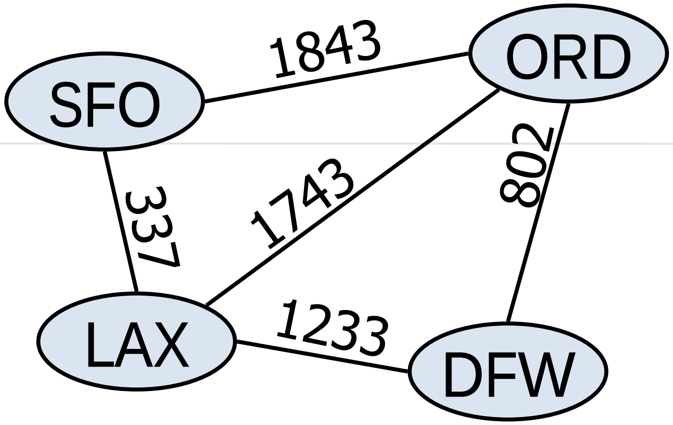
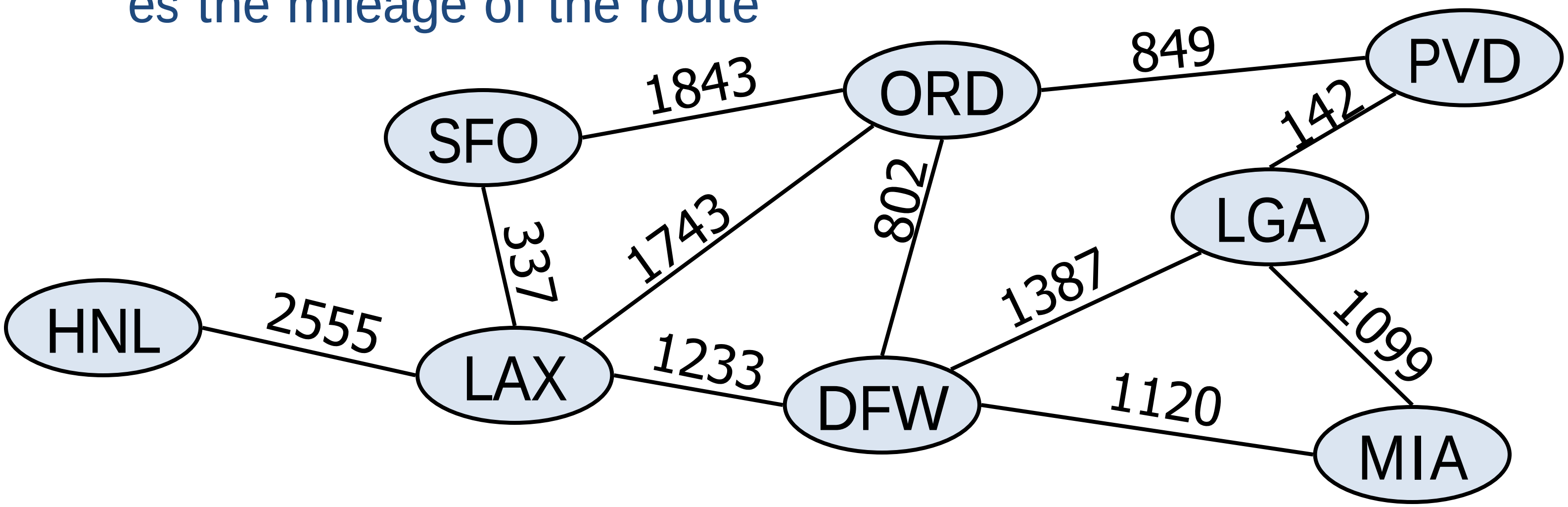
GRAPHS (CH14)



Presentation for use with the textbook Data Structures and Algorithms in Java, 6th edition, by M. T. Goodrich, R. Tamassia, and M. H. Goldwasser, Wiley, 2014

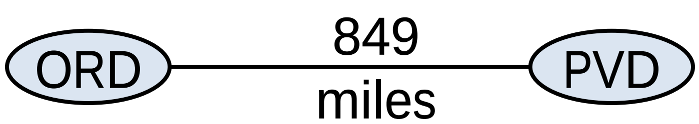
GRAPHS

- A **graph** is a pair (V, E), where
  - V is a set of nodes, called **vertices** (aka nodes)
  - E is a collection of pairs of vertices, called **edges** (aka arcs)
  - Vertices and edges are positions and store elements
- Example:
  - A vertex represents an airport and stores the three-letter airport code
  - An edge represents a flight route between two airports and stor es the mileage of the route



EDGE TYPES

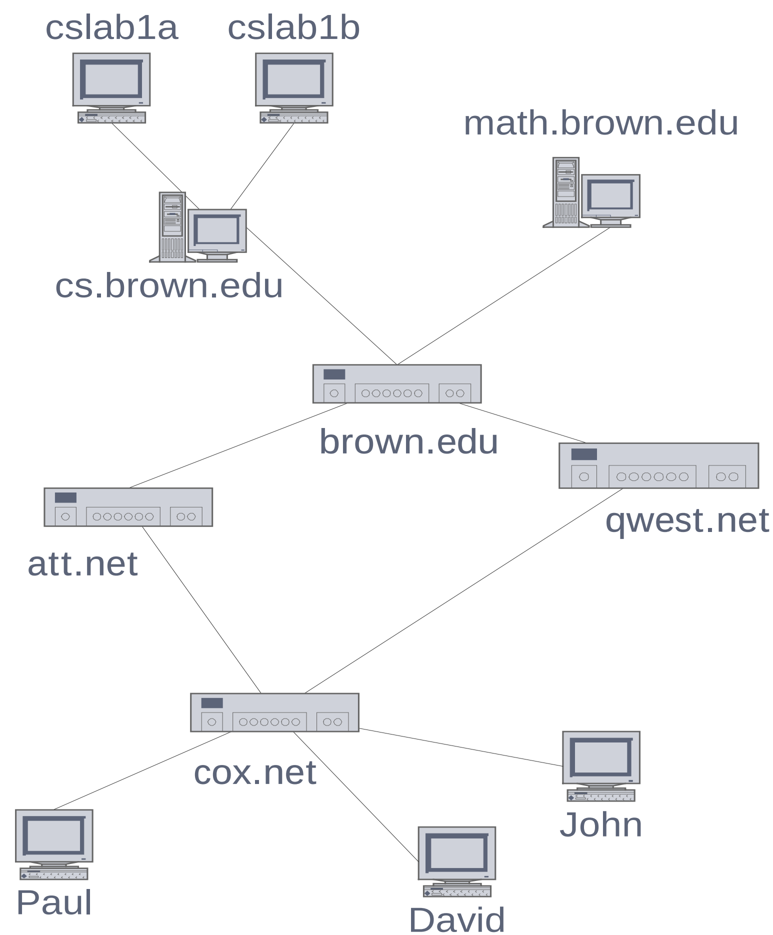
- **Directed edge**
  - ordered pair of vertices (u,v)
  - first vertex u is the origin
  - second vertex v is the destination
  - e.g., a flight
- **Undirected edge**
  - unordered pair of vertices (u,v)
  - e.g., a flight route
- **Directed graph**
  - all the edges are directed
  - e.g., route network
- **Undirected graph**
  - all the edges are undirected
  - e.g., flight network



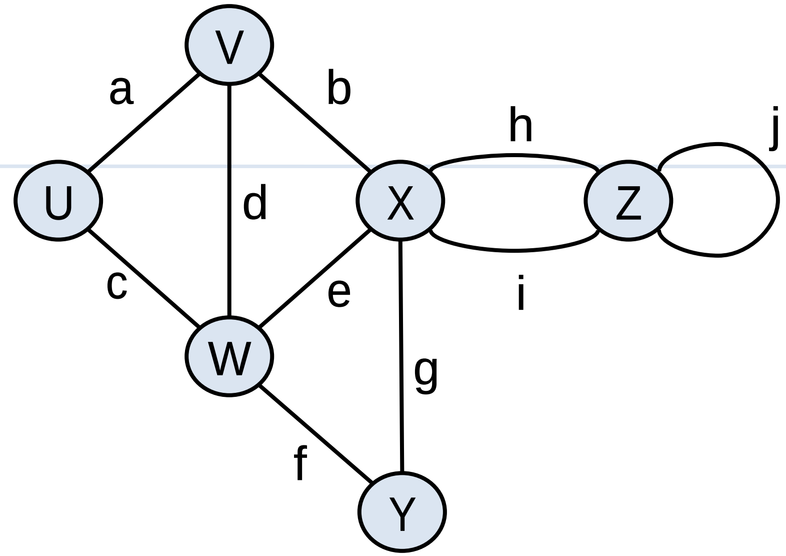
- **Mixed graph :** graph that has both directed and undirected edges

APPLICATIONS

- Electronic circuits
  - Printed circuit board
  - Integrated circuit
- Transportation networks
  - Highway network
  - Flight network
- Computer networks
  - Local area network
  - Internet
  - Web
- Databases
  - Entity-relationship diagram



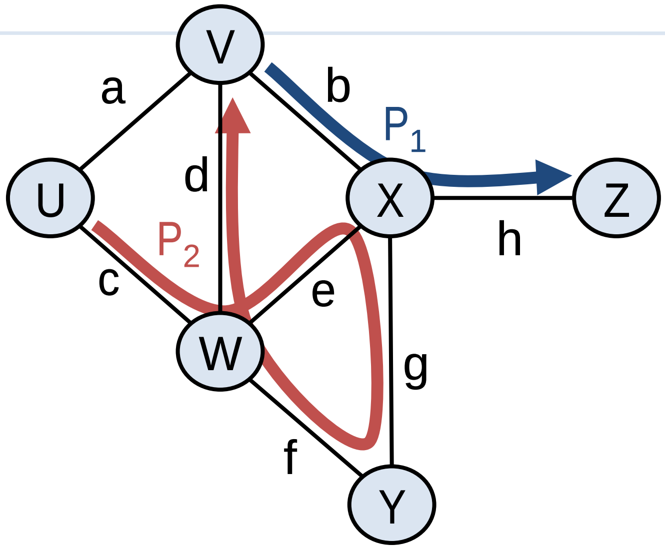
TERMINOLOGY



- End vertices (or endpoints) of an edge
  - U and V are the endpoints of a
- Edges incident on a vertex
  - a, d, and b are incident on V
- Adjacent vertices
  - U and V are adjacent
- **Degree** of a vertex
  - deg(X)= 5; X has degree 5
- Parallel edges (multiple edges)
  - h and i are parallel edges
  - Edges are collections (not sets)
- Self-loop
  - j is a self-loop

- ***outgoing edges*** of a vertex:
  - directed edges whose origin is that vertex.
- ***incoming edges*** of a vertex:
  - directed edges whose destination is that vertex.
- ***in-degree*** & ***out-degree*** of a vertex *v*
  - the number of the incoming and outgoing edges of *v*,
  - Denoted indeg(*v*) and outdeg(*v*)

TERMINOLOGY (CONT.)

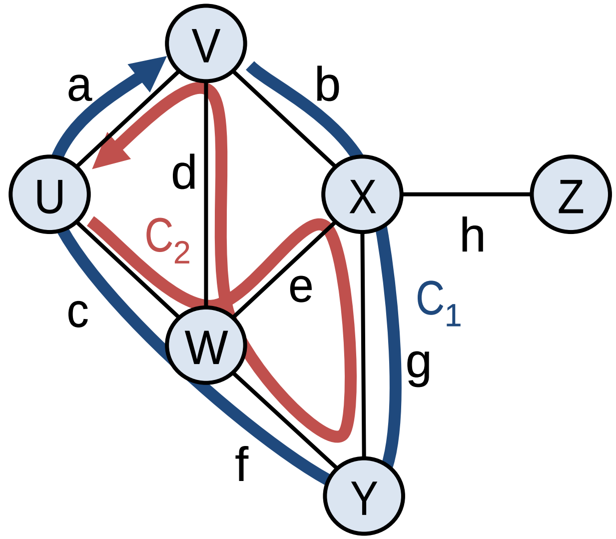


- **Path**
  - sequence of alternating vertices and edges
  - begins with a vertex
  - ends with a vertex
  - each edge is preceded and followed by its endpoints
- Simple path
  - path such that all its vertices and edges are distinct
- Examples
  - P_1=(V,b,X,h,Z) is a simple path
  - P_2=(U,c,W,e,X,g,Y,f,W,d,V) is a path that is **not** simple

- Graphs are said to be ***simple*** if they do not have parallel edges or self-loops
- Most graphs are simple; we will assume that a graph is simple unless otherwise specified

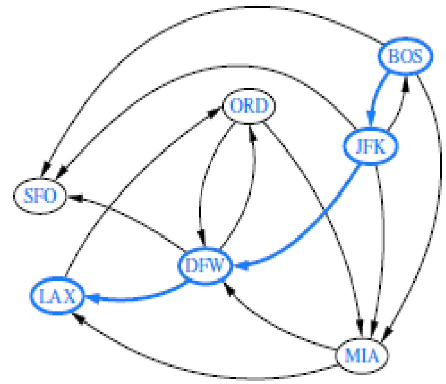
TERMINOLOGY (CONT.)

- **Cycle**
  - circular sequence of alternating vertices and edges
  - each edge is preceded and followed by its endpoints
- Simple cycle
  - cycle such that all its vertices and edges are distinct, except for the first and the last
- Examples
  - C_1=(V,b,X,g,Y,f,W,c,U,a,↵) is a simple cycle
  - C_2=(U,c,W,e,X,g,Y,f,W,d,V,a,↵) is a cycle that is **not** simple

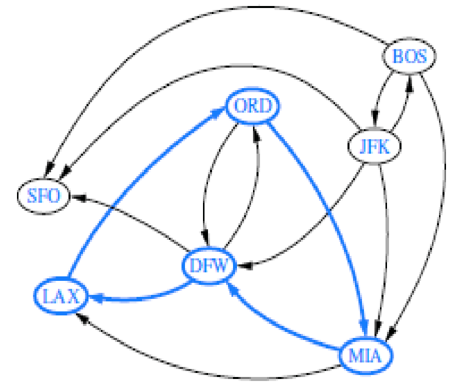


TERMINOLOGY (CONT.)

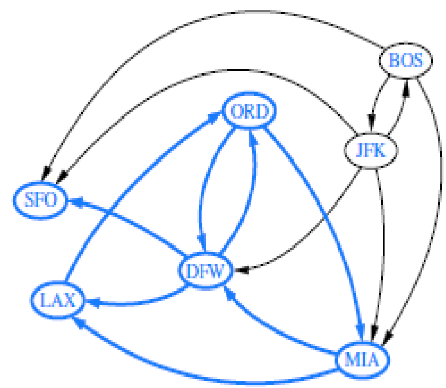
- Given vertices *u* and *v* of a (directed) graph *G*,
- *u* ***reaches*** *v*, and that *v* is ***reachable*** from *u*, if *G* has a (directed) path from *u* to *v*.
- ***reachability* :**
  - undirected graph ***reachability*** is **symmetric**, that is to say, *u* reaches *v* if an only if *v* reaches *u*.
  - directed graph ***reachability*** is **asymmetric**, it is possible that *u* reaches *v* but *v* does not reach *u*,



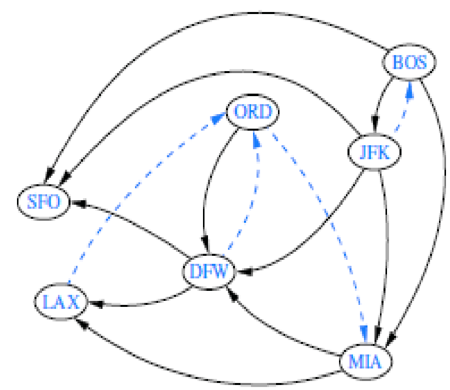
a directed path



strongly connected subgraph



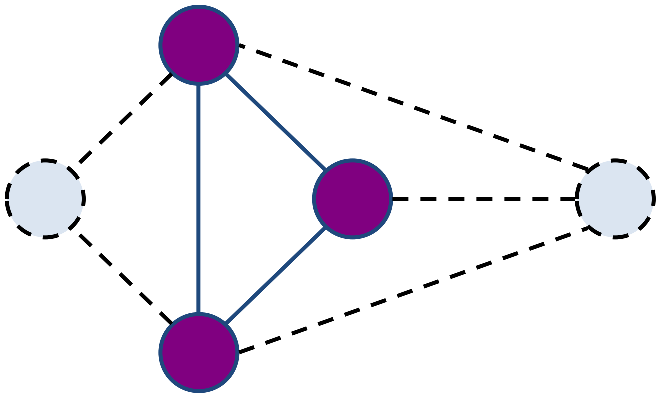
subgraph of the vertices and edges reachable from ORD



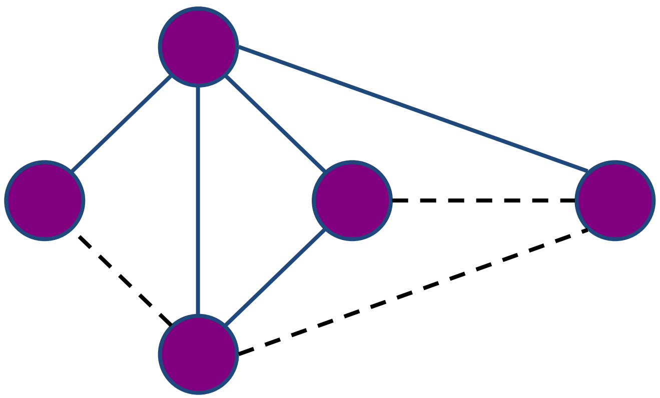
removal of the dashed edges results in a directed acyclic graph

SUBGRAPHS

- A subgraph S of a graph G is a graph such that
  - The vertices of S are a subset of the vertices of G
  - The edges of S are a subset of the edges of G
- A spanning subgraph of G is a subgraph that contains all the vertices of G



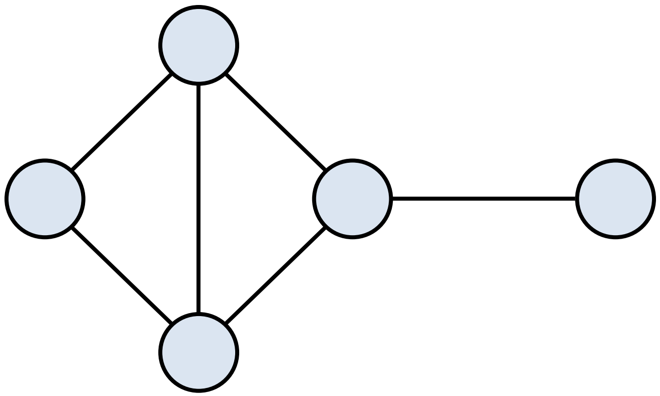
Subgraph



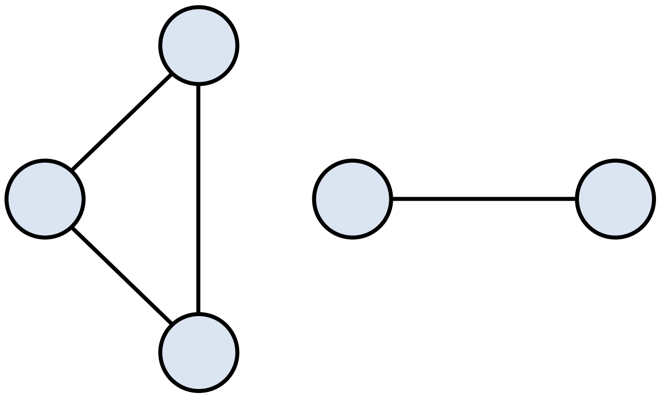
Spanning subgraph

CONNECTIVITY

- A graph is ***connected*** if, for any two vertices, there is a path between them.
- A directed graph *G* is ***strongly connected*** if for any two vertices *u* and *v* of *G*, *u* reaches *v* and *v* reaches *u*.
- A connected component of a graph G is a maximal connected subgraph of G



Connected graph

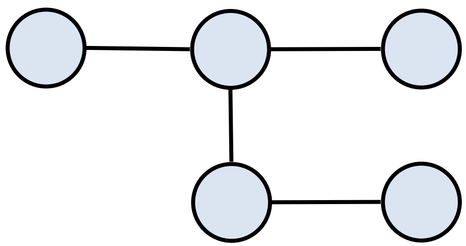


Non connected graph with two connected components

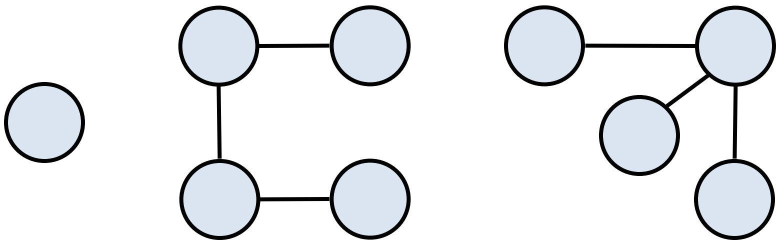
TREES AND FORESTS

- A (free) **tree** is an undirected graph T such that
  - T is connected
  - T has no cycles

  This definition of tree is different from the one of a rooted tree
- A **forest** is an undirected graph without cycles
- The connected components of a forest are trees



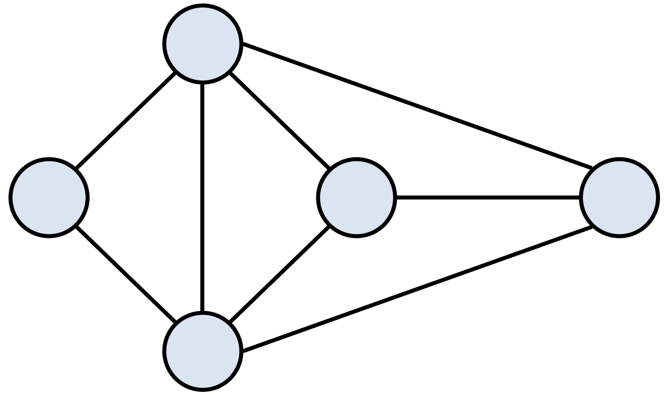
Tree



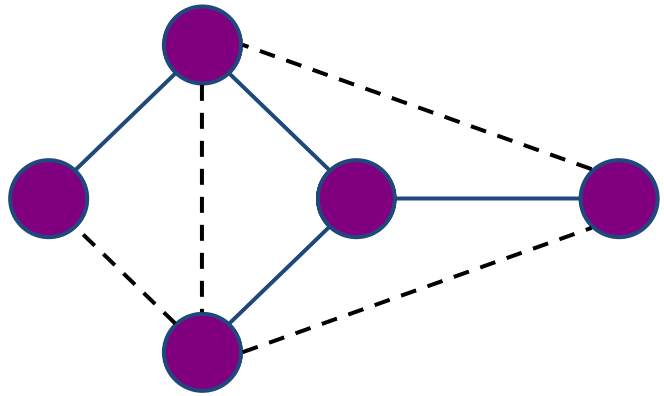
Forest

SPANNING TREES AND FORESTS

- A **spanning tree** of a connected graph is a spanning subgraph that is a tree
- A spanning tree is not unique unless the graph is a tree
- A spanning forest of a graph is a spanning subgraph that is a forest



Graph



Spanning tree

PROPERTIES

Property 1: If *G* is a graph with *m* edges and vertex set *V*, then

$$\sum_{v \text{ in } V} \deg(v) = 2m$$

Proof: each edge is counted twice

Property 2: If *G* is a directed graph with *m* edges and vertex set *V*, then

$$\sum_{v \text{ in } V} \text{indeg}(v) = \sum_{v \text{ in } V} \text{outdeg}(v) = m$$

Property 3: Let *G* be a simple graph with *n* vertices and *m* edges. If *G* is undirected, then

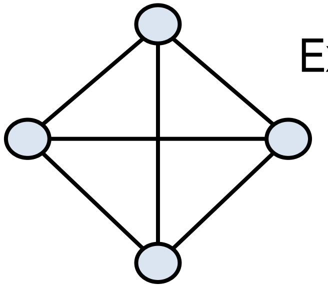
$$m \le n(n-1)/2$$

Proof: each vertex has degree at most $(n-1)$

=> A simple graph with *n* vertices has $O(n^2)$ edges.

Notation

n	number of vertices
m	number of edges
$\deg(v)$	degree of vertex v



Example

- $n = 4$
- $m = 6$
- $\deg(v) = 3$

Let *G* be an undirected graph

- If *G* is connected, then $m \ge n-1$.
- If *G* is a tree, then $m = n-1$.
- If *G* is a forest, then $m \le n-1$.

VERTICES AND EDGES

- A **graph** is a collection of **vertices** and **edges**.
- We model the abstraction as a combination of three data types: Vertex, Edge, and Graph.
- A **Vertex** is a lightweight object that stores an arbitrary element provided by the user (e.g., an airport code)
  - We assume it supports a method, element(), to retrieve the stored element.
- An **Edge** stores an associated object (e.g., a flight number, travel distance, cost), retrieved with the element( ) method.

GRAPH ADT

either *undirected* or *directed*

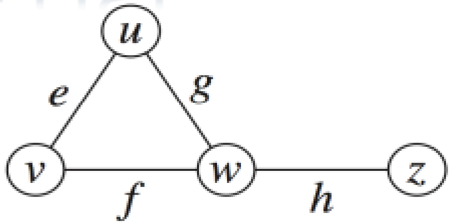
numVertices( ):	Returns the number of vertices of the graph.
vertices( ):	Returns an iteration of all the vertices of the graph.
numEdges( ):	Returns the number of edges of the graph.
edges( ):	Returns an iteration of all the edges of the graph.
getEdge(u, v):	Returns the edge from vertex u to vertex v, if one exists; otherwise return null. For an undirected graph, there is no difference between getEdge(u, v) and getEdge(v, u).
endVertices(e):	Returns an array containing the two endpoint vertices of edge e. If the graph is directed, the first vertex is the origin and the second is the destination.
opposite(v, e):	For edge e incident to vertex v, returns the other vertex of the edge; an error occurs if e is not incident to v.
outDegree(v):	Returns the number of outgoing edges from vertex v.
inDegree(v):	Returns the number of incoming edges to vertex v. For an undirected graph, this returns the same value as does outDegree(v).
outgoingEdges(v):	Returns an iteration of all outgoing edges from vertex v.
incomingEdges(v):	Returns an iteration of all incoming edges to vertex v. For an undirected graph, this returns the same collection as does outgoingEdges(v).
insertVertex(x):	Creates and returns a new Vertex storing element x.
insertEdge(u, v, x):	Creates and returns a new Edge from vertex u to vertex v, storing element x; an error occurs if there already exists an edge from u to v.
removeVertex(v):	Removes vertex v and all its incident edges from the graph.
removeEdge(e):	Removes edge e from the graph.

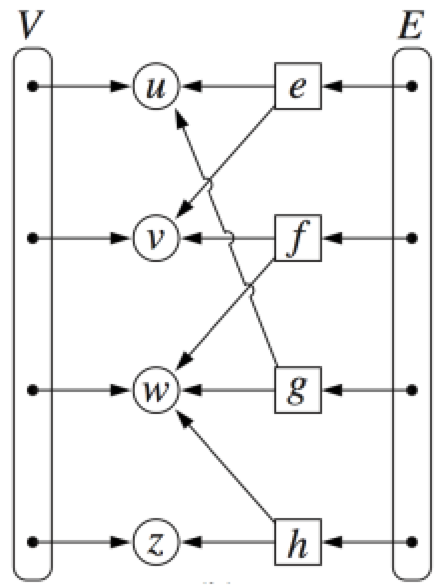
DATA STRUCTURES FOR GRAPHS

- In an ***edge list***, we maintain an unordered list of all edges.
  - This minimally suffices, but there is no efficient way to locate a particular edge (u,v), or the set of all edges incident to a vertex v.
- In an ***adjacency list***, we additionally maintain, for each vertex, a separate list containing those edges that are incident to the vertex.
  - This organization allows us to more efficiently find all edges incident to a given vertex.
- An ***adjacency map*** is similar to an adjacency list, but the secondary container of all edges incident to a vertex is organized as a map, rather than as a list, with the adjacent vertex serving as a key.
  - This allows more efficient access to a specific edge (u,v), for example, in $O(1)$ expected time with hashing.
- An ***adjacency matrix*** provides worst-case $O(1)$ access to a specific edge (u,v) by maintaining an $n \times n$ matrix, for a graph with n vertices.
  - Each slot is dedicated to storing a reference to the edge (u,v) for a particular pair of vertices u and v; if no such edge exists, the slot will store null.

