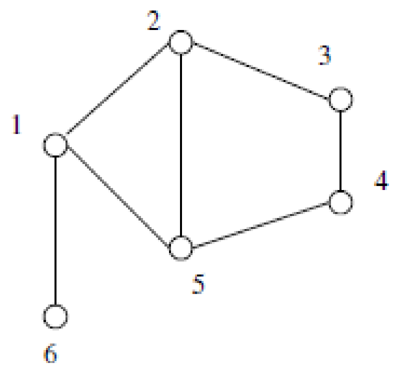
- In BFS vertices are discovered in order of increasing distance from the root, so this tree has a very important property.
- The unique tree path from the root to any node $x \in V$ uses the smallest number of edges (or equivalently, intermediate nodes) possible on any *root-to-x* path in the graph.

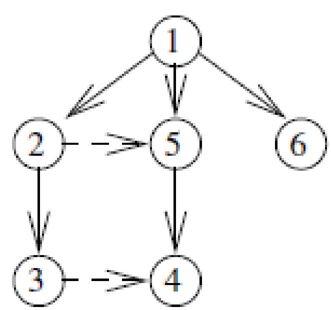
RECURSION AND PATH FINDING

- We can reconstruct this path by following the chain of ancestors from x to the root.
- Note that we have to work backward.
  - We cannot find the path from the root to x, since that does not follow the direction of the parent pointers.
  - Instead, we must find the path from x to the root.

FINDING PATH EXAMPLE

```
find path(int start, int end, int parents[])
{
        if ((start == end) || (end == -1))
                printf("%d",start);
        else {
                find_path(start,parents[end],parents);
                printf(" %d",end);
        }
}
```





vertex	1	2	3	4	5	6
parent	-1	1	2	5	1	1

For the shortest path from 1 to 4, upper-right corner, this parent relation yields the path *{*1*,* 5*,* 4*}*.

BFS APPLICATION 1: CONNECTED COMPONENTS

- A graph is *connected* if there is a path between any tw o vertices.
- The *connected components* of an undirected graph is a maximal set of vertices such that there is a path between every pair of vertices.
- The components are separate "pieces" of the graph s uch that there is no connection between the pieces

- Many seemingly complicated problems reduce to finding or counting connected components.
  - EX> Testing whether a puzzle such as Rubik's cube or the 15-puzzle can be solved from any position is really asking whether the graph of legal configurations is connected.
- Connected components can be found using BFS
  - Anything we discover during a BFS must be part of the same connected component.
  - Repeat the search from any undiscovered vertex (if one exists) to define the next component, until all vertices have been found

IMPLEMENTATION

```
connected_components(graph *g)
{
        int c =0;                    /* component number */
        int i;                       /* counter */

        initialize_search(g);
        for (i=1; i<=g->nvertices; i++){
                if (discovered[i] == FALSE) {
                        c = c+1;
                        printf("Component %d:",c);
                        bfs(g,i);
                        printf("\n");
                }
        }
}

process_vertex_early(int v) { printf(" %d",v); }

process_edge(int x, int y) {   }
```

$O(n + m)$

BFS APPLICATION 2: TWO-COLORING GRAPHS

- The *vertex coloring* problem seeks to assign a label (or color) to each vertex of a graph such that no edge links any two vertices of the same color.
- The goal is to use as few colors as possible
- Vertex coloring problems often arise in **scheduling applications**
  - Ex> register allocation in compilers

- A graph is ***bipartite*** if it can be colored without conflicts while using only two colors.
- Bipartite graphs are important because they arise naturally in many applications.
  - For example, consider the “married-to” graph in a hetero sexual world. Men have marry only with women, and vice versa.
  - Thus gender defines a legal two-coloring.

- Solution Scratch (Augmented BFS):
  - Whenever we discover a new vertex, color it the opposite of i ts parent.
  - Check for conflict:
    - We check whether any nondiscovery edge links two vertices of the sa me color.
  - We will have constructed a proper two-coloring whenever we terminate without conflict

FINDING A TWO-COLORING

```
twocolor(graph *g)
{
    int i;                          /* counter */
    for (i=1; i<=(g->nvertices); i++)
            color[i] = UNCOLORED;
    bipartite = TRUE;
    initialize search(&g);
    for (i=1; i<=(g->nvertices); i++){
            if (discovered[i] == FALSE) f
                    color[i] = WHITE;
                    bfs(g,i);
            }
    }
}
```

```
process_edge(int x, int y)
{
    if (color[x] == color[y]) {
            bipartite = FALSE;
            printf("Warning: graph not bipartite, due to (%d,%d)",x,y);
    }
    color[y] = complement(color[x]);
}

complement(int color)
{
    if (color == WHITE) return(BLACK);
    if (color == BLACK) return(WHITE);
    return(UNCOLORED);
}
```

- We can assign the first vertex in any connected component to be whatever color we wish.