DATA STRUCTURES FOR GRAPHS: EDGE LIST

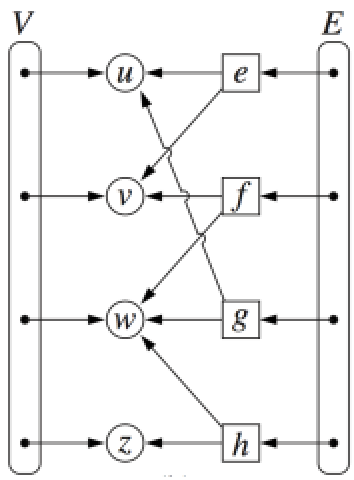
- All vertex objects are stored in an unordered list *V*, and all edge objects are stored in an unordered list *E*.
- Components:
  - Vertex object
    - reference to element v, to support getElement()
    - reference to position in vertex sequence for efficiently removed
  - Edge object
    - reference to element e, to support getElement()
    - References to the origin vertex object & destination vertex object, to support endVertices(e) and opposite e).
    - reference to position in edge sequence sequence for efficiently removed
  - Vertex sequence
    - sequence of vertex objects
  - Edge sequence
    - sequence of edge objects





space usage is $O(n + m)$

PERFORMANCE OF THE EDGE LIST STRUCTURE



space usage is $O(n+m)$

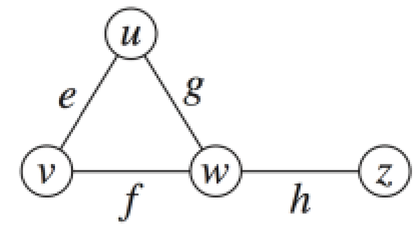
Method	Running Time
numVertices( ), numEdges( )	$O(1)$
vertices( )	$O(n)$
edges( )	$O(m)$
getEdge(u, v), outDegree(v), outgoingEdges(v)	$O(m)$
insertVertex(x), insertEdge(u, v, x), removeEdge(e)	$O(1)$
removeVertex(v)	$O(m)$

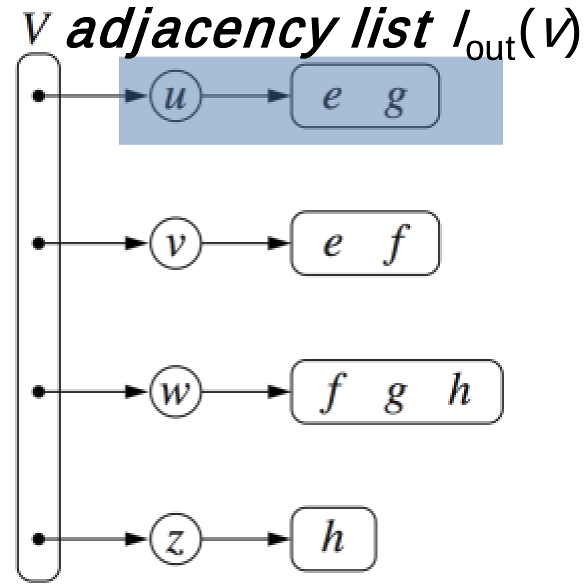
Exhaustive inspection of all edges needed.

when a vertex v is removed from the graph, all edges incident to v must also be removed

DATA STRUCTURES FOR GRAPHS: ADJACENCY LIST

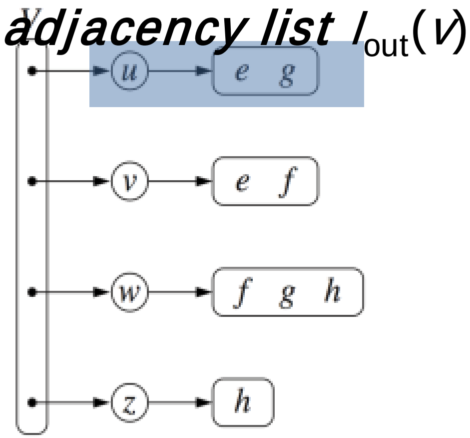
- Adds extra information to the edge list structure that supports direct access to the incident edges
  - For each vertex *v*, we maintain a collection *I*(*v*), called ***incidence collection*** of *v*
- Components:
  - Incidence sequence for each vertex
    - sequence of references to edge objects of incident edges
  - Augmented edge objects
    - references to associated positions in incidence sequences of end vertices





positional list to represent *V*

PERFORMANCE OF THE ADJACENCY LIST STRUCTURE



assuming that the primary collection V and E, and all secondary collections $I(v)$ are implemented with doubly linked lists.

using $O(n+m)$ space

Method	Running Time
numVertices( ), numEdges( )	$O(1)$
vertices( )	$O(n)$
edges( )	$O(m)$
getEdge(u, v)	$O(\min(\deg(u), \deg(v)))$
outDegree(v), inDegree(v)	$O(1)$
outgoingEdges(v), incomingEdges(v)	$O(\deg(v))$
insertVertex(x), insertEdge(u, v, x)	$O(1)$
removeEdge(e)	$O(1)$
removeVertex(v)	$O(\deg(v))$

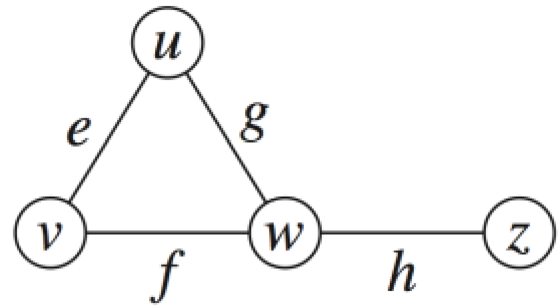
search through either $I(u)$ or $I(v)$

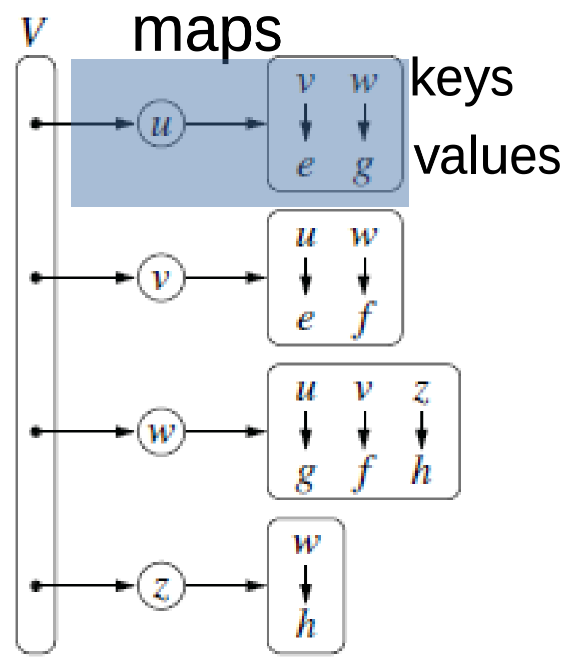
based on use of $I(v)$.

DATA STRUCTURES FOR GRAPHS: ADJACENCY MAP

- use a hash-based map to implement $I(v)$ for each vertex v.
- let the opposite endpoint of each incident edge serve as a key in the map, with the edge structure serving as the value
- getEdge(u, v) method can be implemented in ***expected*** $O(1)$ time

space usage is $O(n+m)$

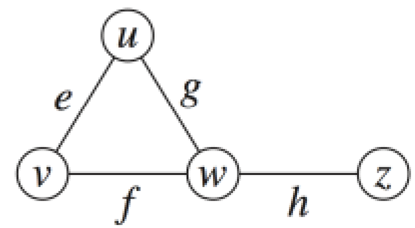


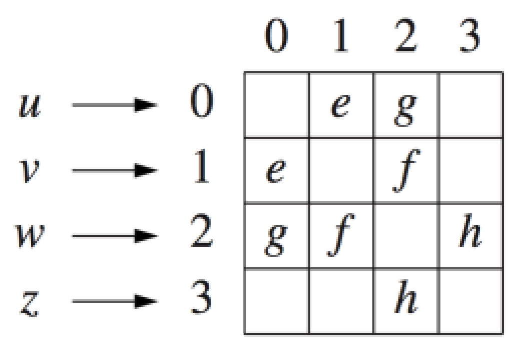


DATA STRUCTURES FOR GRAPHS: ADJACENCY MATRIX

- *adjacency matrix* *A* allows us to locate an edge between a given pair of vertices in ***worst-case*** **O(1) time**.
- cell *A*[*i*][*j*] holds a reference to the edge (*u*,*v*), if it exists, where *u* is the vertex with index *i* and *v* is the vertex with index *j*
- Edge list structure
- Augmented vertex objects
  - Integer key (index) associated with vertex
- 2D-array adjacency array
  - Reference to edge object for adjacent verti ces
  - Null for non nonadjacent vertices
- The “old fashioned” version just has 0 for no e dge and 1 for edge

$O(n^2)$ space usage





PERFORMANCE: SIMPLE GRAPH

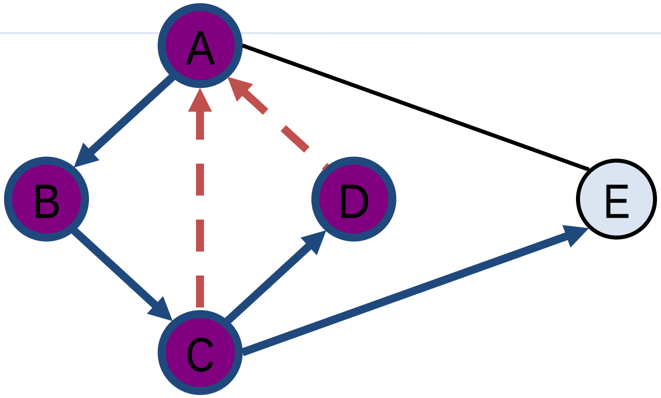
Method	Edge List	Adj. List	Adj. Map	Adj. Matrix
numVertices( )	$O(1)$	$O(1)$	$O(1)$	$O(1)$
numEdges( )	$O(1)$	$O(1)$	$O(1)$	$O(1)$
vertices( )	$O(n)$	$O(n)$	$O(n)$	$O(n)$
edges( )	$O(m)$	$O(m)$	$O(m)$	$O(m)$
getEdge(u, v)	$O(m)$	$O(\min(d_u, d_v))$	$O(1)$ exp.	$O(1)$
outDegree(v) inDegree(v)	$O(m)$	$O(1)$	$O(1)$	$O(n)$
outgoingEdges(v) incomingEdges(v)	$O(m)$	$O(d_v)$	$O(d_v)$	$O(n)$
insertVertex(x)	$O(1)$	$O(1)$	$O(1)$	$O(n^2)$
removeVertex(v)	$O(m)$	$O(d_v)$	$O(d_v)$	$O(n^2)$
insertEdge(u, v, x)	$O(1)$	$O(1)$	$O(1)$ exp.	$O(1)$
removeEdge(e)	$O(1)$	$O(1)$	$O(1)$ exp.	$O(1)$

adjacency matrix uses $O(n^2)$ space, while all other structures use $O(n+m)$ space

JAVA IMPLEMENTATION OF *ADJACENCY MAP*

- Positional lists to represent each of the primary lists *V* and *E*
- use a hash-based map to represent the secondary incidence map *I*(*v*) for each vertex *v* in V
  - each vertex maintains two different map references: outgoing and incoming.
  - Directed graphs: initialized to two distinct map instances, representing $I_{out}(v)$ and $I_{in}(v)$, respectively.
  - Undirected graph: assign both outgoing and incoming as aliases to a single map instance.
- For details of the code: please look at the book.

GRAPH TRAVERSALS:
DEPTH-FIRST SEARCH



GRAPH TRAVERSAL

- A *traversal* is a systematic procedure for exploring a graph by examining all of its vertices and edges.
- A traversal is efficient if it visits all the vertices and edges in time proportional to their number, that is, in linear time.
- We will look at two efficient graph traversal algorithms
  - *depth-first search (DFS)*
  - *breadth-first search (BFS)*

DEPTH-FIRST SEARCH

- A DFS traversal of a graph G
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G
- The DFS process naturally identifies what is known as the ***depth-first search tree*** rooted at a starting vertex s.
- DFS on a graph with n vertices and m edges takes $O(n+m)$ time
- DFS can be further extended to solve other graph problems
  - Find and report a path between two given vertices
  - Find a cycle in the graph
- Depth-first search is to graphs what Euler tour is to binary trees

DFS ALGORITHM FROM A VERTEX

Algorithm DFS(G, u):

Input: A graph G and a vertex u of G

Output: A collection of vertices reachable from u, with their discovery edges

Mark vertex u as visited.

for each of u's outgoing edges, $e = (u, v)$ **do**

  if vertex v has not been visited **then**

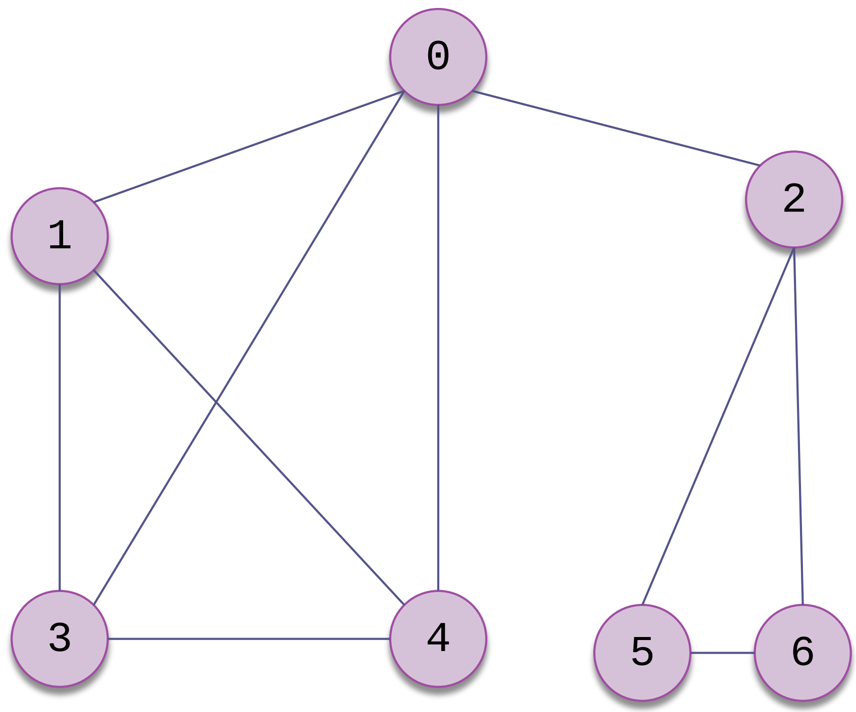
    Record edge e as the discovery edge for vertex v.

    Recursively call DFS(G, v).

JAVA IMPLEMENTATION

```
/** Performs depth-first search of Graph g starting at Vertex u. */
public static <V,E> void DFS(Graph<V,E> g, Vertex<V> u,
                Set<Vertex<V>> known, Map<Vertex<V>,Edge<E>> forest) {
  known.add(u);                                  // u has been discovered
  for (Edge<E> e : g.outgoingEdges(u)) {         // for every outgoing edge from u
    Vertex<V> v = g.opposite(u, e);
    if (!known.contains(v)) {
      forest.put(v, e);                          // e is the tree edge that discovered v
      DFS(g, v, known, forest);                  // recursively explore from v
    }
  }
}
```

Example of a Depth-First Search

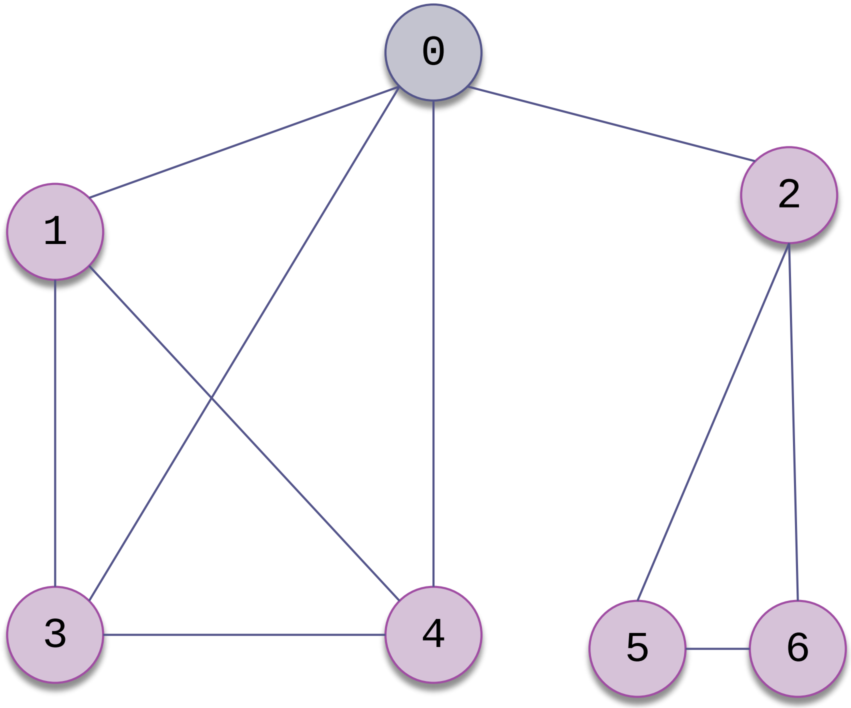


Example of a Depth-First Search (cont.)

Mark 0 as being visited

Discovery (Visit) order:
0

Finish order:

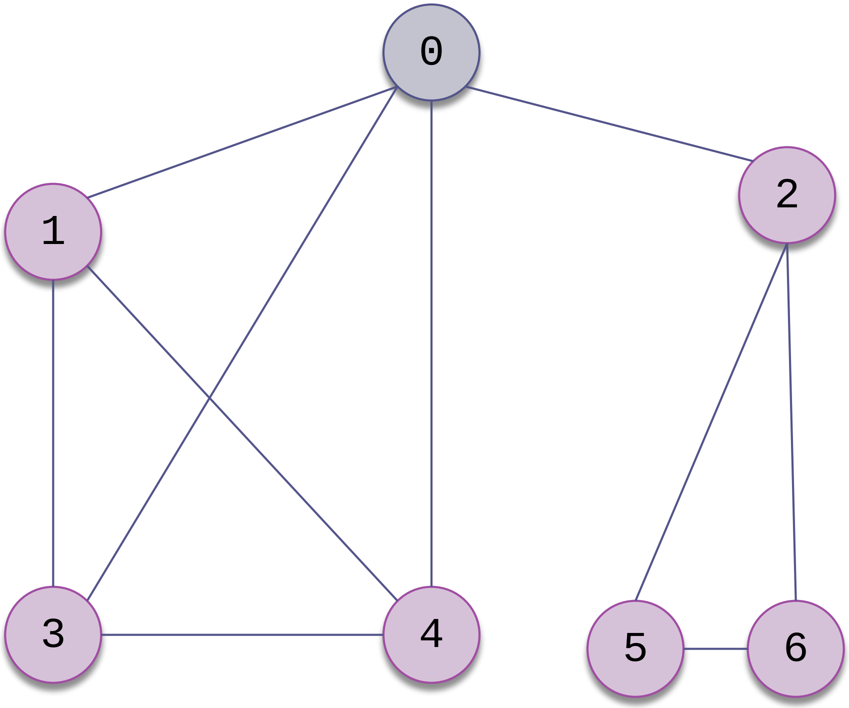


Example of a Depth-First Search (cont.)

Choose an adjacent vertex that is not being visited

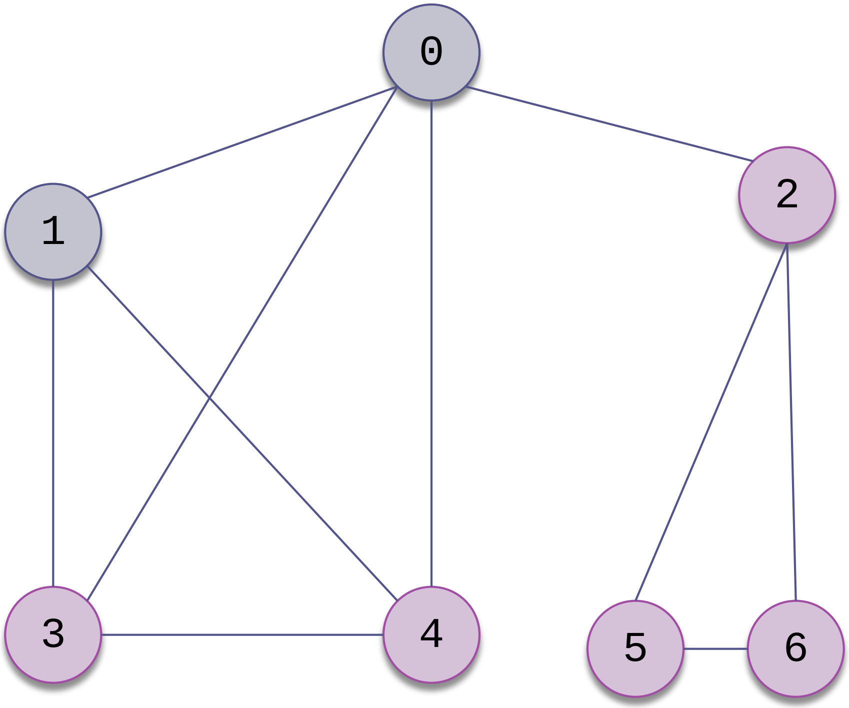
Discovery (Visit) order:
0

Finish order:



Example of a Depth-First Search (cont.)

Choose an adjacent vertex that is not being visited



Discovery (Visit) order:
0, 1

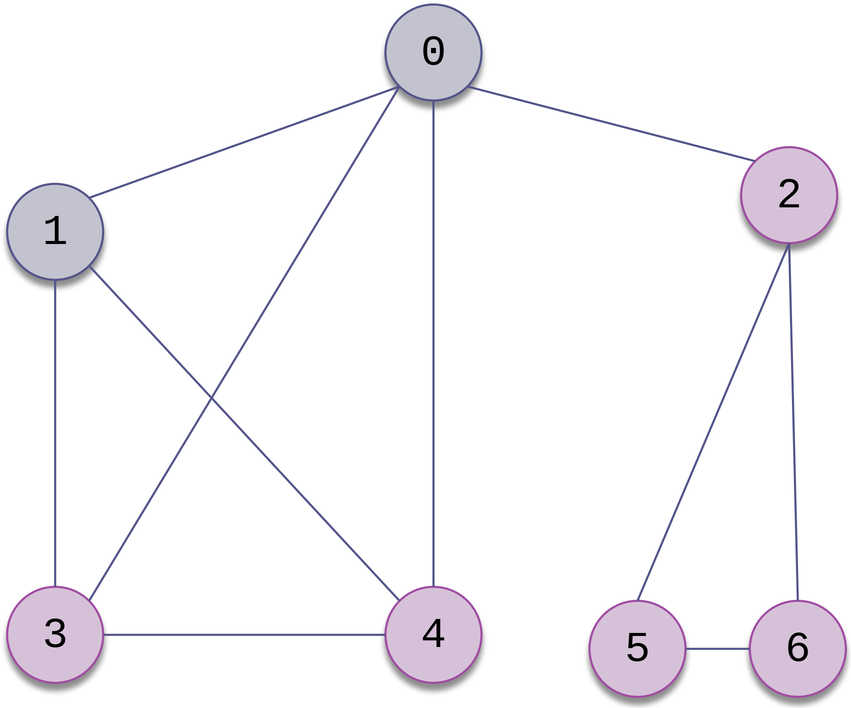
Finish order:

Example of a Depth-First Search (cont.)

(Recursively) choose an adjacent vertex that is not being visited

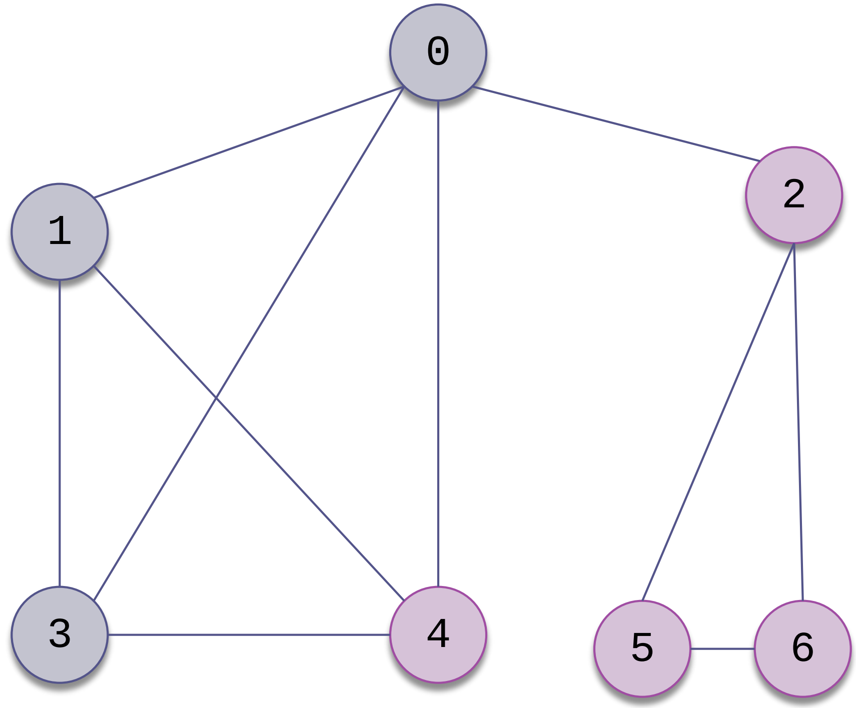
Discovery (Visit) order:
0, 1, 3

Finish order:



Example of a Depth-First Search (cont.)

(Recursively) choose an adjacent vertex that is not being visited



Discovery (Visit) order:
0, 1, 3

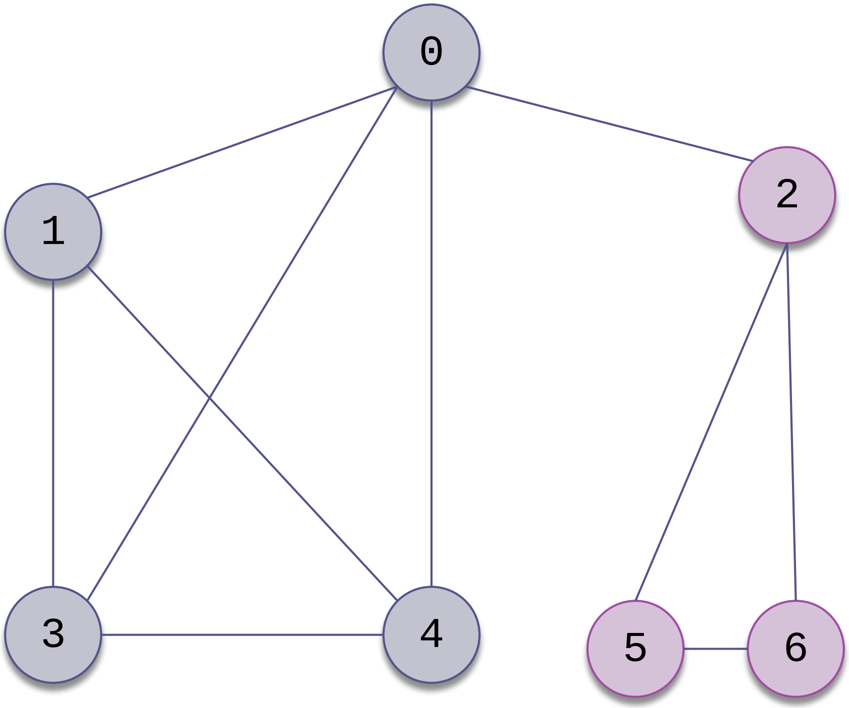
Finish order:

Example of a Depth-First Search (cont.)

(Recursively) choose an adjacent vertex that is not being visited

Discovery (Visit) order:
0, 1, 3, 4

Finish order:

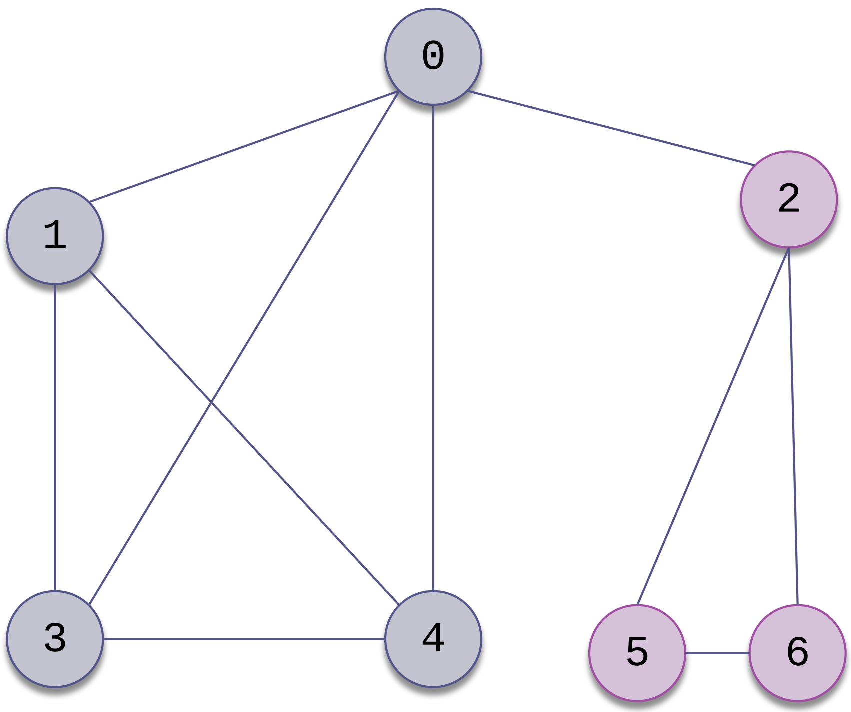


Example of a Depth-First Search (cont.)

There are no vertices adjacent to 4 that are not being visited

Discovery (Visit) order:
0, 1, 3, 4

Finish order:

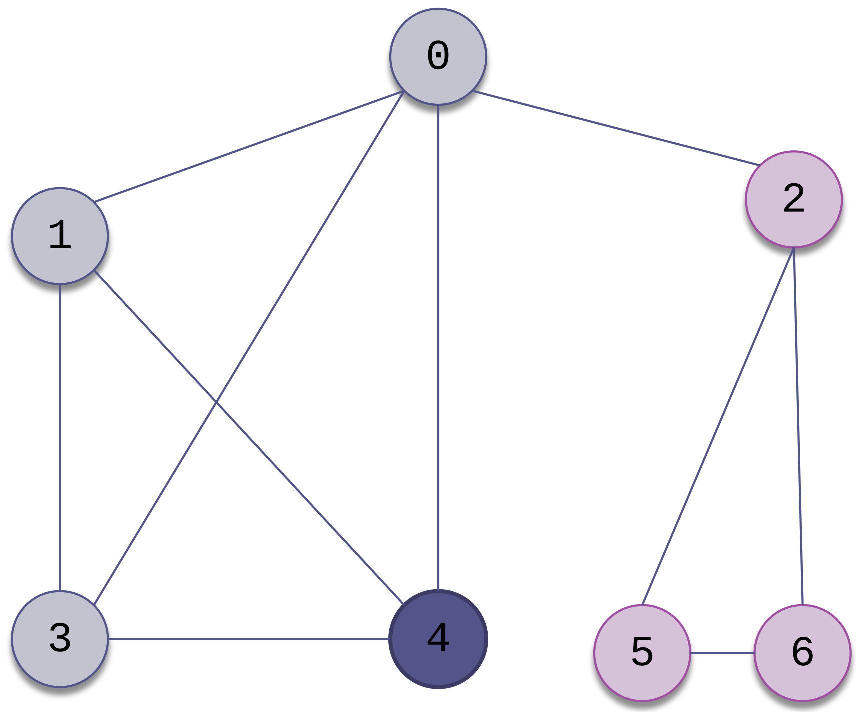


Example of a Depth-First Search (cont.)

Mark 4 as visited

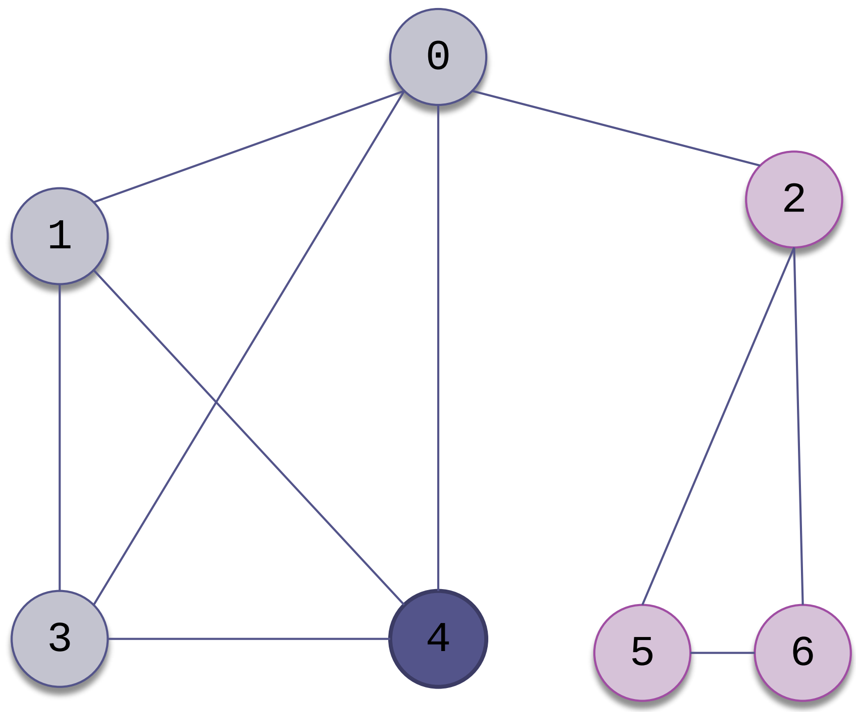
Discovery (Visit) order:
0, 1, 3, 4

Finish order:
4



Example of a Depth-First Search (cont.)

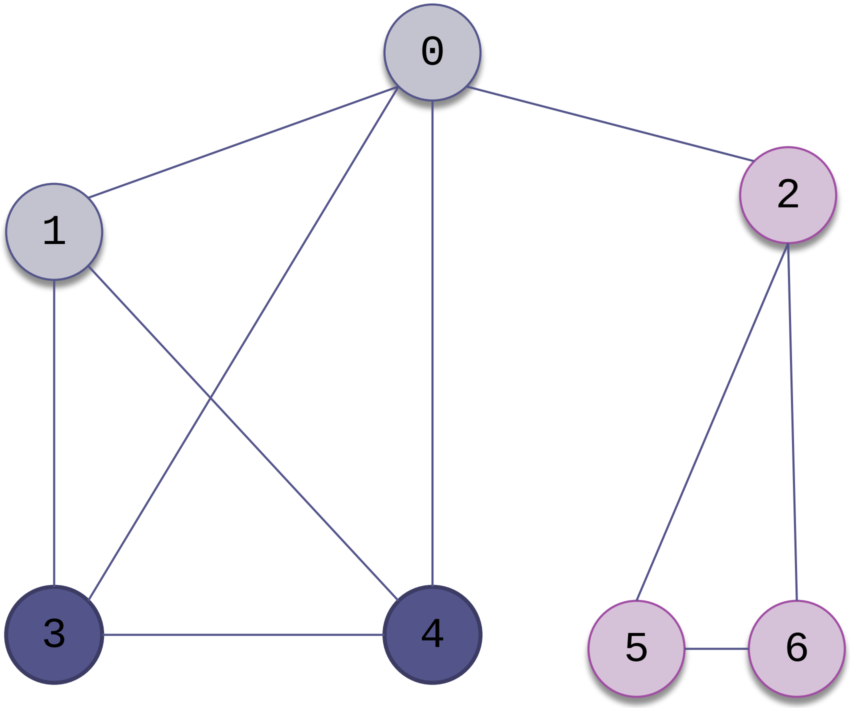
Return from the recursion to 3; all adjacent nodes to 3 are being visited



Finish order:
4

Example of a Depth-First Search (cont.)

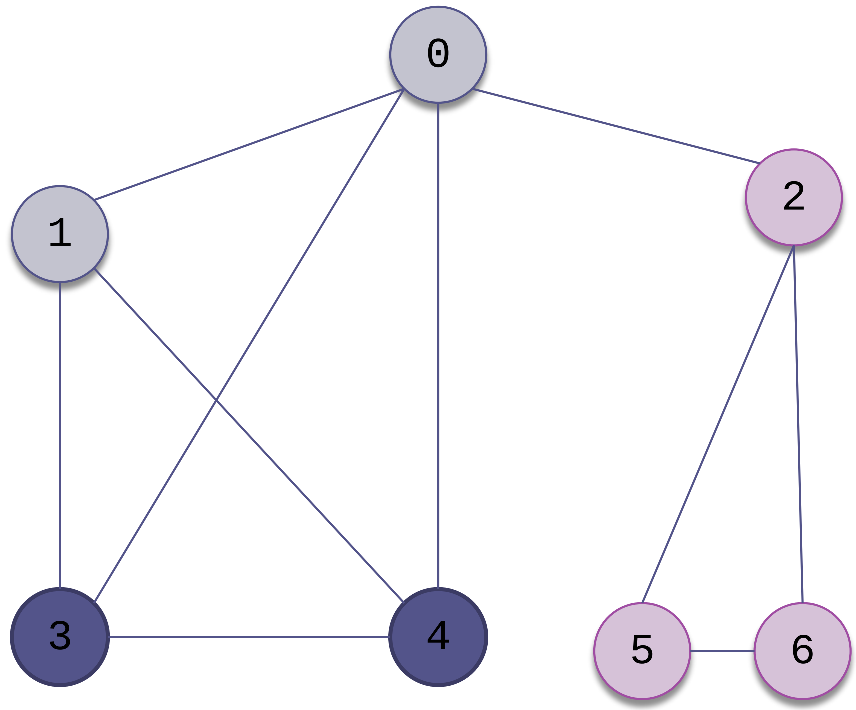
Mark 3 as visited



Finish order:
4, 3

Example of a Depth-First Search (cont.)

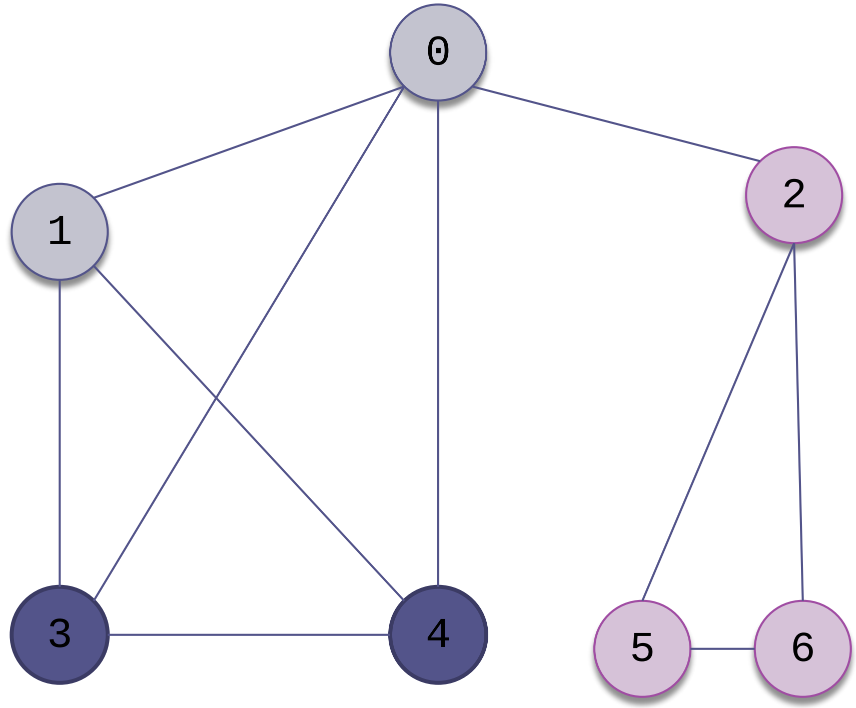
Return from the recursion to 1



Finish order:
4, 3

Example of a Depth-First Search (cont.)

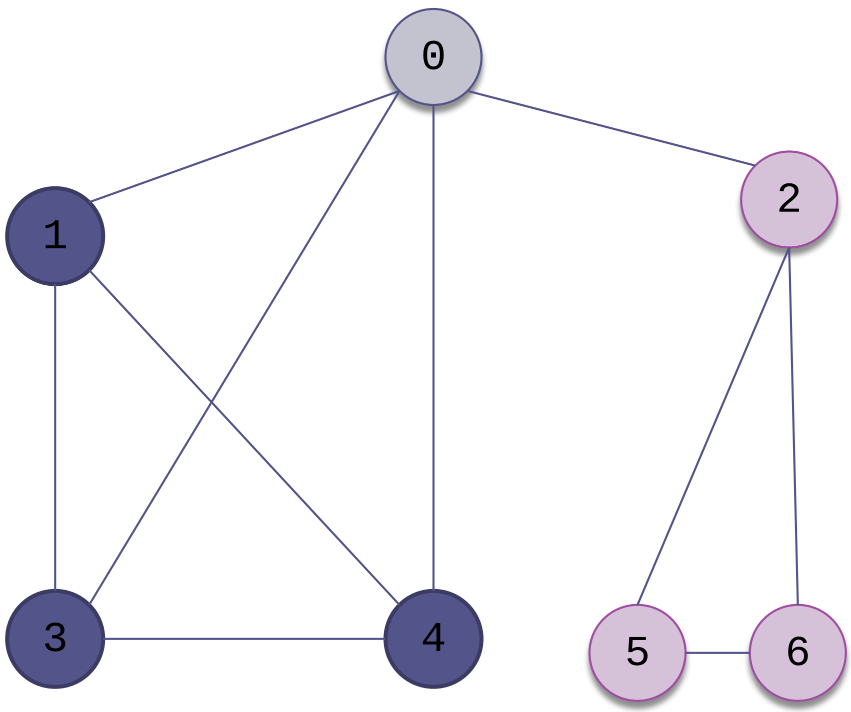
All vertices adjacent to 1 are being visited



Finish order:
4, 3

Example of a Depth-First Search (cont.)

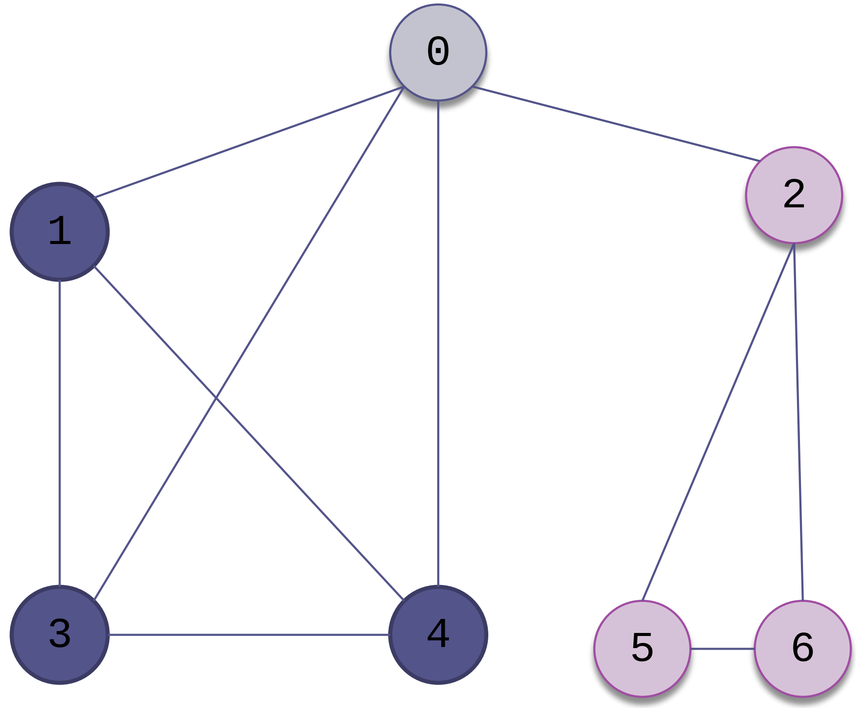
Mark 1 as visited



Finish order:
4, 3, 1

Example of a Depth-First Search (cont.)

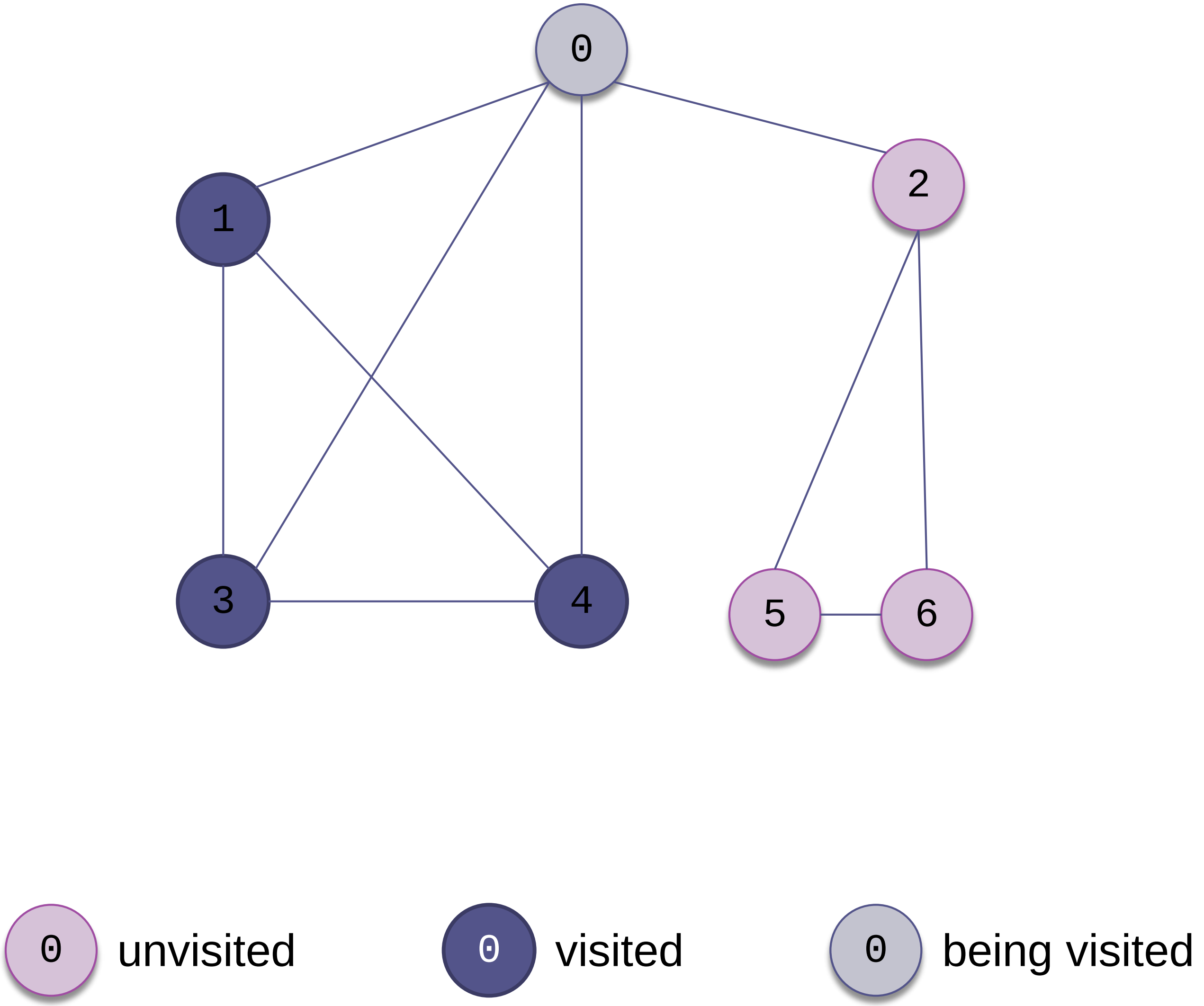
Return from the recursion to 0



Finish order:
4, 3, 1

Example of a Depth-First Search (cont.)

2 is adjacent to 1 and is not being visited



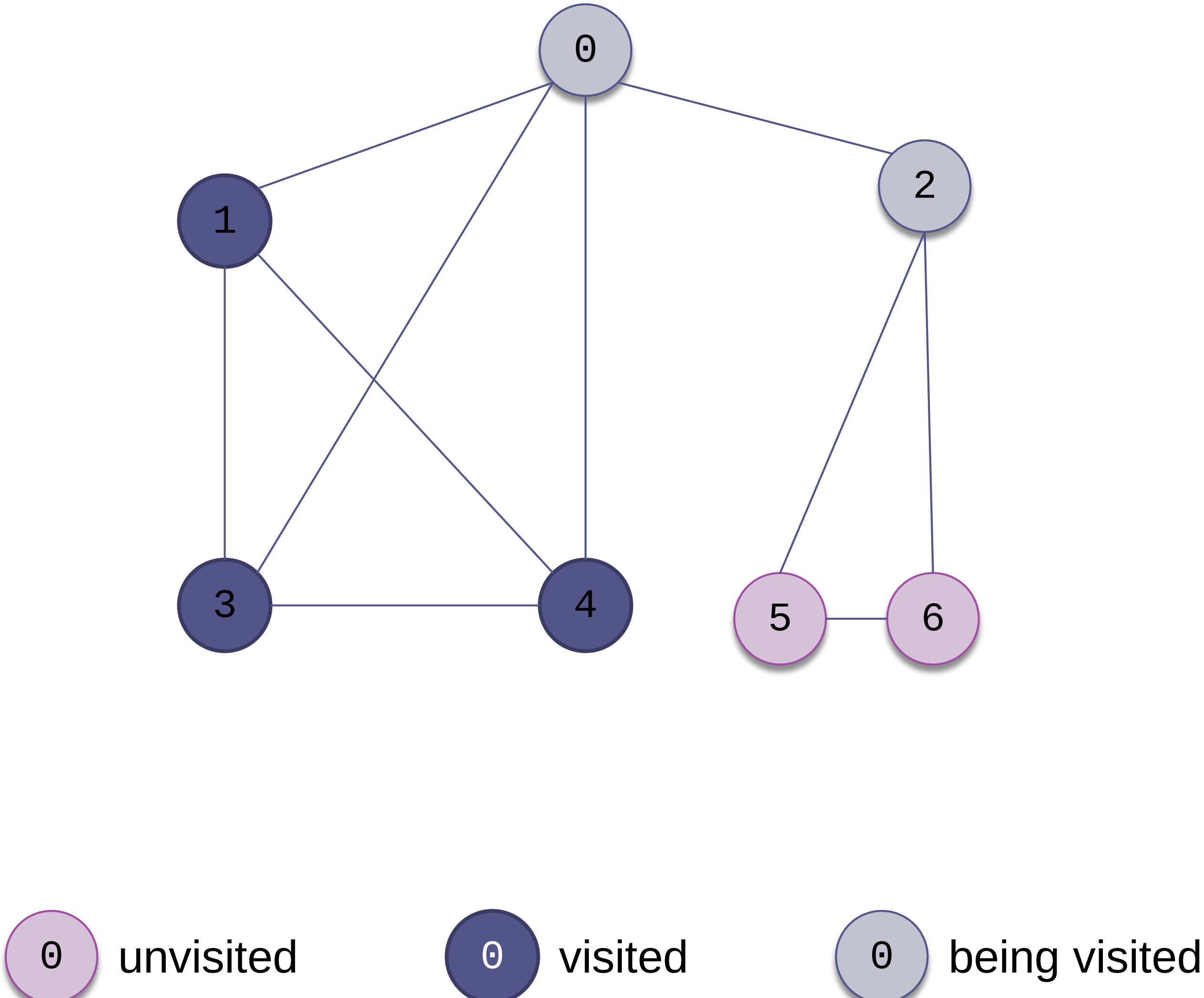
Finish order:
4, 3, 1

Example of a Depth-First Search (cont.)

2 is adjacent to 1 and is not being visited

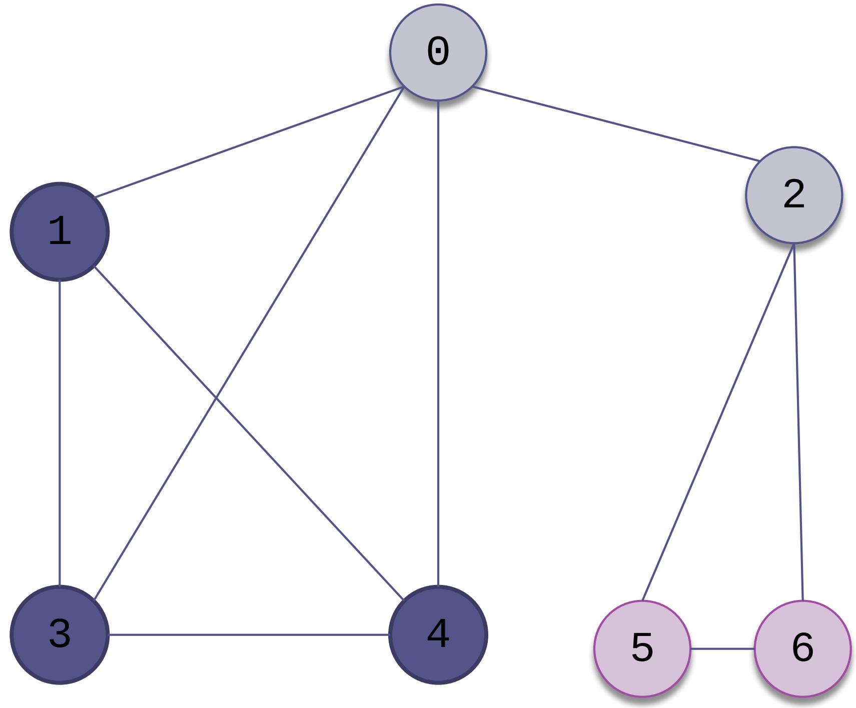
Discovery (Visit) order:
0, 1, 3, 4, 2

Finish order:
4, 3, 1



Example of a Depth-First Search (cont.)

5 is adjacent to 2 and is not being visited

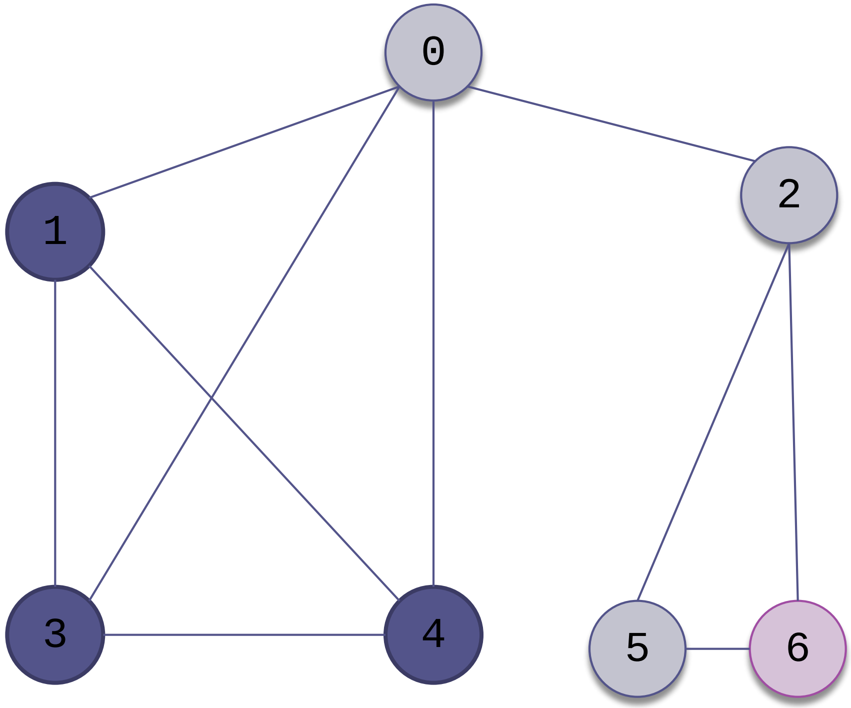


Discovery (Visit) order:
0, 1, 3, 4, 2

Finish order:
4, 3, 1

Example of a Depth-First Search (cont.)

5 is adjacent to 2 and is not being visited

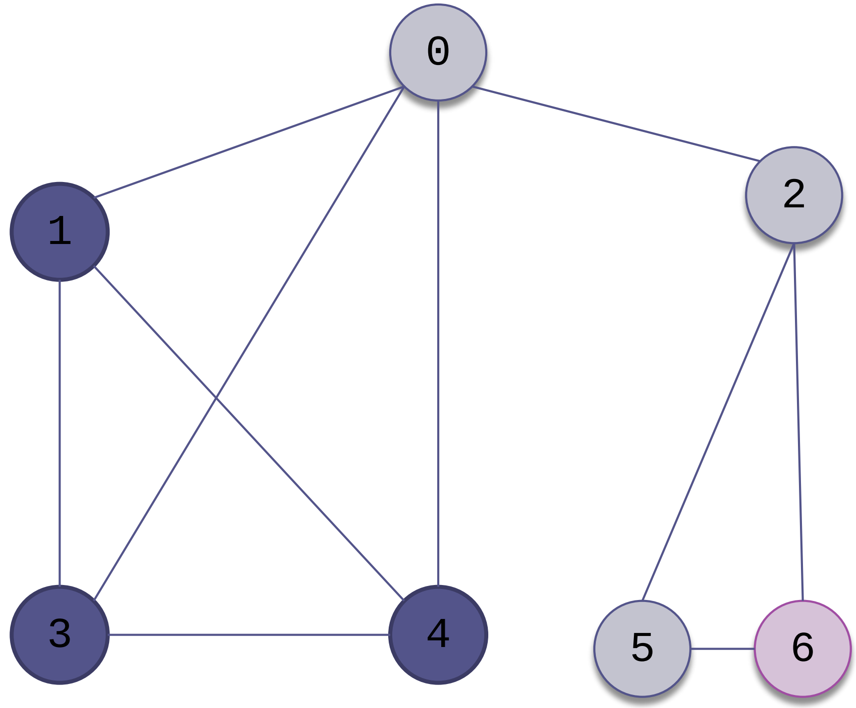


Discovery (Visit) order:
0, 1, 3, 4, 2, 5

Finish order:
4, 3, 1

Example of a Depth-First Search (cont.)

6 is adjacent to 5 and is not being visited

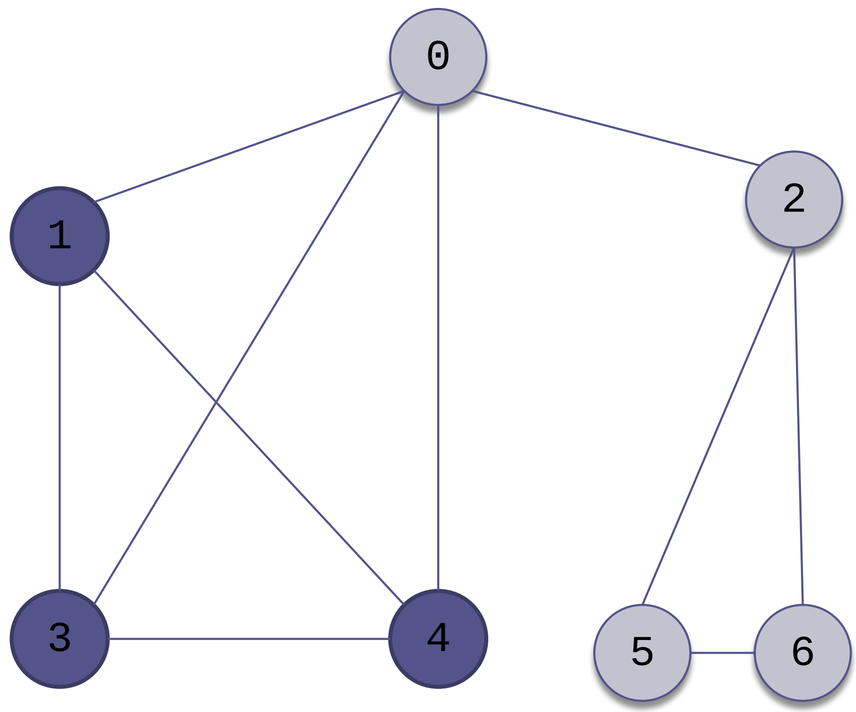


Discovery (Visit) order:
0, 1, 3, 4, 2, 5

Finish order:
4, 3, 1

Example of a Depth-First Search (cont.)

6 is adjacent to 5 and is not being visited

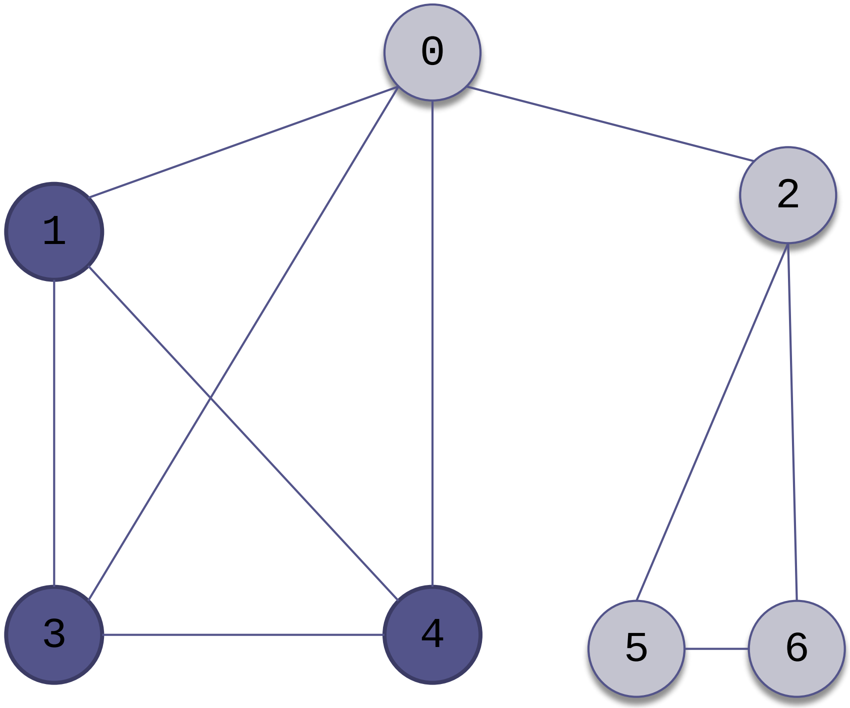


Discovery (Visit) order:
0, 1, 3, 4, 2, 5, 6

Finish order:
4, 3, 1

Example of a Depth-First Search (cont.)

There are no vertices adjacent to 6 not being visited; mark 6 as visited

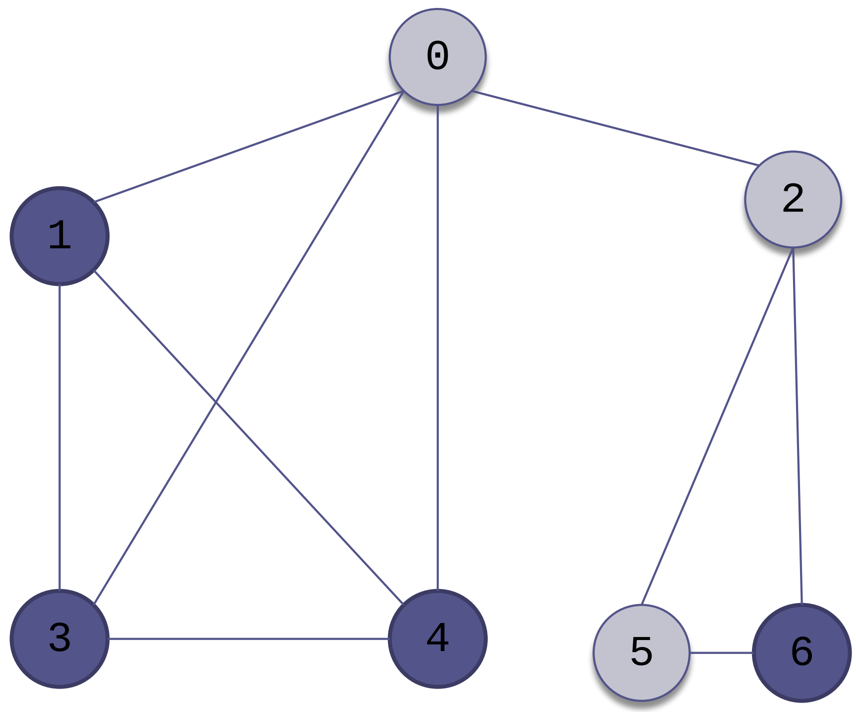


Discovery (Visit) order:
0, 1, 3, 4, 2, 5, 6

Finish order:
4, 3, 1

Example of a Depth-First Search (cont.)

There are no vertices adjacent to 6 not being visited; mark 6 as visited

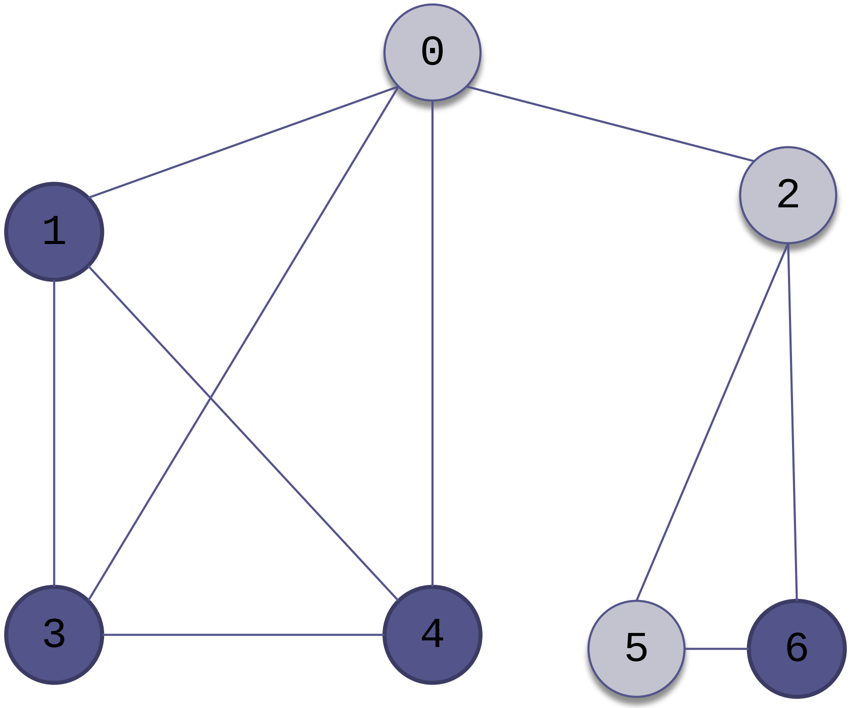


Discovery (Visit) order:
0, 1, 3, 4, 2, 5, 6

Finish order:
4, 3, 1, 6

Example of a Depth-First Search (cont.)

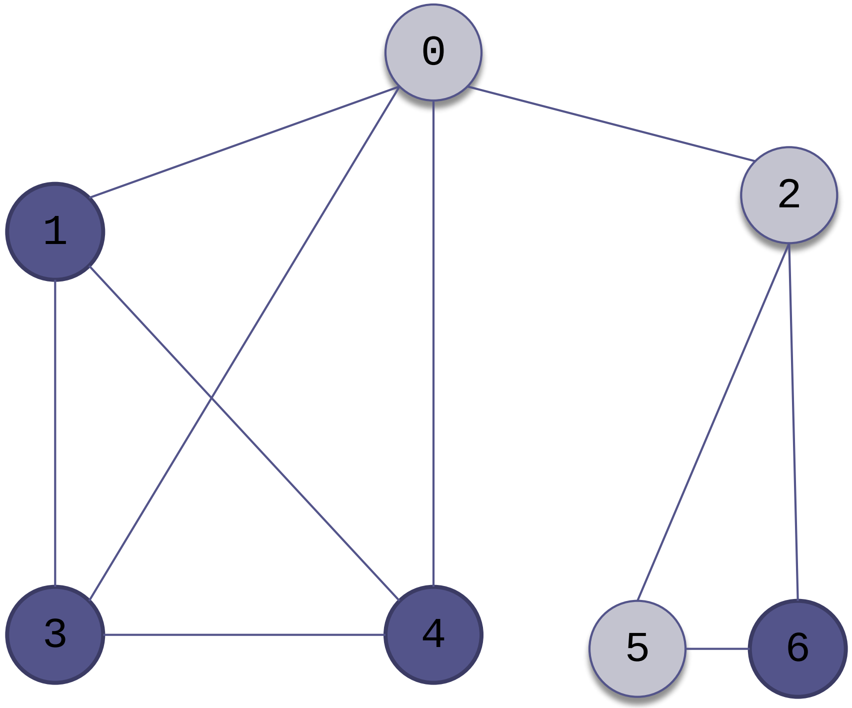
Return from the recursion to 5



Finish order:
4, 3, 1, 6

Example of a Depth-First Search (cont.)

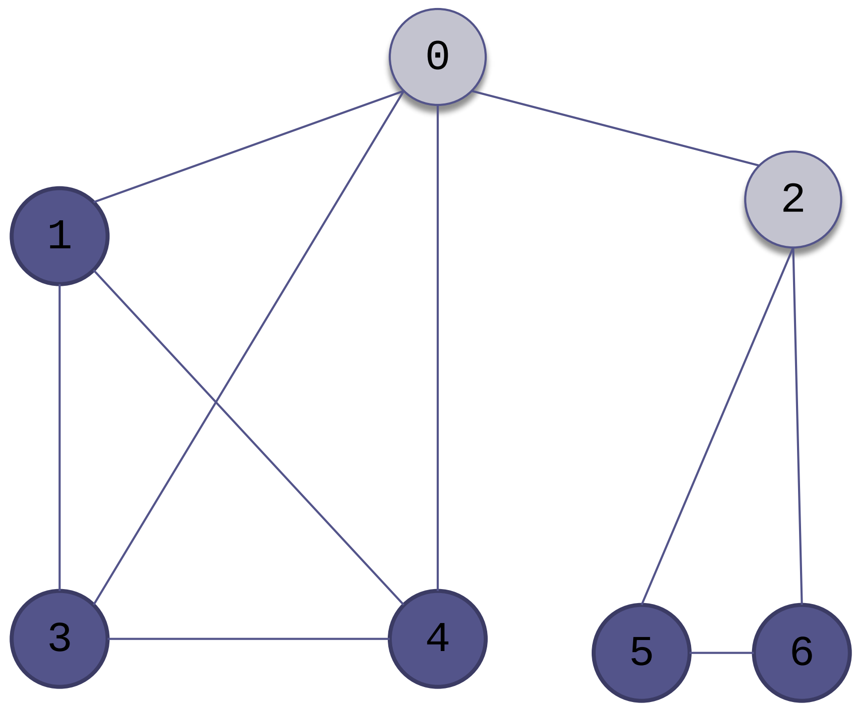
Mark 5 as visited



Finish order:
4, 3, 1, 6

Example of a Depth-First Search (cont.)

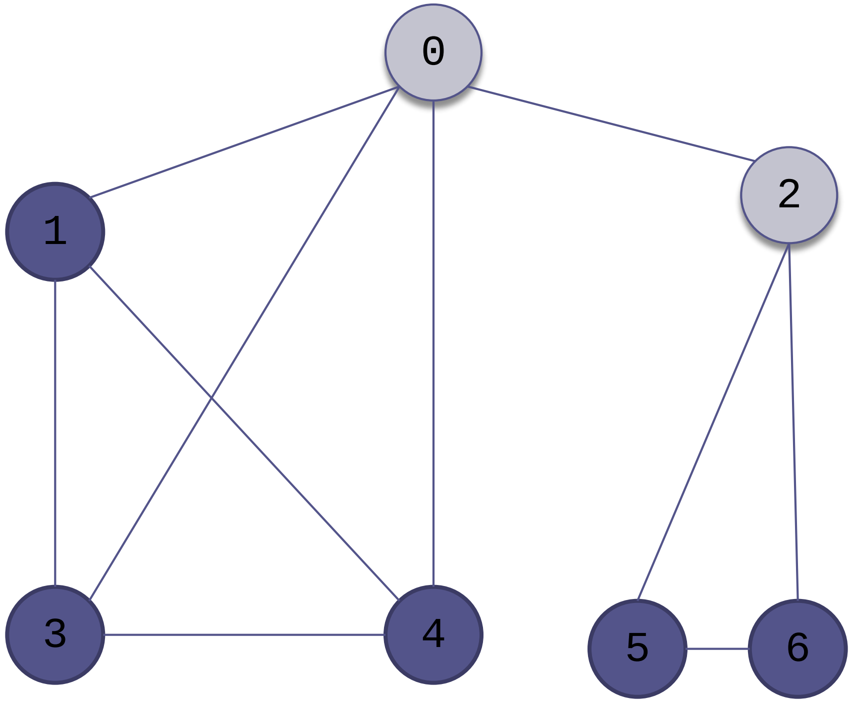
Mark 5 as visited



Finish order:
4, 3, 1, 6, 5

Example of a Depth-First Search (cont.)

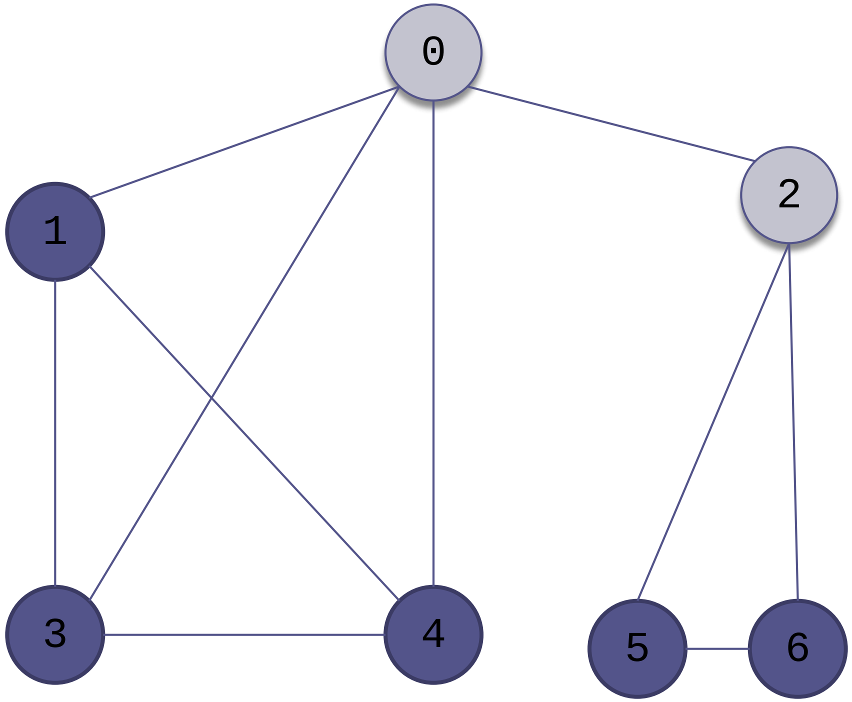
Return from the recursion to 2



Finish order:
4, 3, 1, 6, 5

Example of a Depth-First Search (cont.)

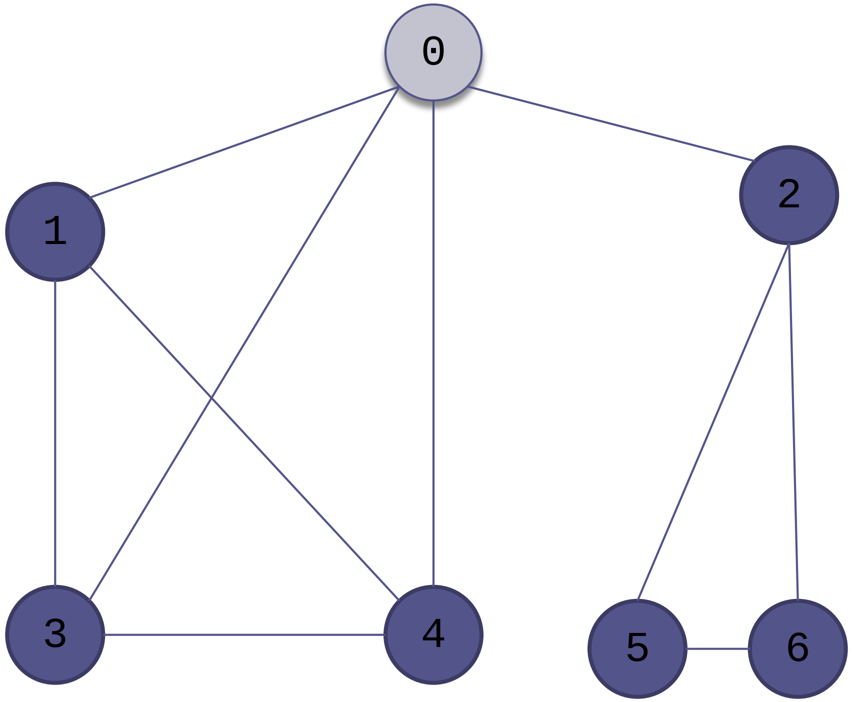
Mark 2 as visited



Finish order:
4, 3, 1, 6, 5

Example of a Depth-First Search (cont.)

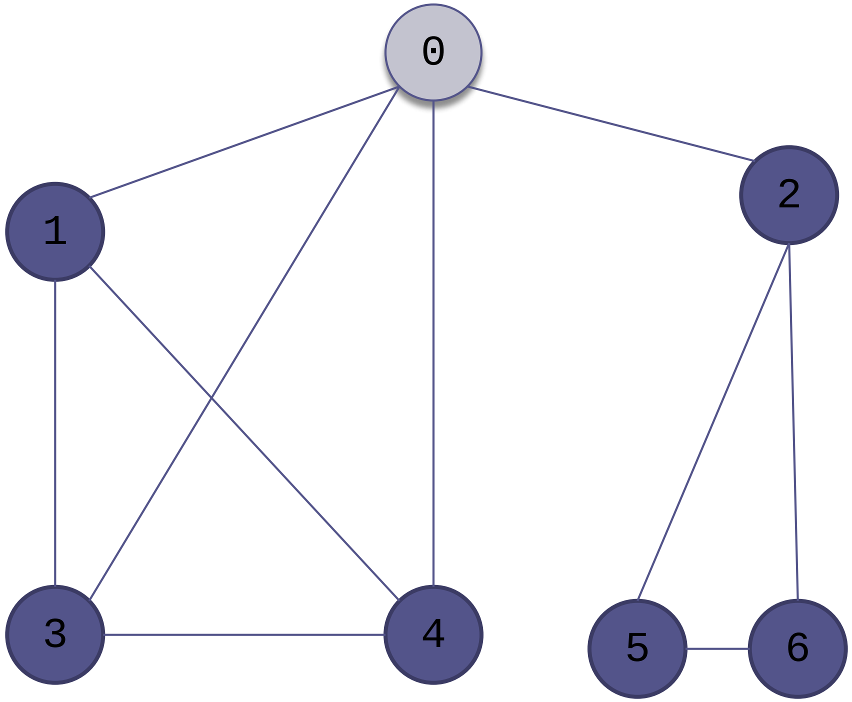
Mark 2 as visited



Finish order:
4, 3, 1, 6, 5, 2

Example of a Depth-First Search (cont.)

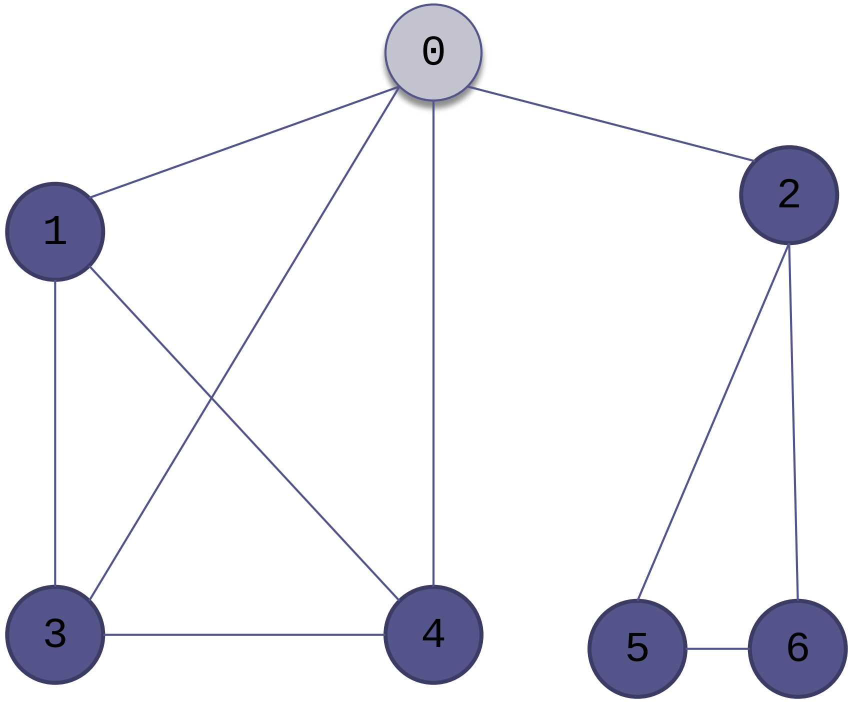
Return from the recursion to 0



Finish order:
4, 3, 1, 6, 5, 2

Example of a Depth-First Search (cont.)

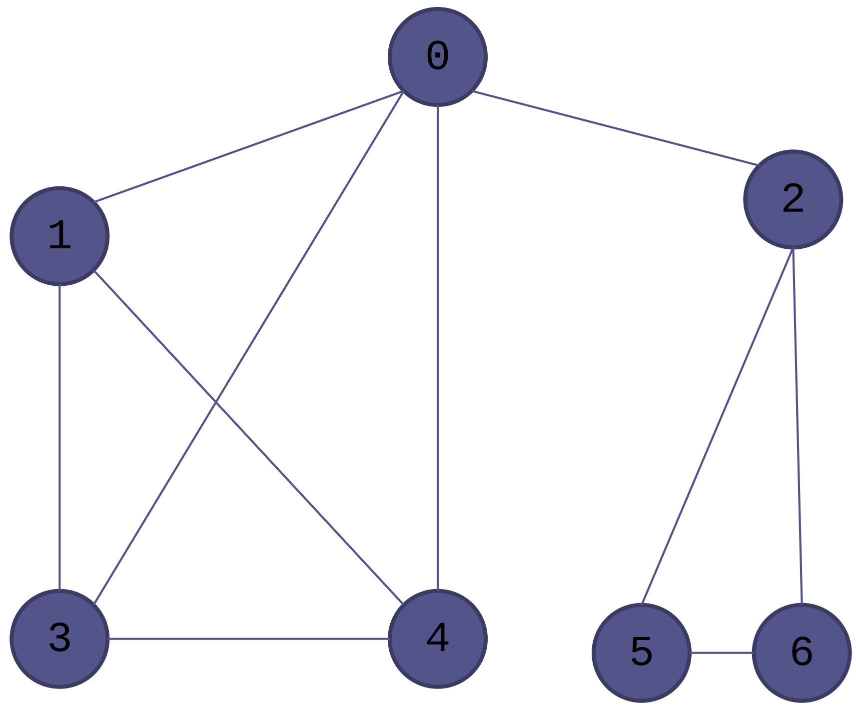
There are no nodes adjacent to 0 not being visited



Finish order:
4, 3, 1, 6, 5, 2

Example of a Depth-First Search (cont.)

Mark 0 as visited



Discovery (Visit) order:
0, 1, 3, 4, 2, 5, 6, 0

Finish order:
4, 3, 1, 6, 5, 2, 0

0 unvisited 0 visited 0 being visited

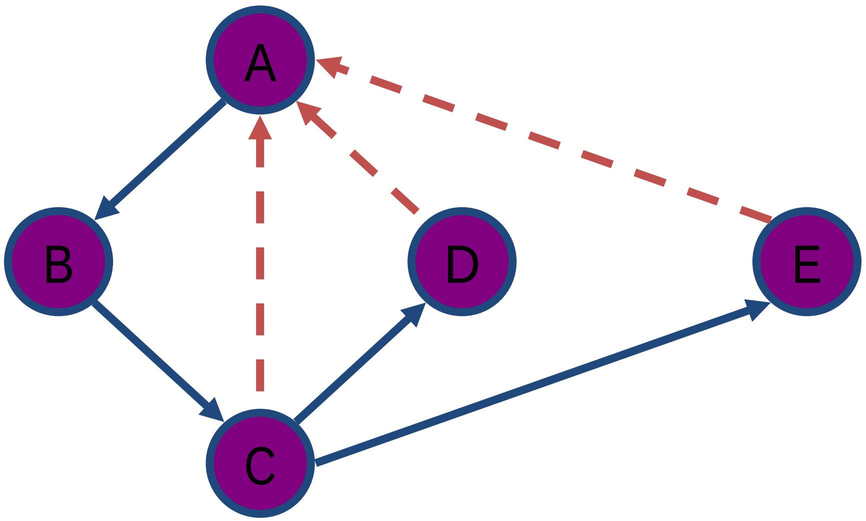
PROPERTIES OF DFS

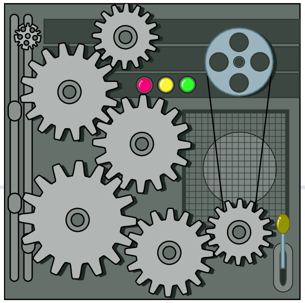
Property 1

DFS*(*G*, *v*)** visits all the vertices and edges in the connected component of ***v

Property 2

The discovery edges labeled by ***DFS*(*G*, *v*)** form a spanning tree of the connected component of ***v***

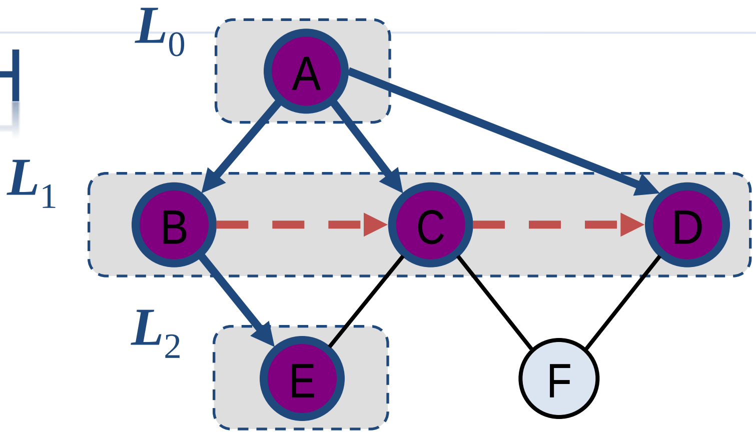




ANALYSIS OF DFS

- Setting/getting a vertex/edge label takes $O(1)$ time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED (Finished)
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or BACK
- Method incidentEdges is called once for each vertex
- DFS runs in $O(n + m)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$

GRAPH TRAVERSALS:
BREADTH-FIRST SEARCH



BREADTH-FIRST SEARCH

- A BFS traversal of a graph G
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G
- BFS on a graph with n vertices and m edges takes $O(n + m)$ time
- BFS can be further extended to solve other graph problems
  - Find and report a path with the minimum number of edges between two given vertices
  - Find a simple cycle, if there is one

BFS ALGORITHM

- The algorithm uses a mechanism for setting and getting "labels" of vertices and edges

Algorithm *BFS*(*G*)
    Input graph *G*
    Output labeling of the edges and partition of the vertices of *G*

```
for all u ∈ G.vertices()
    setLabel(u, UNEXPLORED)
for all e ∈ G.edges()
    setLabel(e, UNEXPLORED)
for all v ∈ G.vertices()
    if getLabel(v) = UNEXPLORED
        BFS(G, v)
```

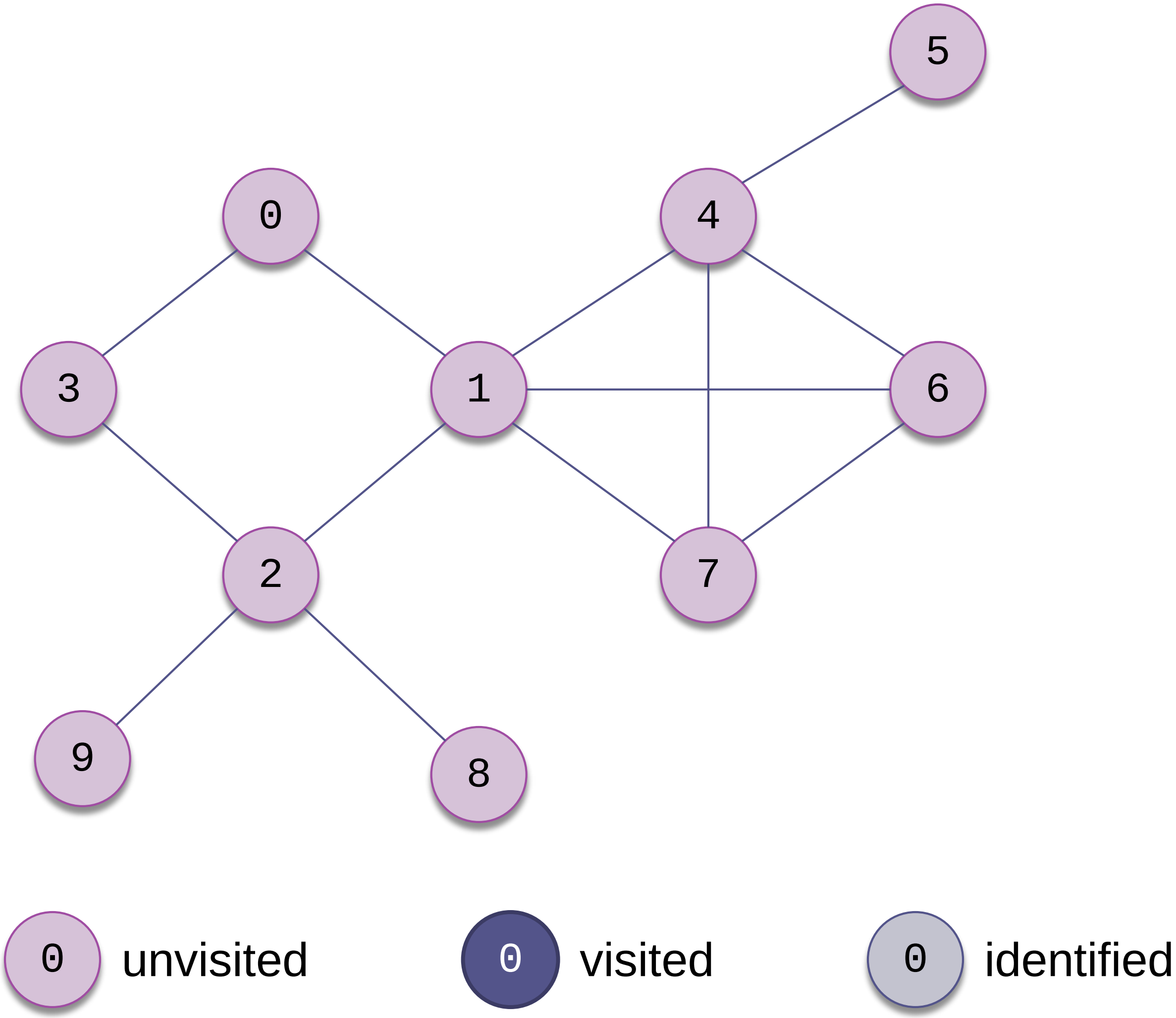
Algorithm *BFS*(*G*, *s*)

```
L_0 ← new empty sequence
L_0.addLast(s)
setLabel(s, VISITED)
i ← 0
while ¬L_i.isEmpty()
    L_{i+1} ← new empty sequence
    for all v ∈ L_i.elements()
        for all e ∈ G.incidentEdges(v)
            if getLabel(e) = UNEXPLORED
                w ← opposite(v,e)
                if getLabel(w) = UNEXPLORED
                    setLabel(e, DISCOVERY)
                    setLabel(w, VISITED)
                    L_{i+1}.addLast(w)
                else
                    setLabel(e, CROSS)
    i ← i +1
```

JAVA IMPLEMENTATION

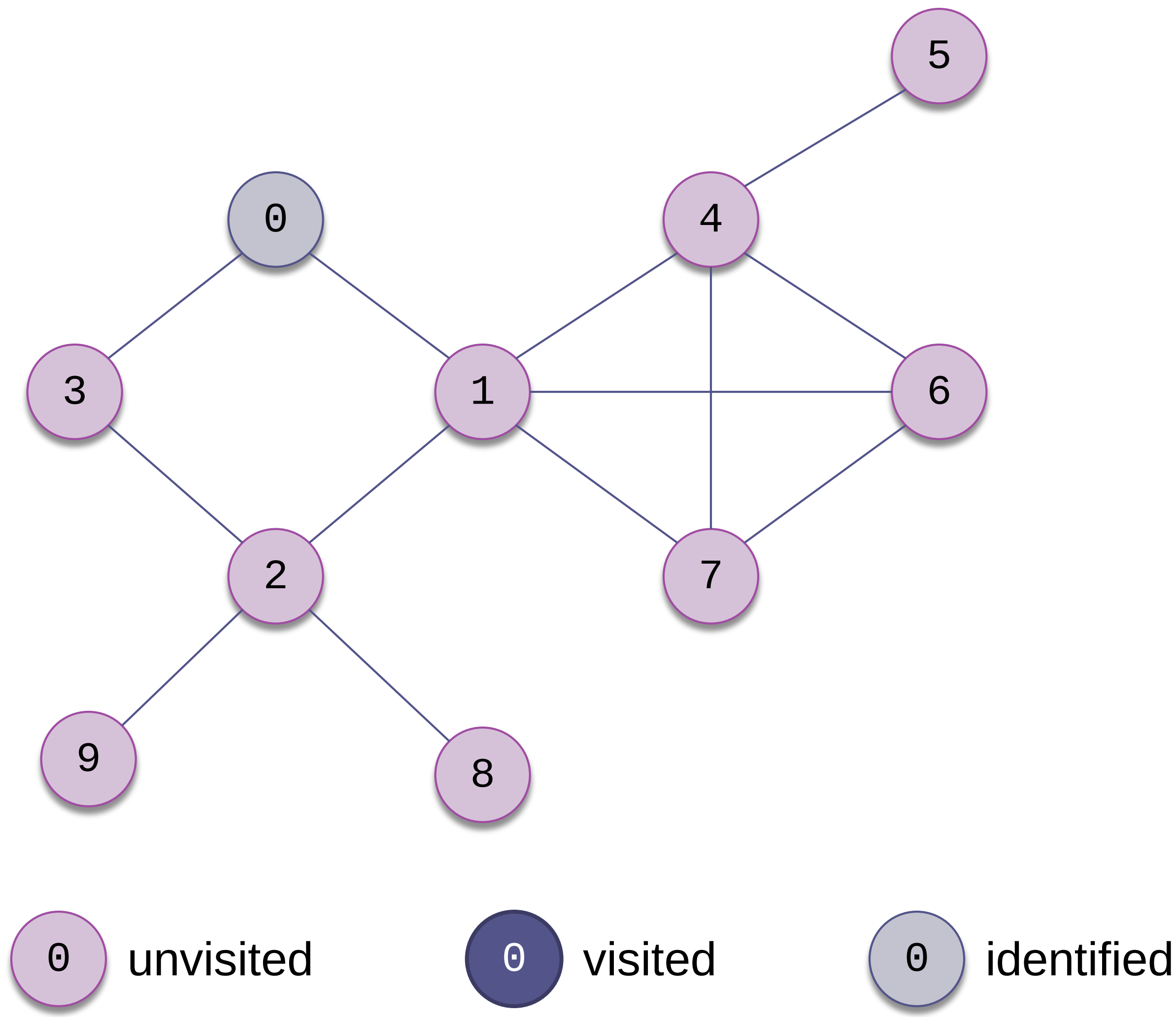
```
/** Performs breadth-first search of Graph g starting at Vertex u. */
public static <V,E> void BFS(Graph<V,E> g, Vertex<V> s,
                    Set<Vertex<V>> known, Map<Vertex<V>,Edge<E>> forest) {
  PositionalList<Vertex<V>> level = new LinkedPositionalList<>( );
  known.add(s);
  level.addLast(s);                        // first level includes only s
  while (!level.isEmpty( )) {
    PositionalList<Vertex<V>> nextLevel = new LinkedPositionalList<>( );
    for (Vertex<V> u : level)
      for (Edge<E> e : g.outgoingEdges(u)) {
        Vertex<V> v = g.opposite(u, e);
        if (!known.contains(v)) {
          known.add(v);
          forest.put(v, e);                // e is the tree edge that discovered v
          nextLevel.addLast(v);            // v will be further considered in next pass
        }
      }
    level = nextLevel;                     // relabel 'next' level to become the current
  }
}
```

Example of a Breadth-First Search



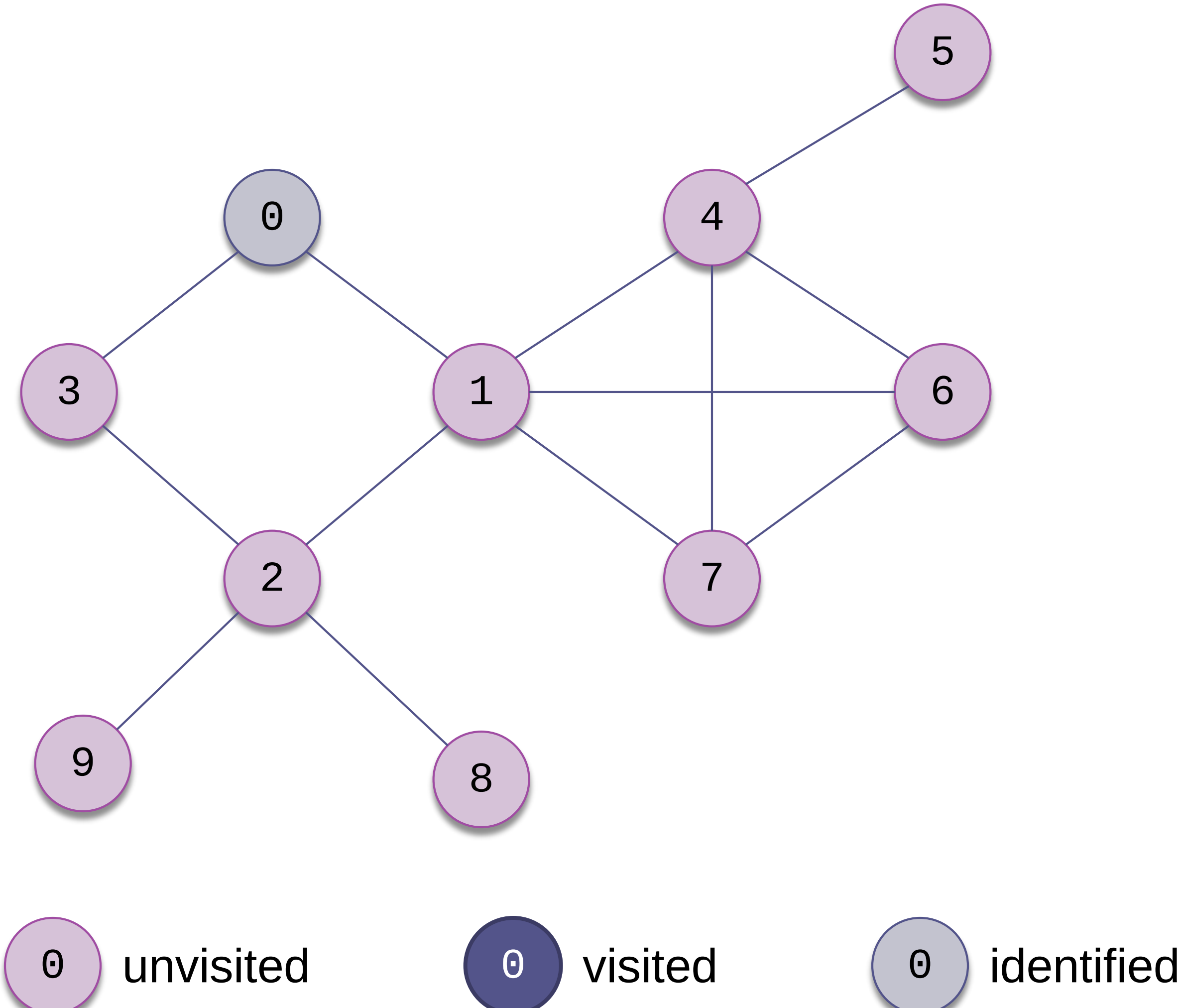
Example of a Breadth-First Search (cont.)

Identify the start node



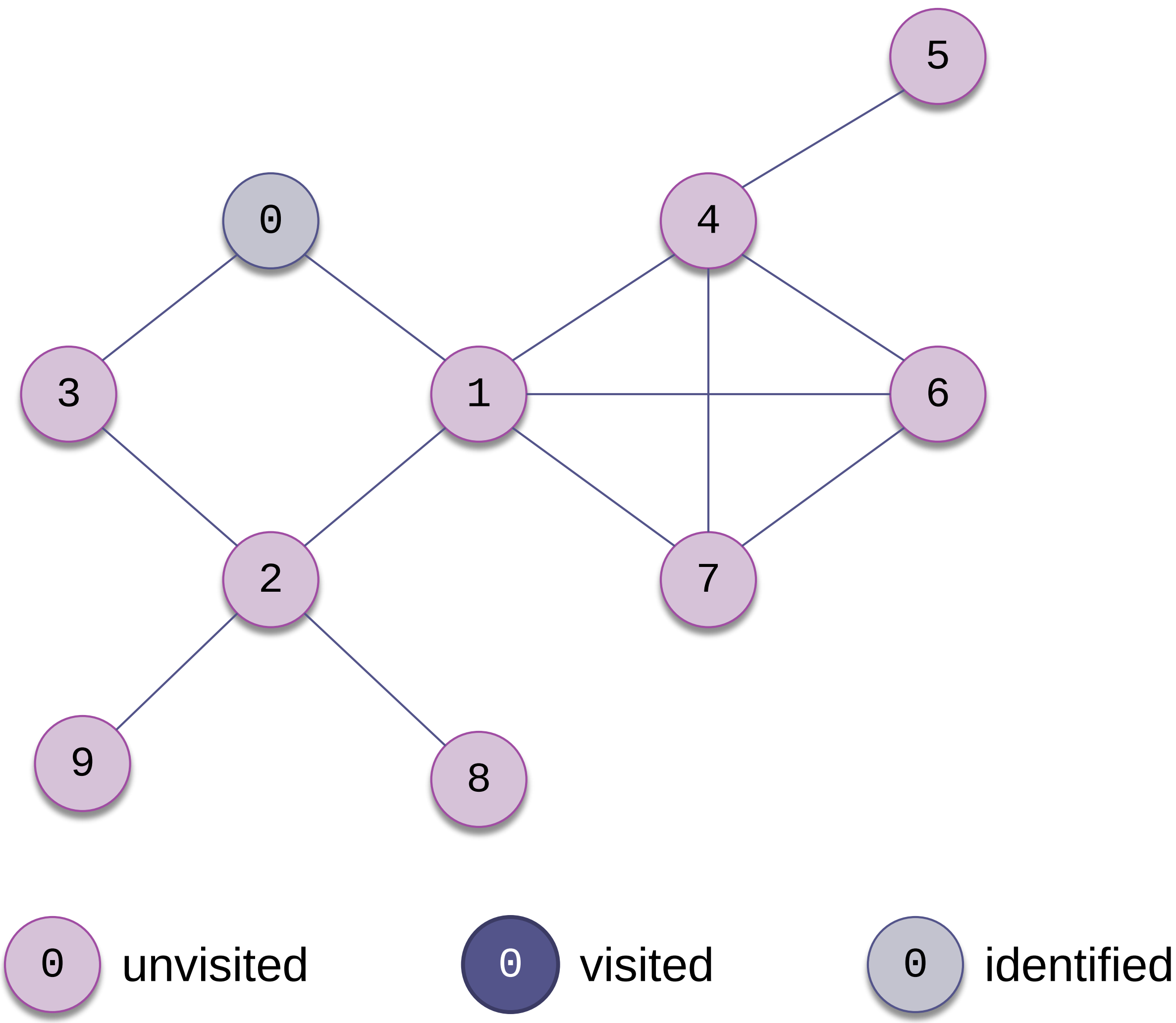
Example of a Breadth-First Search (cont.)

While visiting it, we can identify its adjacent nodes



Example of a Breadth-First Search (cont.)

We identify its adjacent nodes and add them to a queue of identified nodes



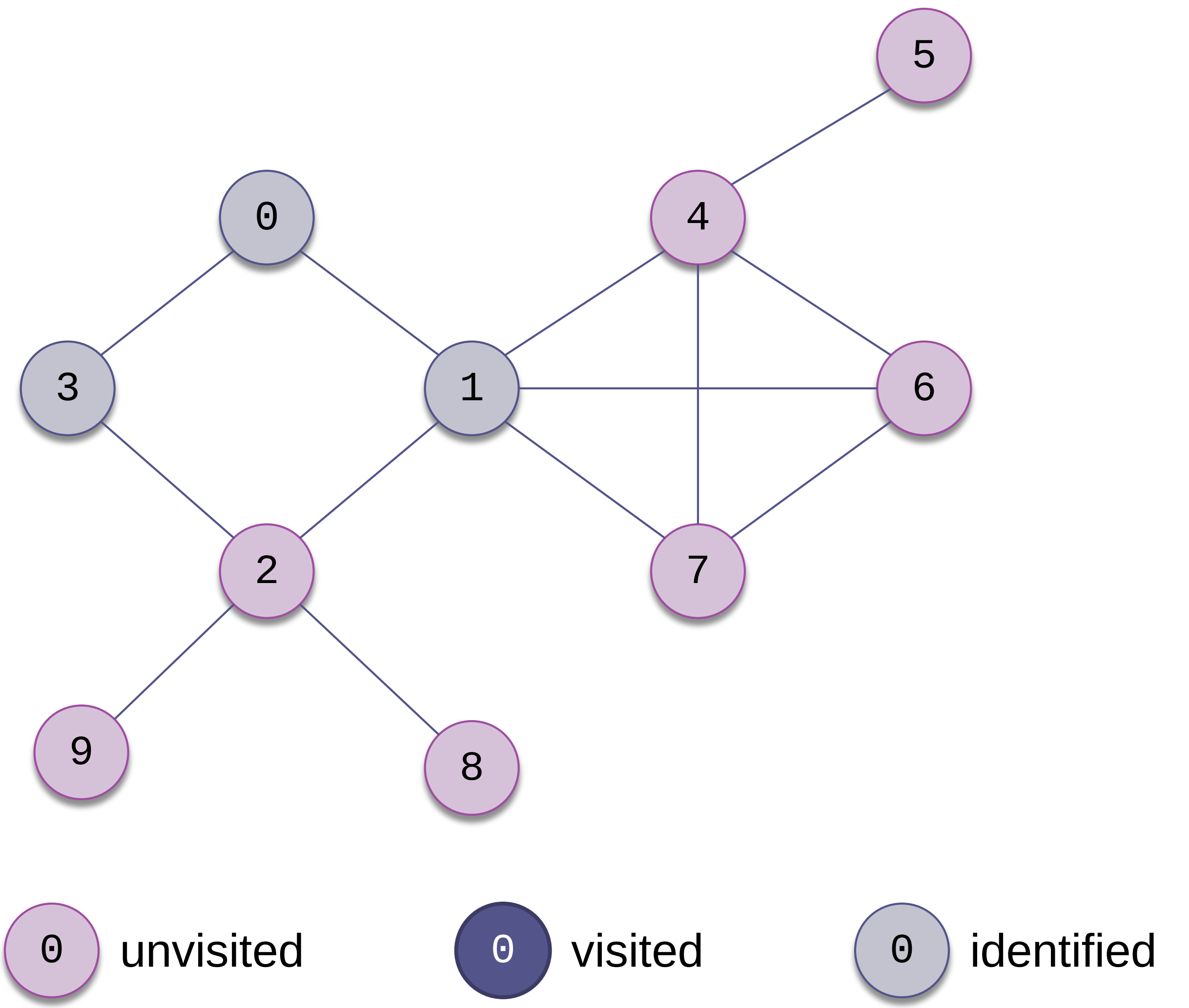
Visit sequence:
0

Example of a Breadth-First Search (cont.)

We identify its adjacent nodes and add them to a queue of identified nodes

Queue:
1, 3

Visit sequence:
0

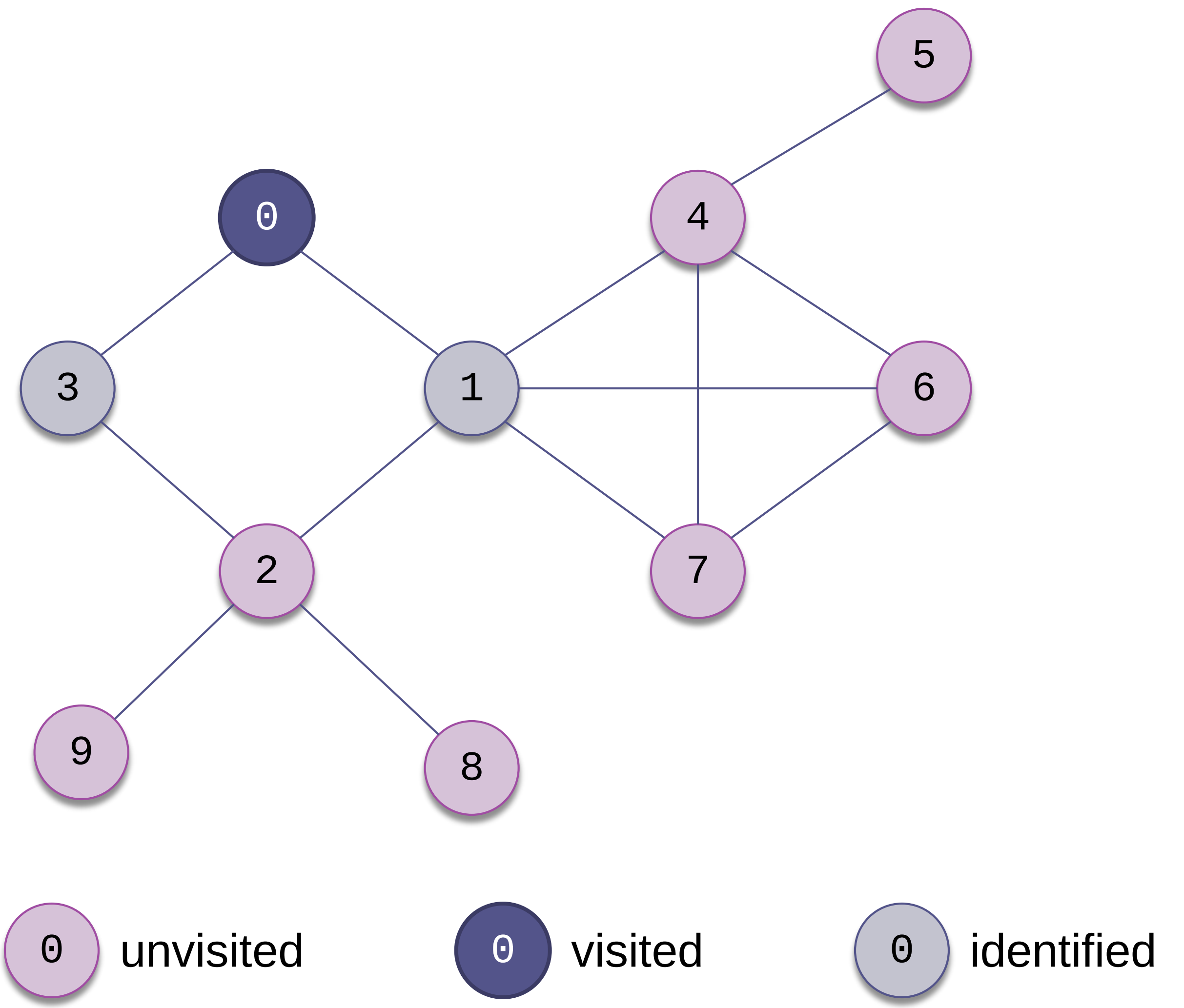


Example of a Breadth-First Search (cont.)

We color the node as visited

Queue:
1, 3

Visit sequence:
0

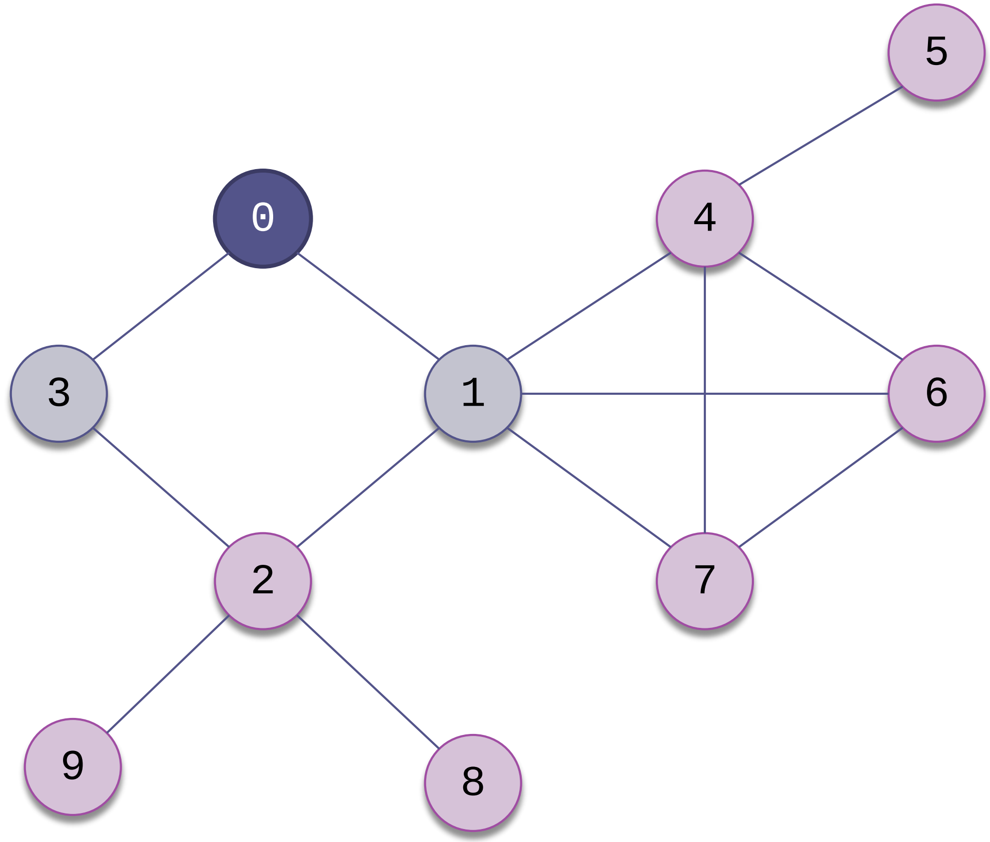


Example of a Breadth-First Search (cont.)

The queue determines which nodes to visit next

Queue:
1, 3

Visit sequence:
0

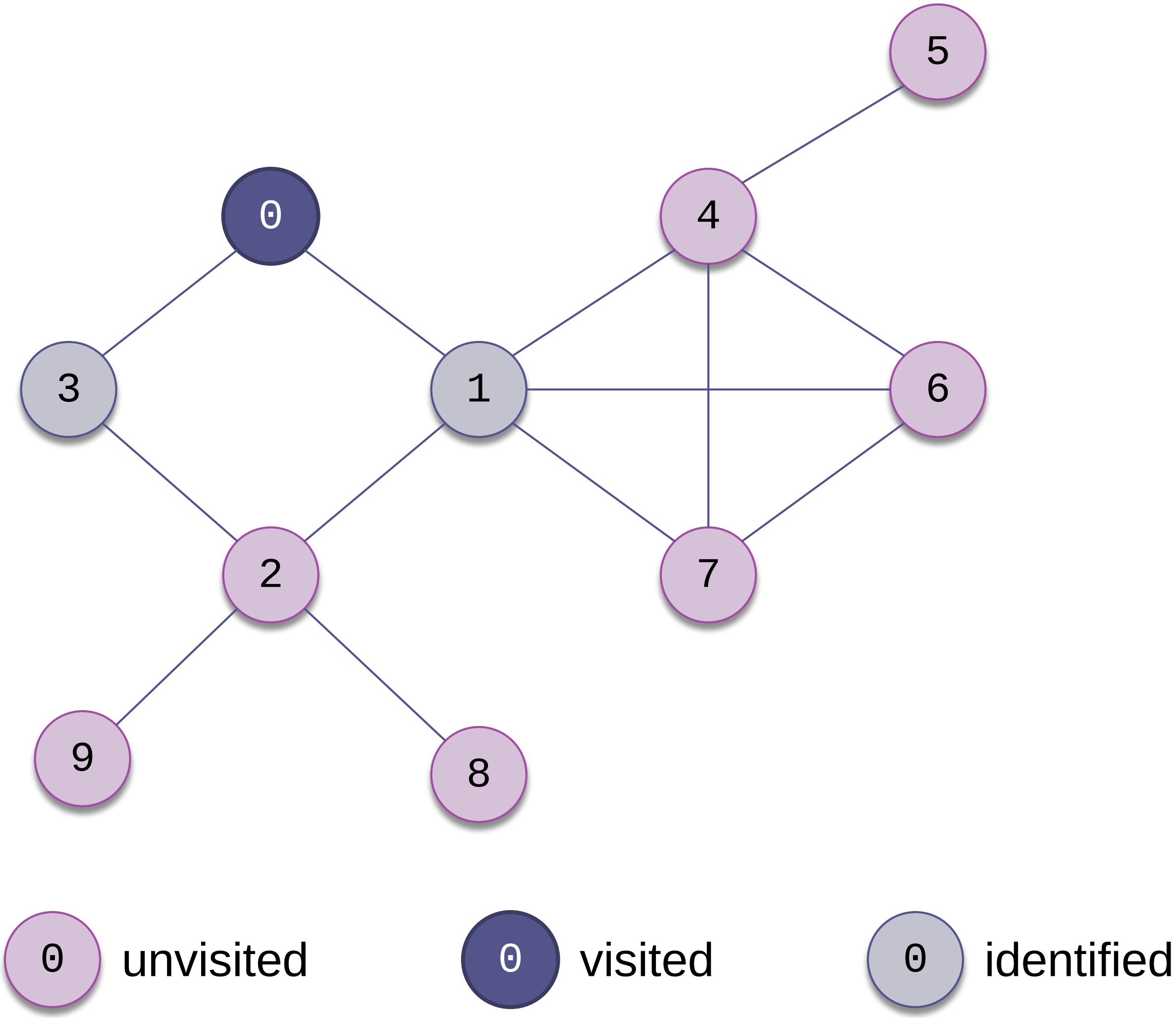


Example of a Breadth-First Search (cont.)

Visit the first node in the queue, 1

Queue:
1, 3

Visit sequence:
0

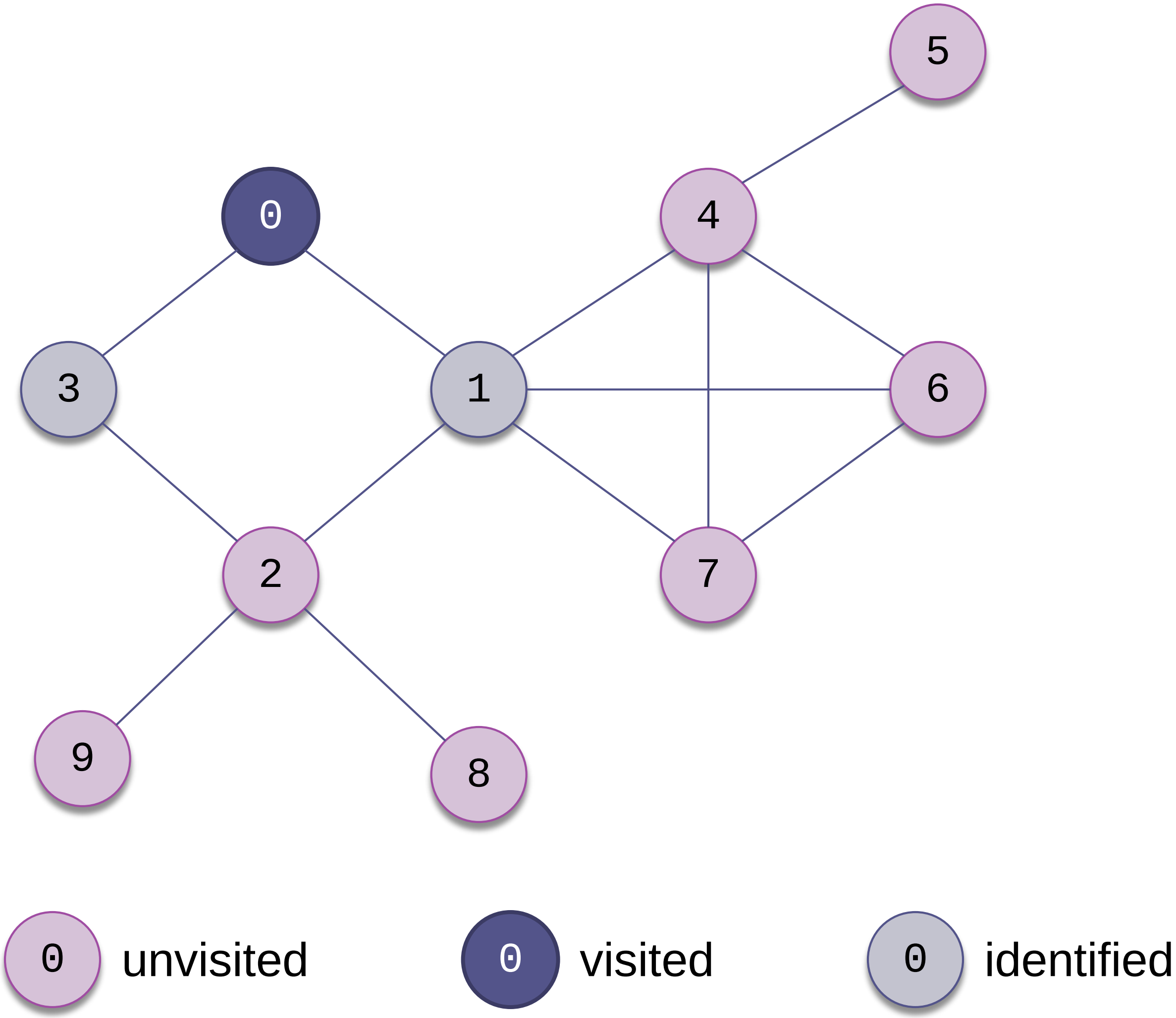


Example of a Breadth-First Search (cont.)

Visit the first node in the queue, 1

Queue:
3

Visit sequence:
0, 1

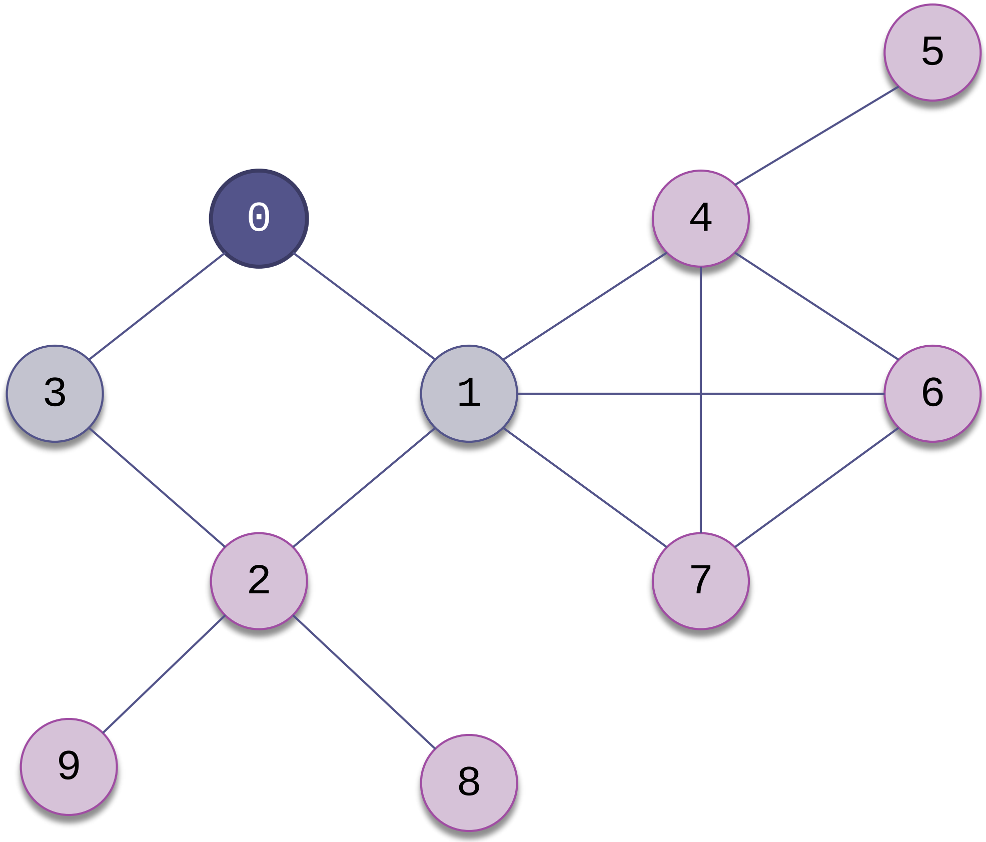


Example of a Breadth-First Search (cont.)

Select all its adjacent nodes that have not been visited or identified

Queue:
3

Visit sequence:
0, 1

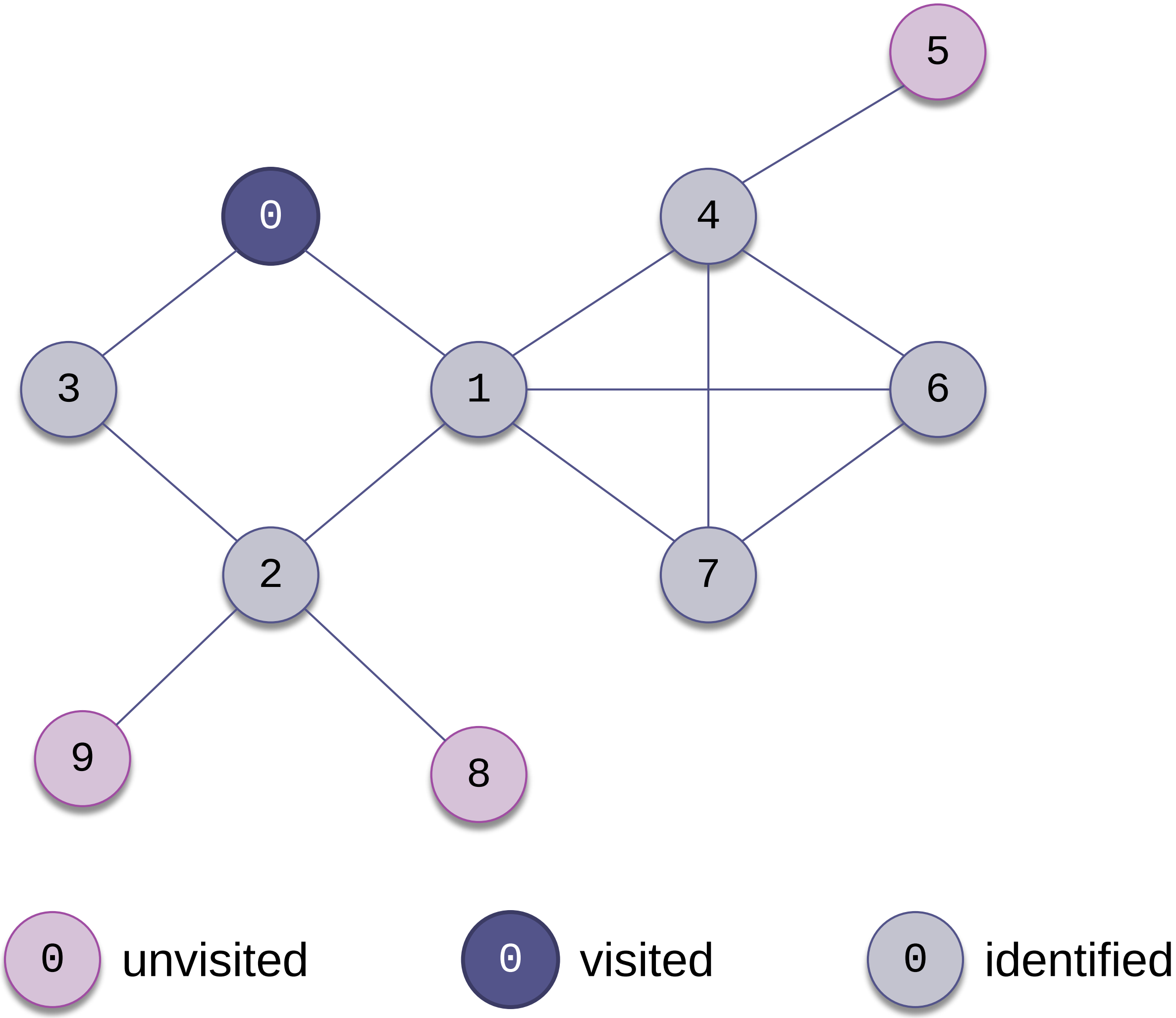


Example of a Breadth-First Search (cont.)

Select all its adjacent nodes that have not been visited or identified

Queue:
3, 2, 4, 6, 7

Visit sequence:
0, 1

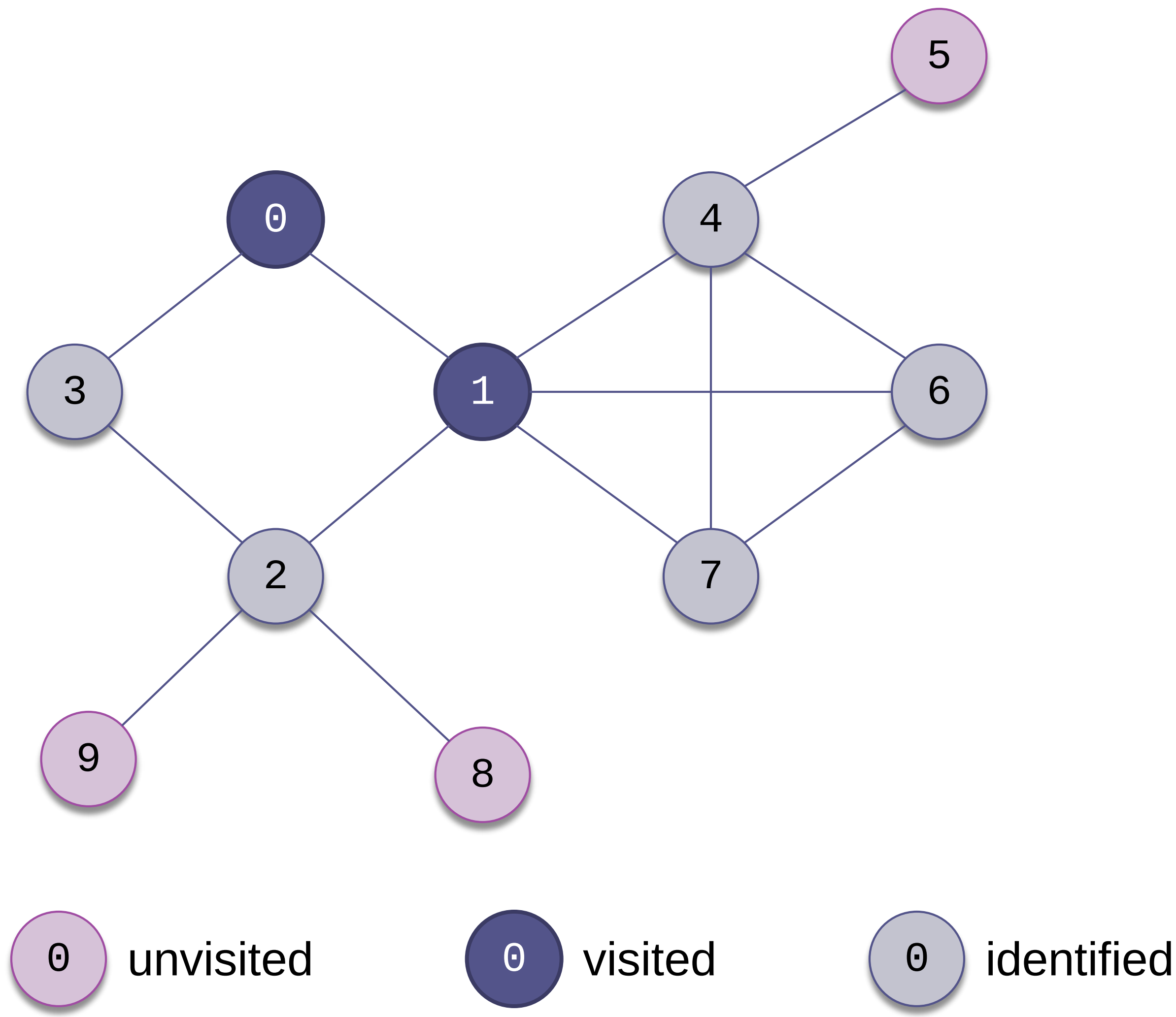


Example of a Breadth-First Search (cont.)

Now that we are done with 1, we color it as visited

Queue:
3, 2, 4, 6, 7

Visit sequence:
0, 1

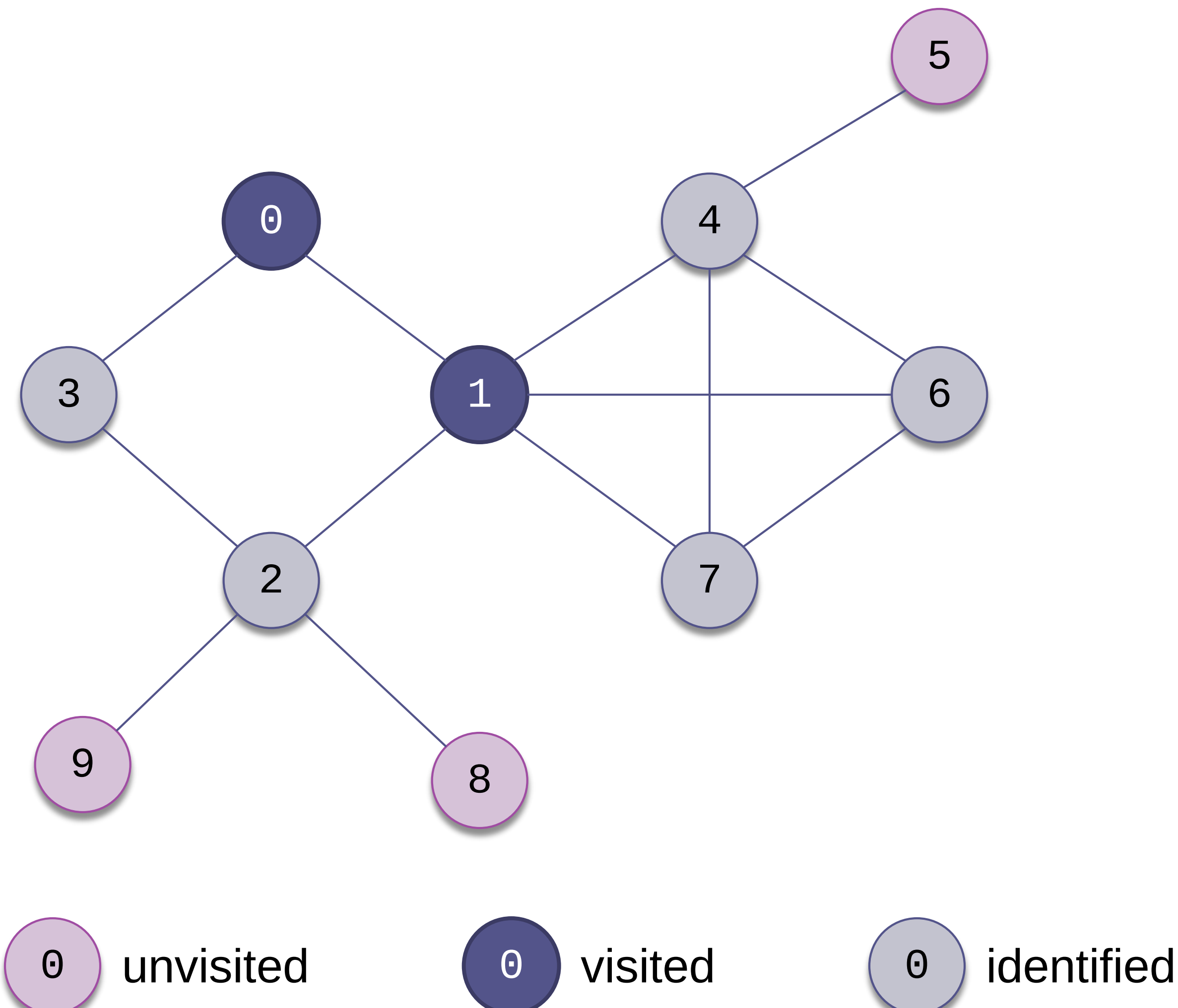


Example of a Breadth-First Search (cont.)

and then visit the next node in the queue, 3 (which was identified in the first selection)

Queue:
3, 2, 4, 6, 7

Visit sequence:
0, 1

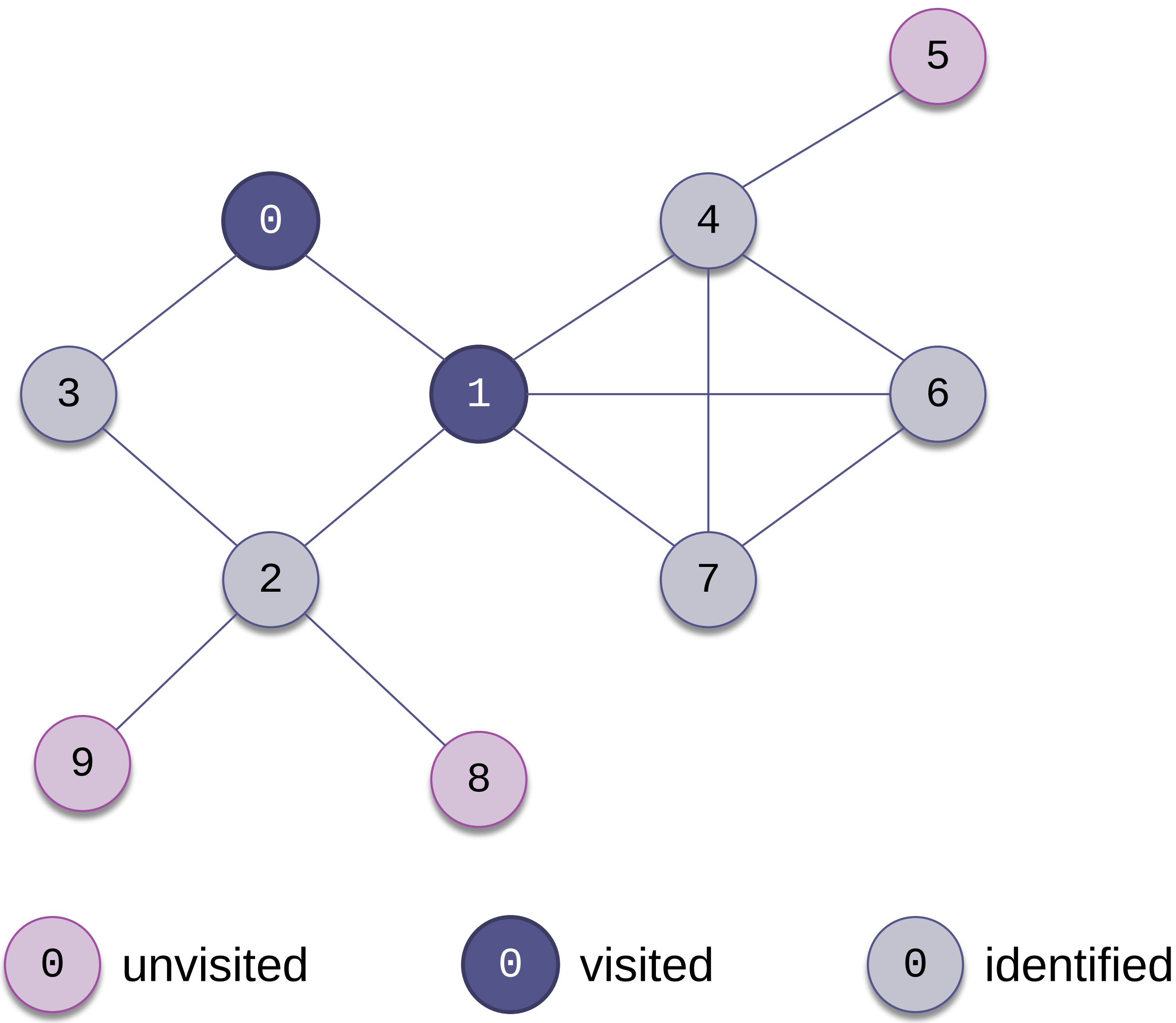


Example of a Breadth-First Search (cont.)

and then visit the next node in the queue, 3 (which was identified in the first selection)

Queue:
2, 4, 6, 7

Visit sequence:
0, 1, 3

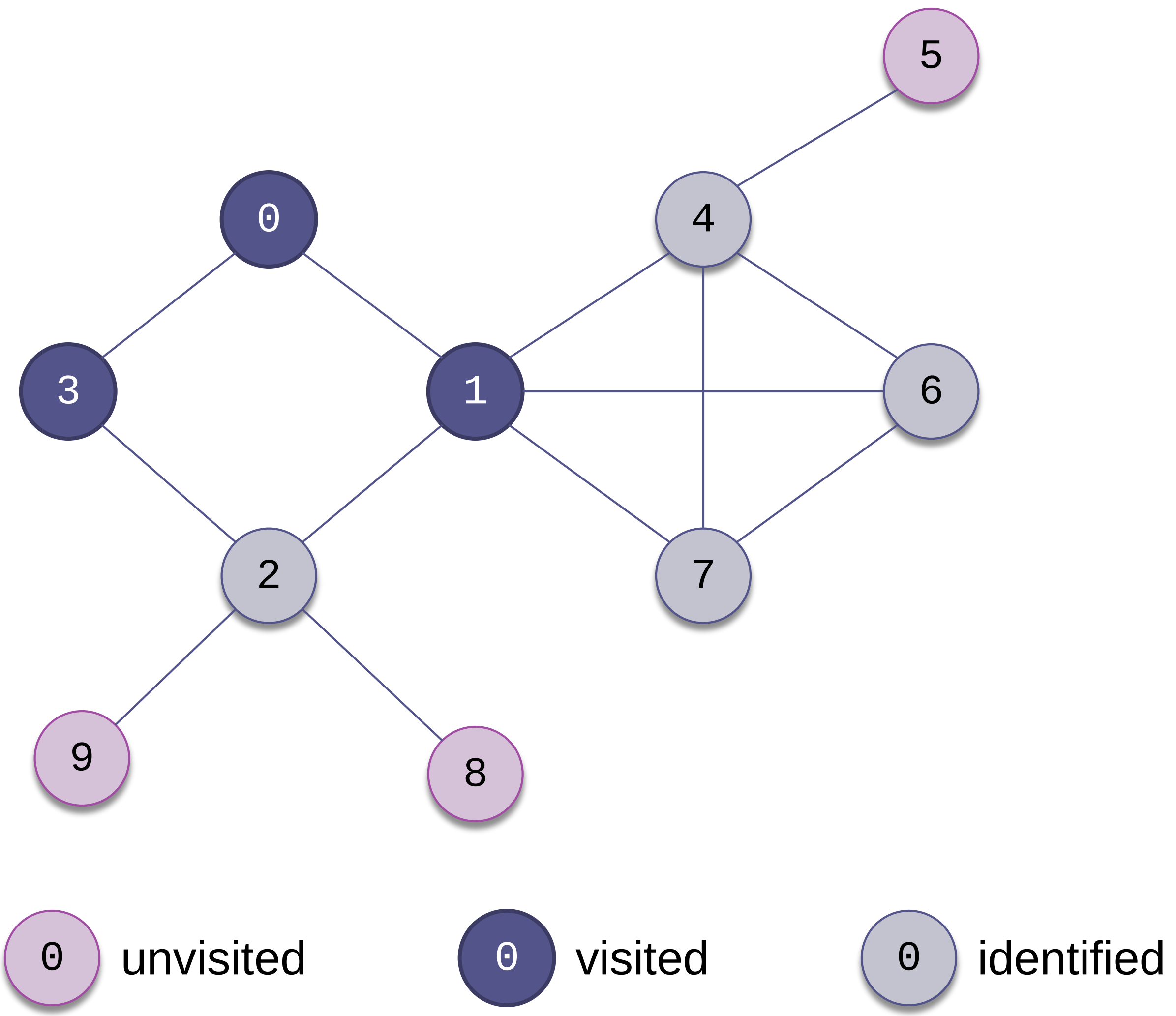


Example of a Breadth-First Search (cont.)

3 has two adjacent vertices. 0 has already been visited and 2 has already been identified. We are done with 3

Queue:
2, 4, 6, 7

Visit sequence:
0, 1, 3

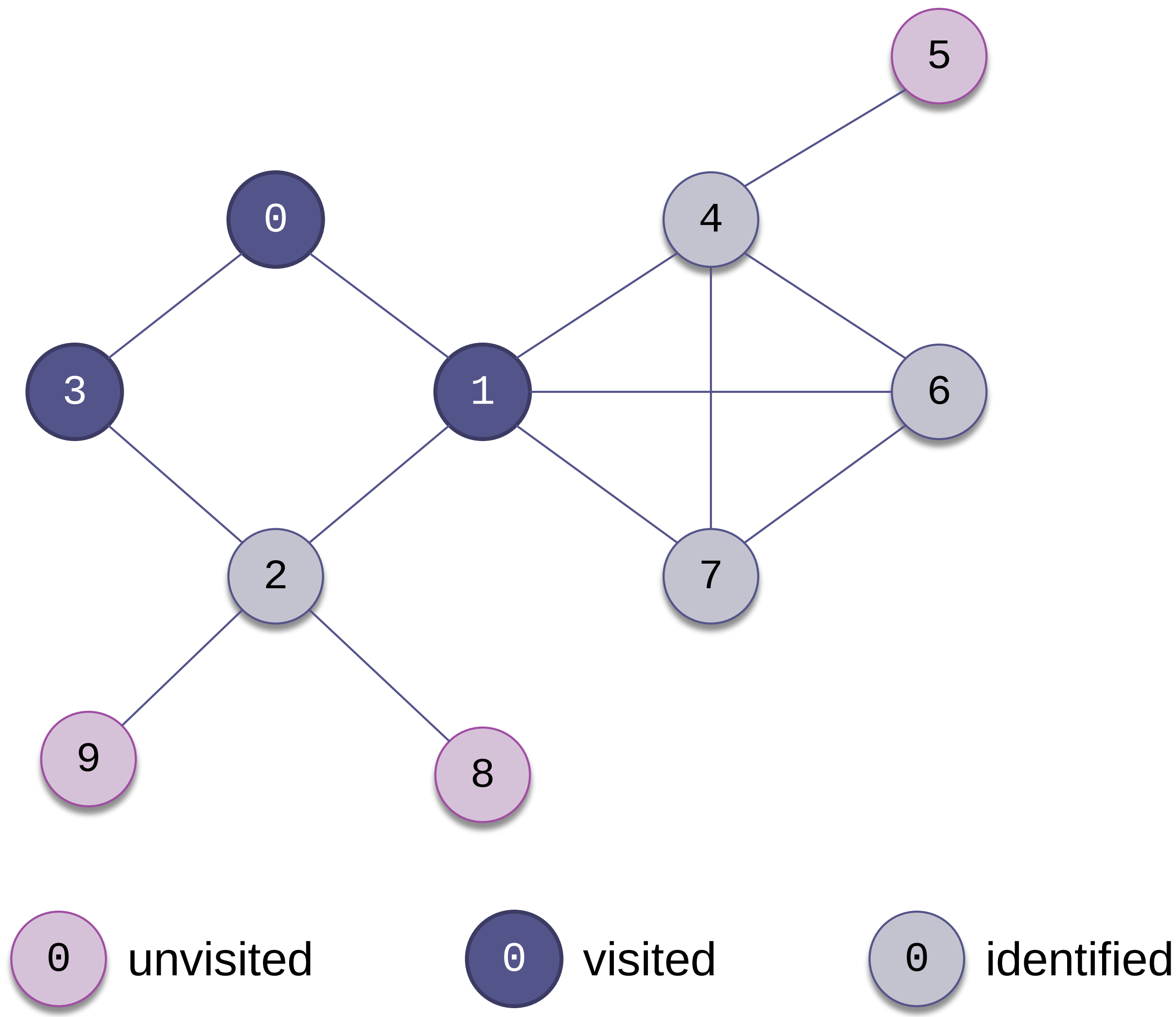


Example of a Breadth-First Search (cont.)

The next node in the queue is 2

Queue:
2, 4, 6, 7

Visit sequence:
0, 1, 3

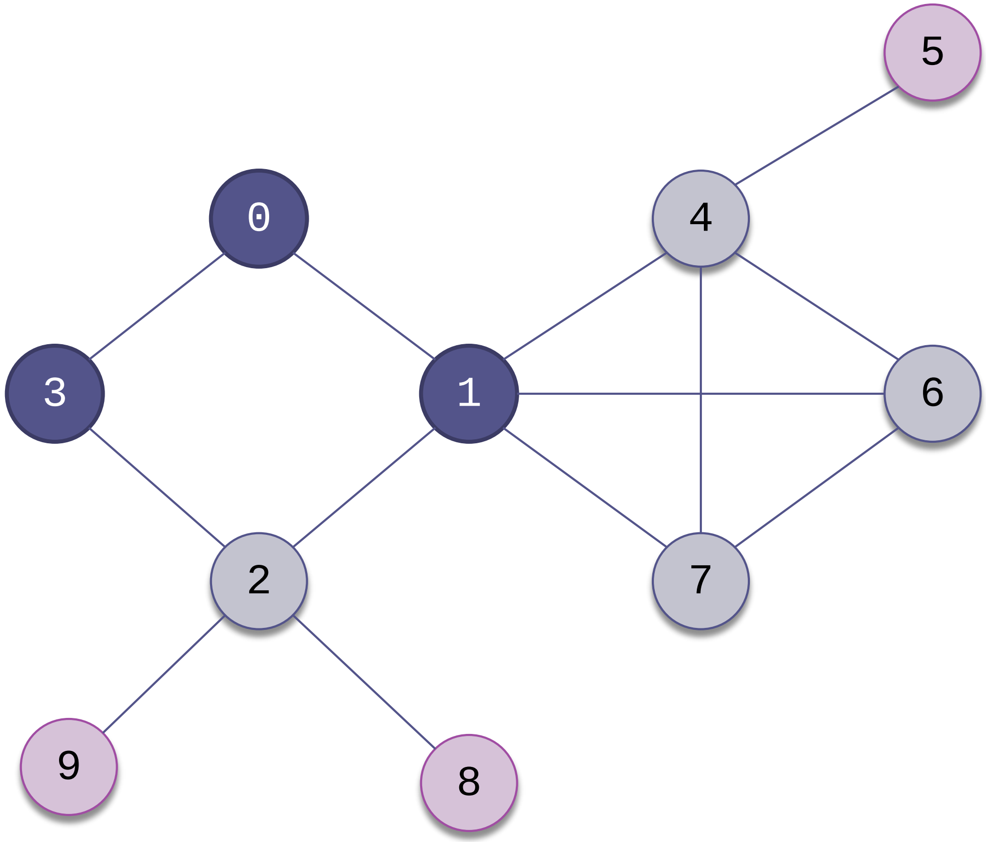


Example of a Breadth-First Search (cont.)

The next node in the queue is 2

Queue:
4, 6, 7

Visit sequence:
0, 1, 3, 2

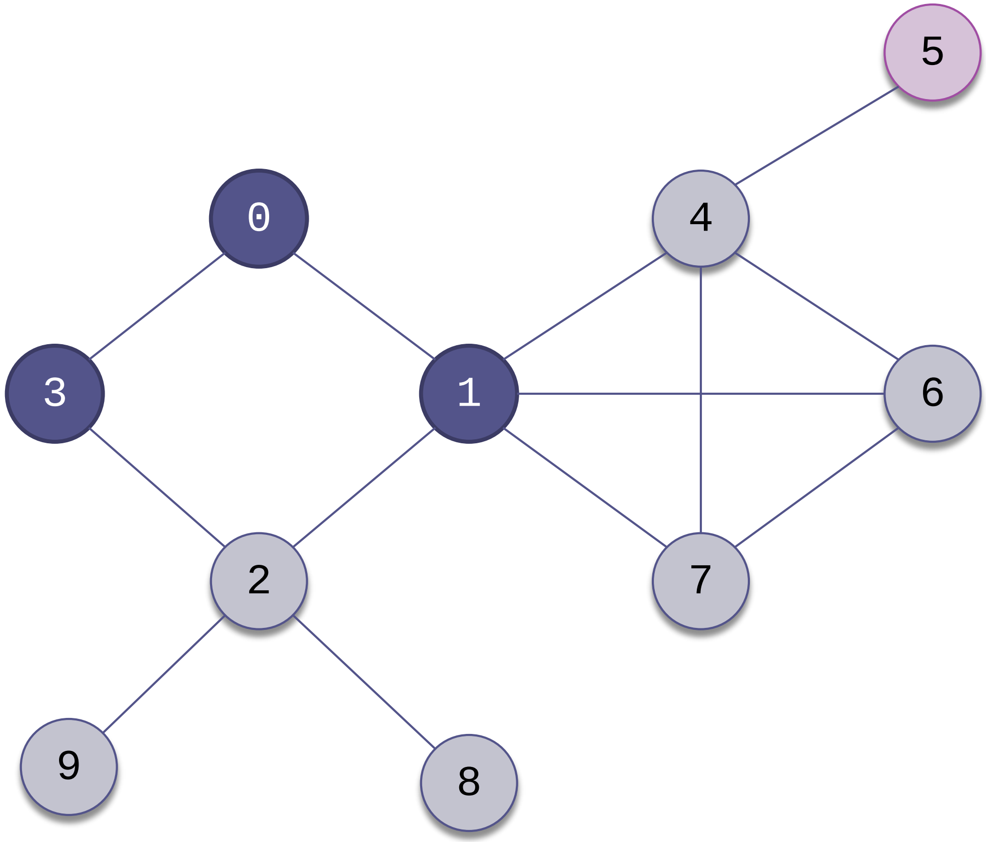


Example of a Breadth-First Search (cont.)

8 and 9 are the only adjacent vertices not already visited or identified

Queue:
4, 6, 7, 8, 9

Visit sequence:
0, 1, 3, 2

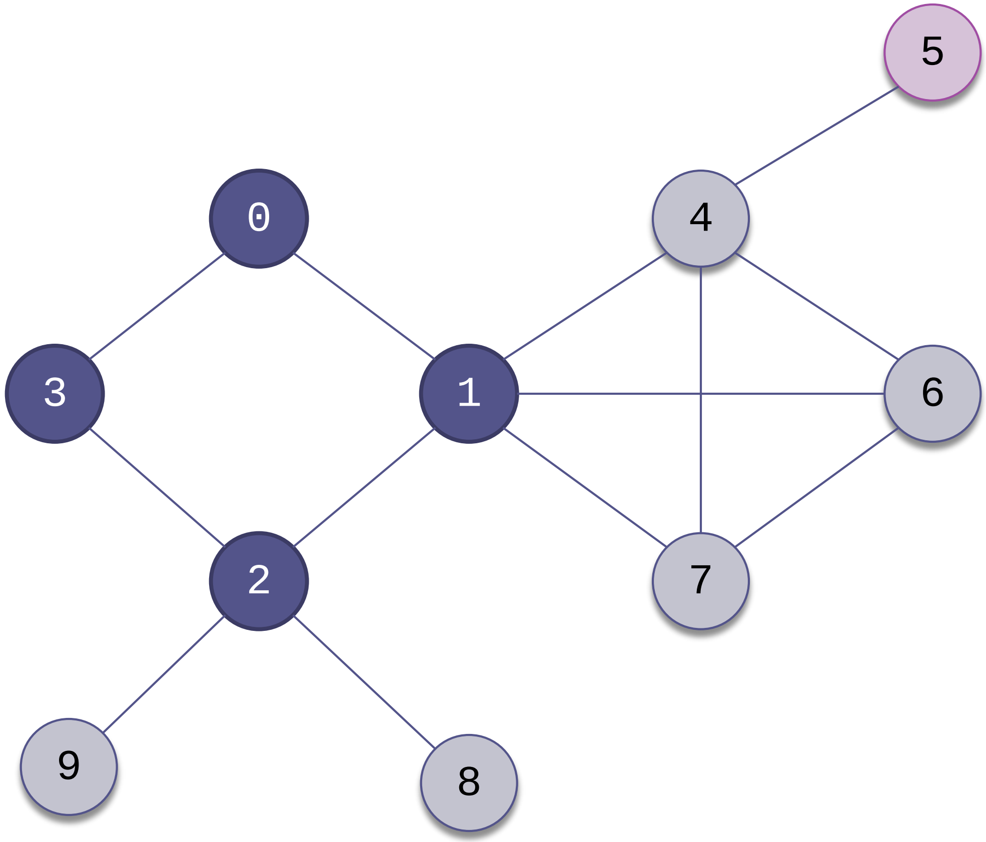


Example of a Breadth-First Search (cont.)

4 is next

Queue:
6, 7, 8, 9

Visit sequence:
0, 1, 3, 2, 4

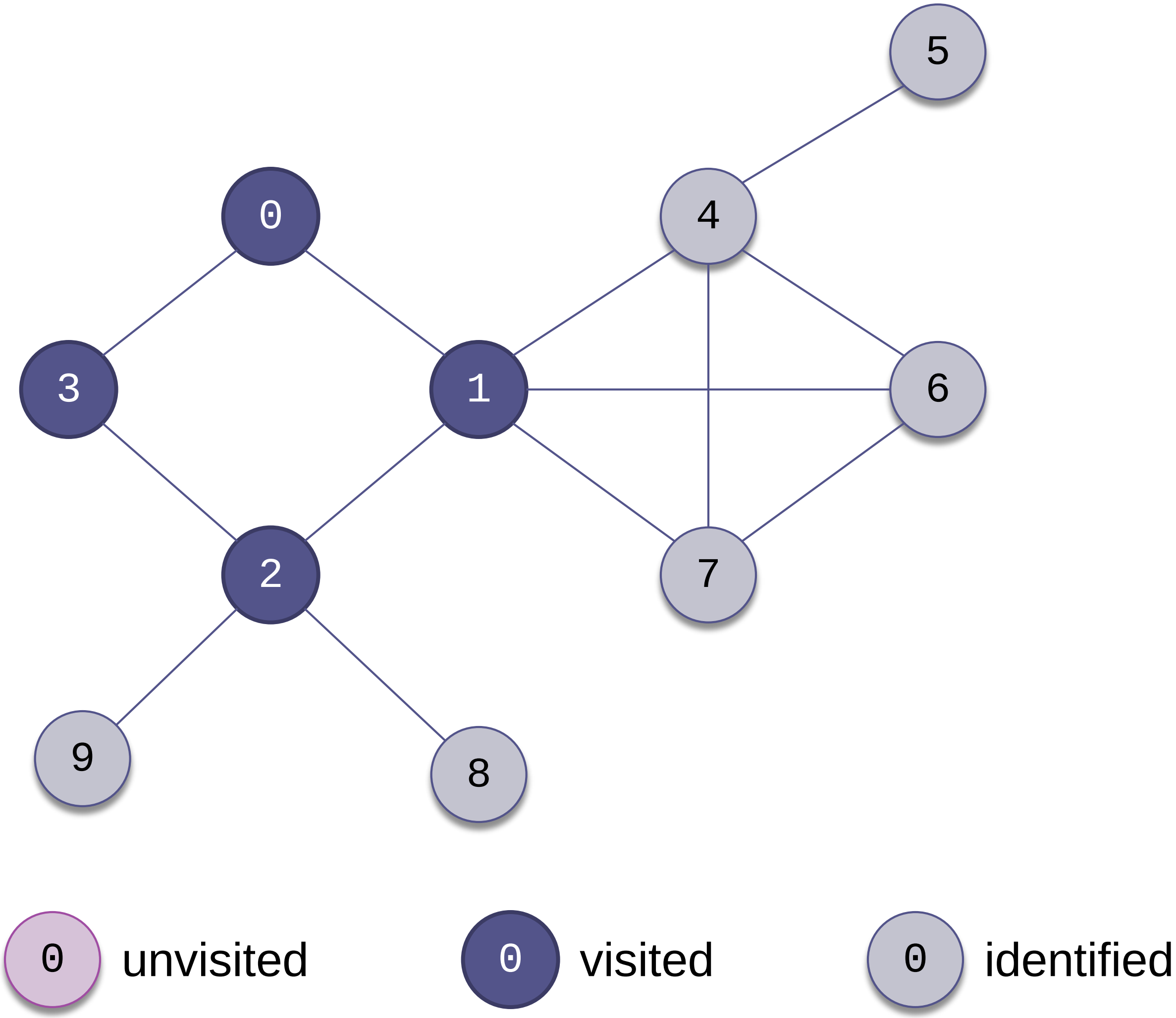


Example of a Breadth-First Search (cont.)

5 is the only vertex not already visited or identified

Queue:
6, 7, 8, 9, 5

Visit sequence:
0, 1, 3, 2, 4

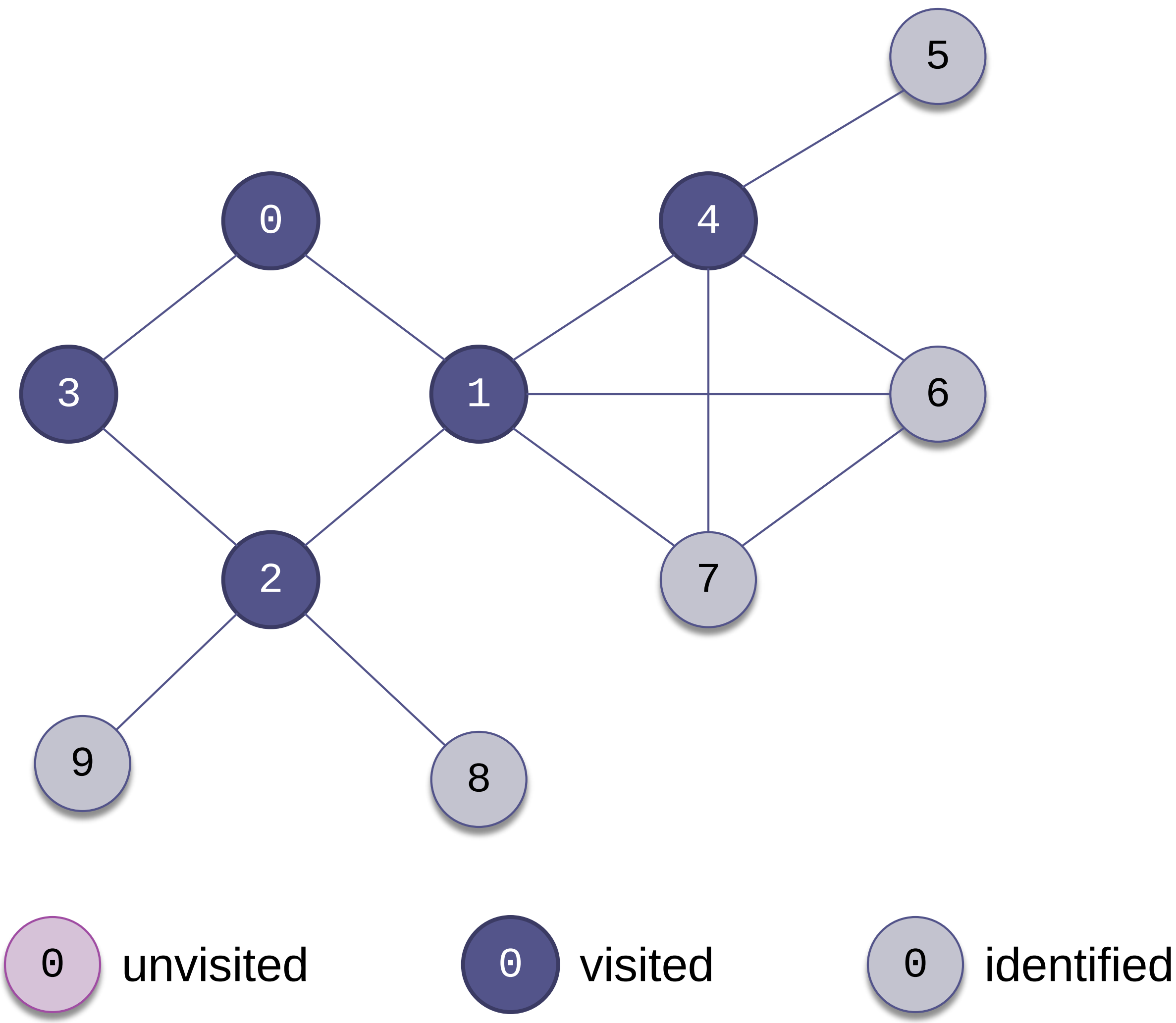


Example of a Breadth-First Search (cont.)

6 has no vertices not already visited or identified

Queue:
7, 8, 9, 5

Visit sequence:
0, 1, 3, 2, 4, 6

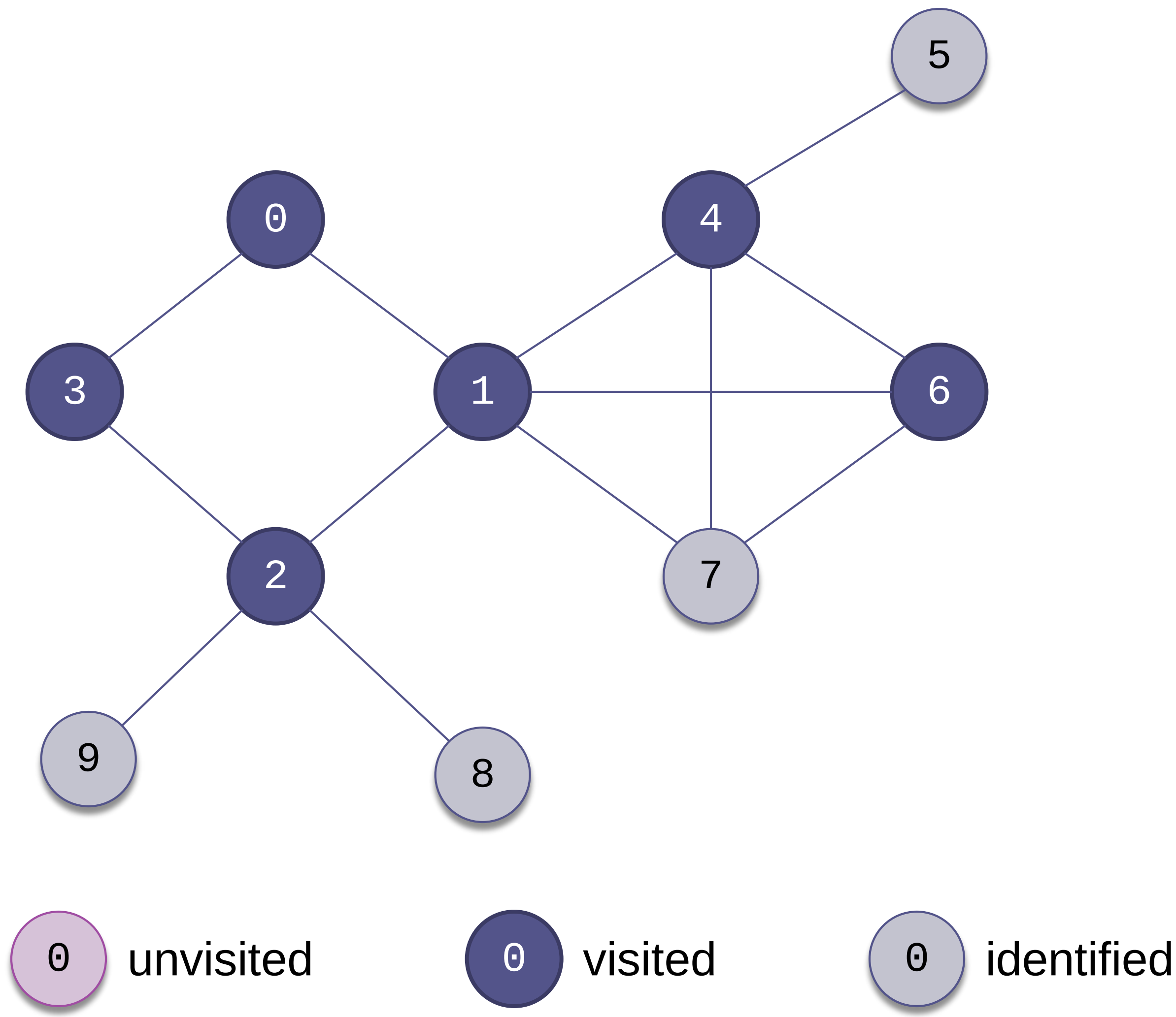


Example of a Breadth-First Search (cont.)

6 has no vertices not already visited or identified

Queue:
7, 8, 9, 5

Visit sequence:
0, 1, 3, 2, 4, 6

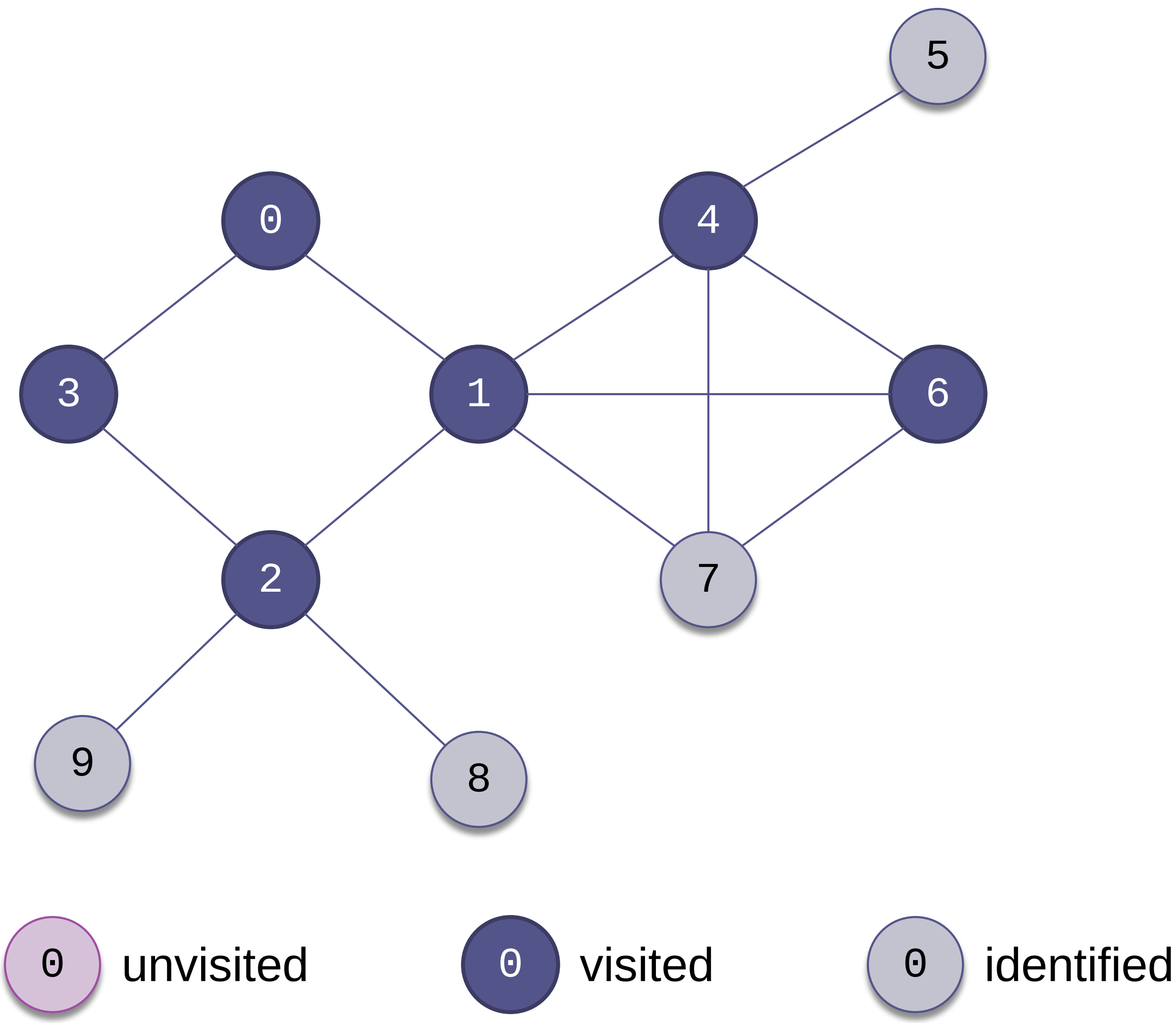


Example of a Breadth-First Search (cont.)

7 has no vertices not already visited or identified

Queue:
8, 9, 5

Visit sequence:
0, 1, 3, 2, 4, 6, 7

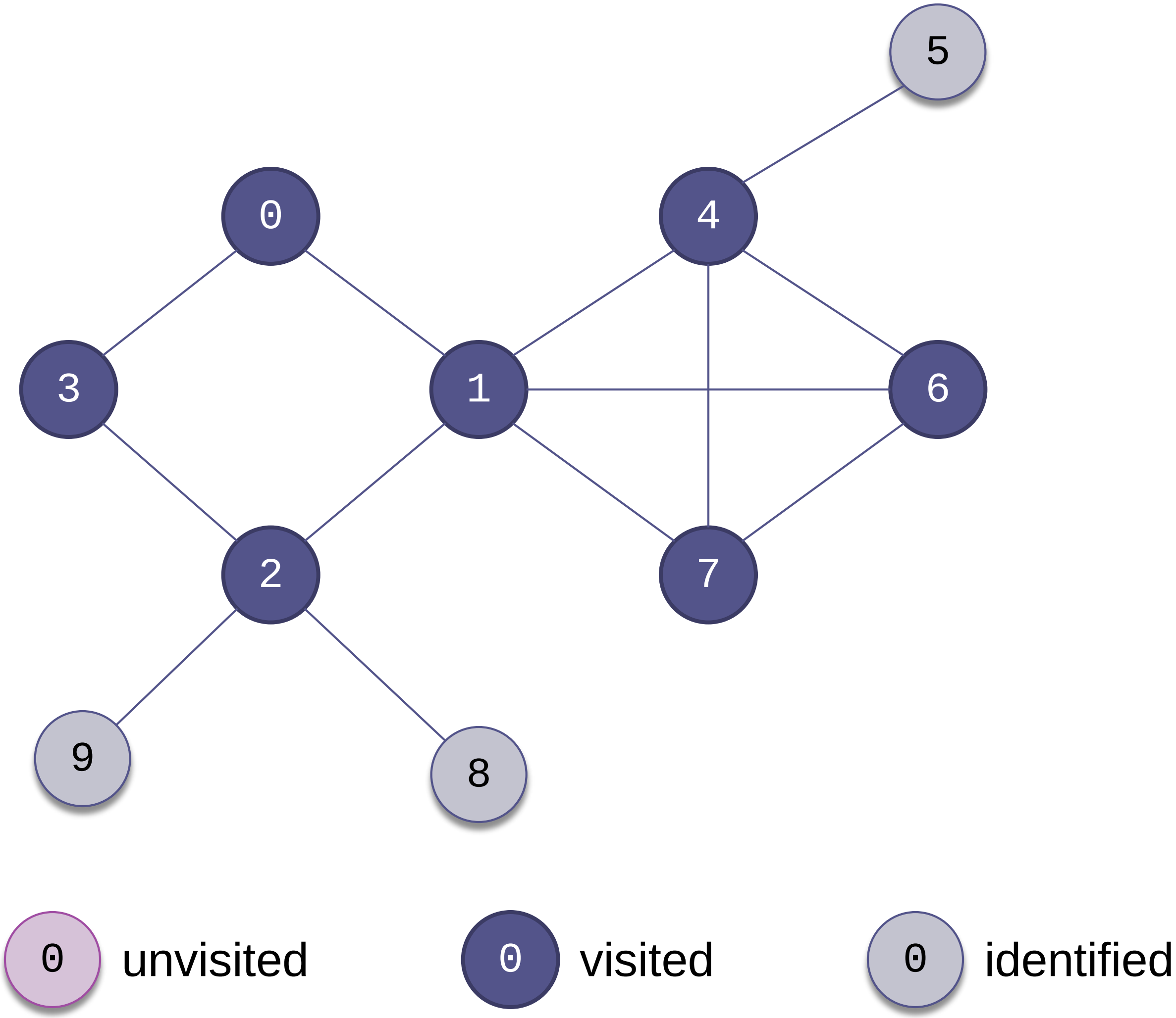


Example of a Breadth-First Search (cont.)

7 has no vertices not already visited or identified

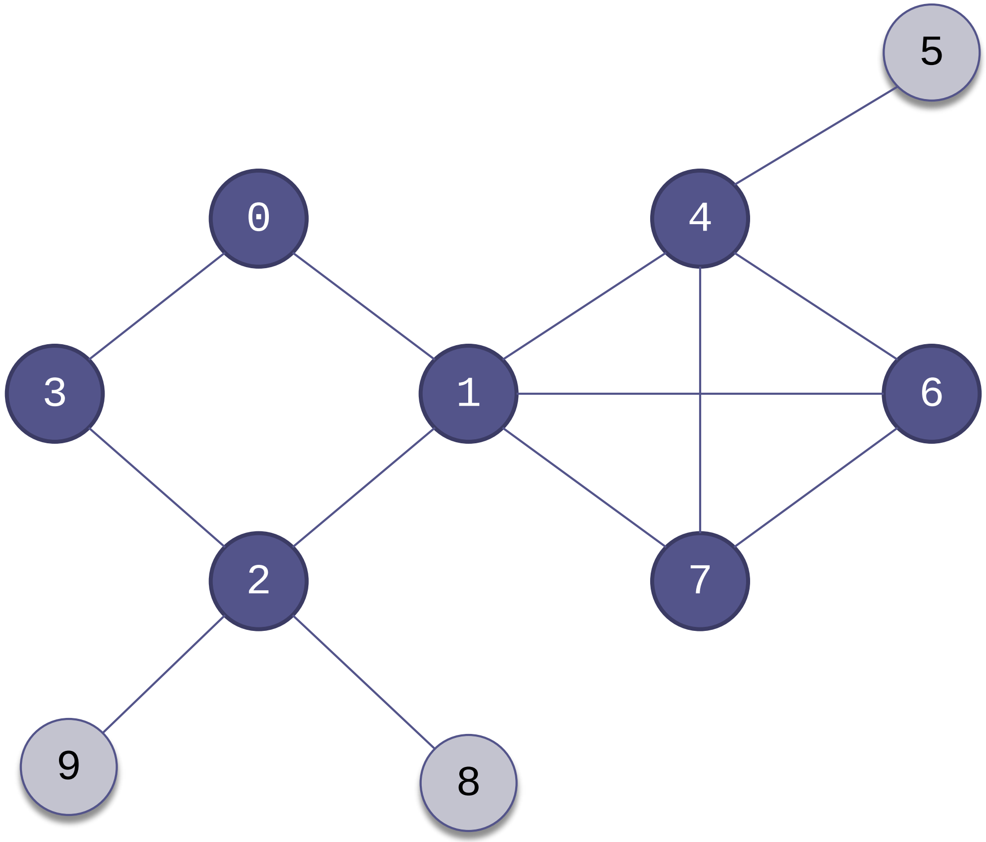
Queue:
8, 9, 5

Visit sequence:
0, 1, 3, 2, 4, 6, 7



Example of a Breadth-First Search (cont.)

We go back to the vertices of 2 and visit them

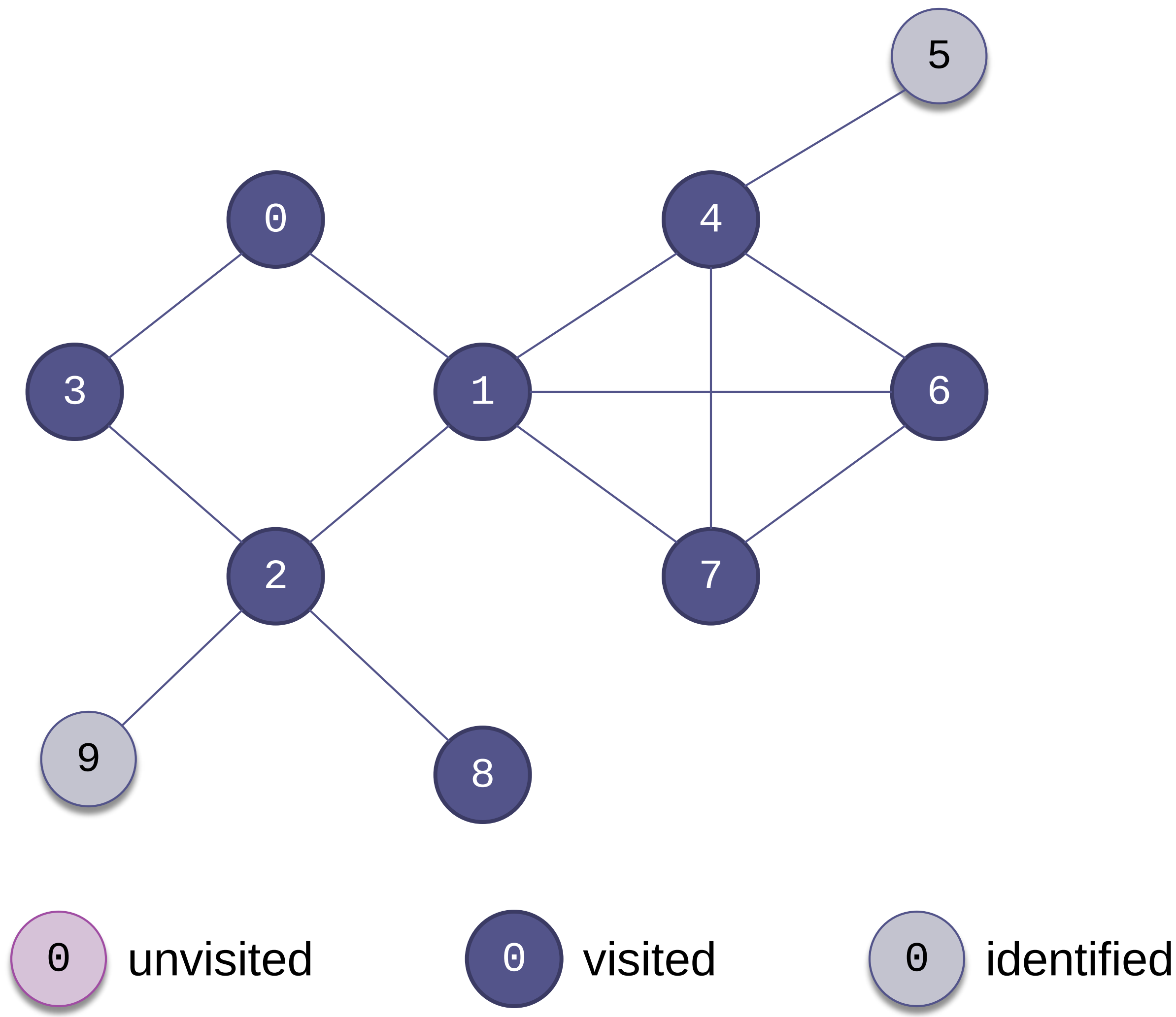


Queue:
8, 9, 5

Visit sequence:
0, 1, 3, 2, 4, 6, 7

Example of a Breadth-First Search (cont.)

8 has no vertices not already visited or identified



Queue:
9, 5

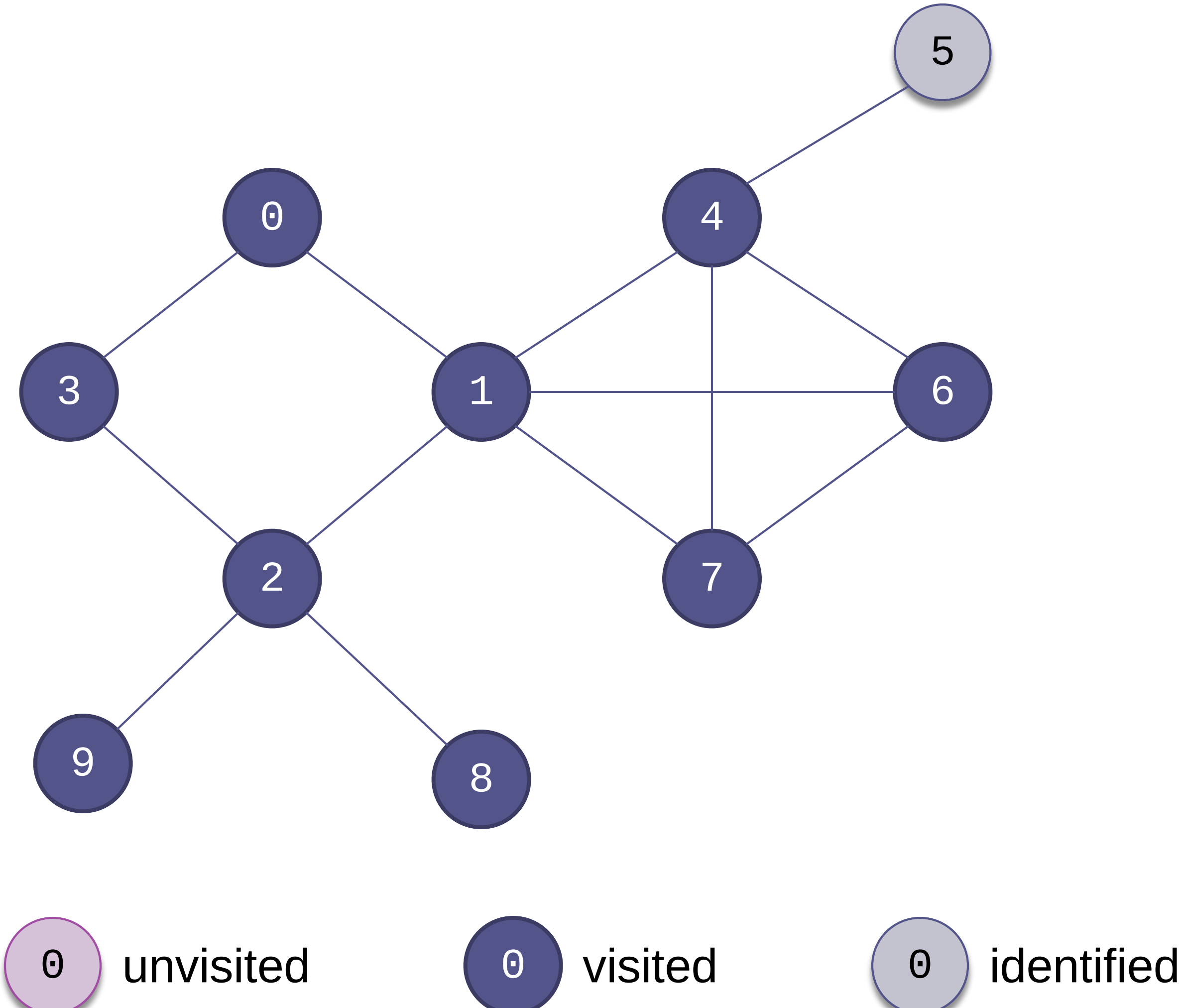
Visit sequence:
0, 1, 3, 2, 4, 6, 7, 8

Example of a Breadth-First Search (cont.)

9 has no vertices not already visited or identified

Queue:
5

Visit sequence:
0, 1, 3, 2, 4, 6, 7, 8, 9

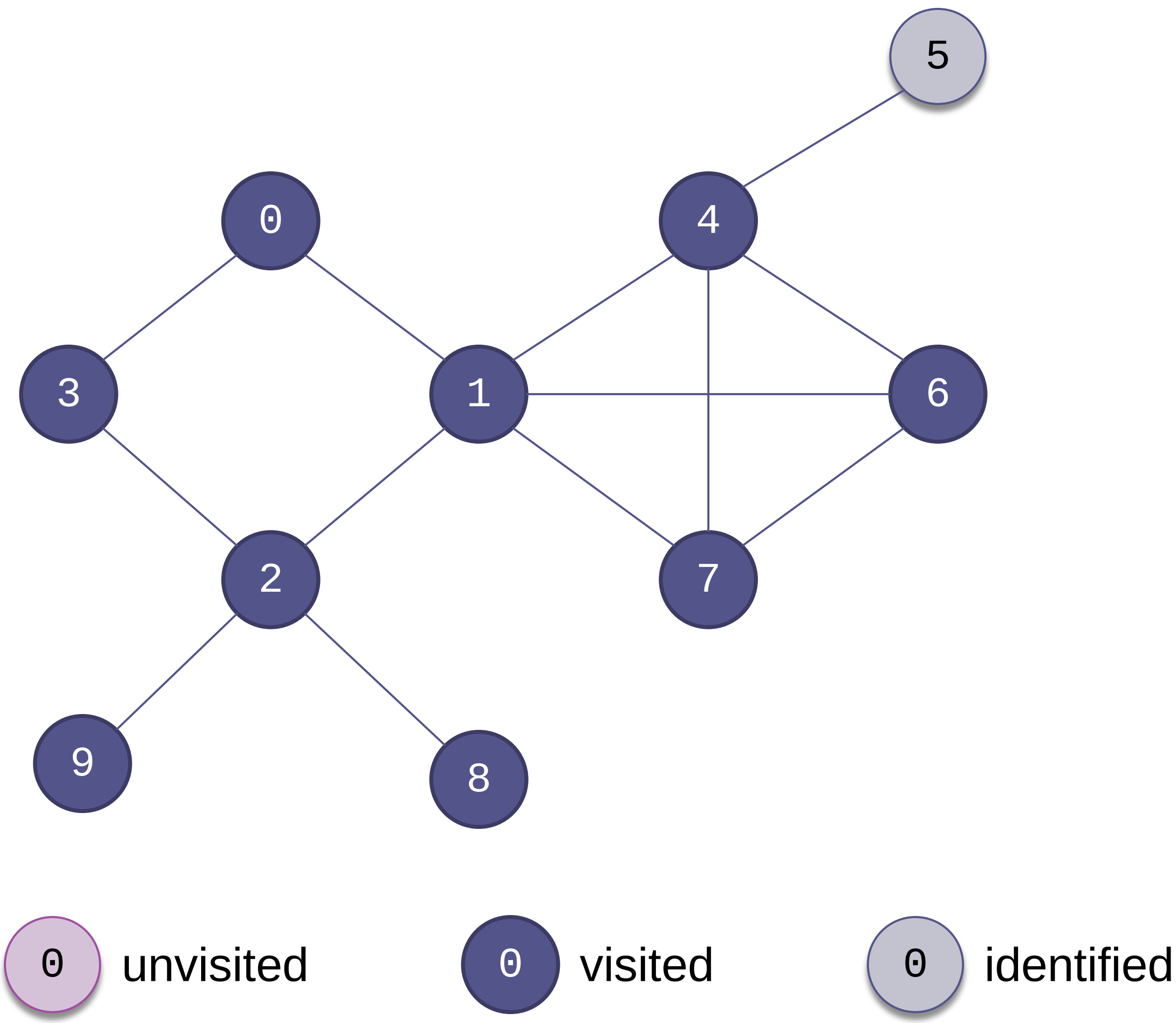


Example of a Breadth-First Search (cont.)

Finally we visit 5

Queue:
5

Visit sequence:
0, 1, 3, 2, 4, 6, 7, 8, 9

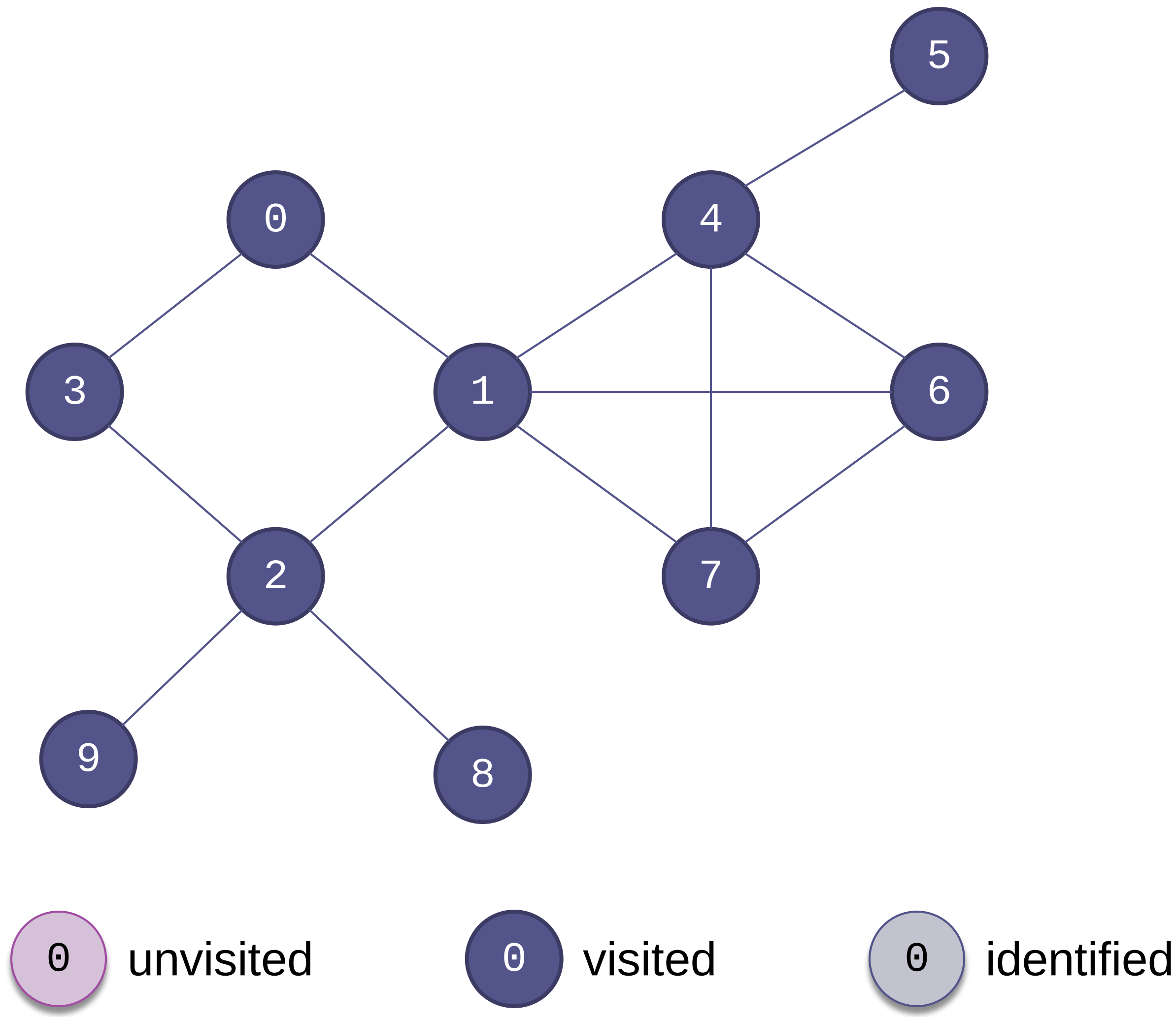


Example of a Breadth-First Search (cont.)

which has no vertices not already visited or identified

Queue:
empty

Visit sequence:
0, 1, 3, 2, 4, 6, 7, 8, 9, 5

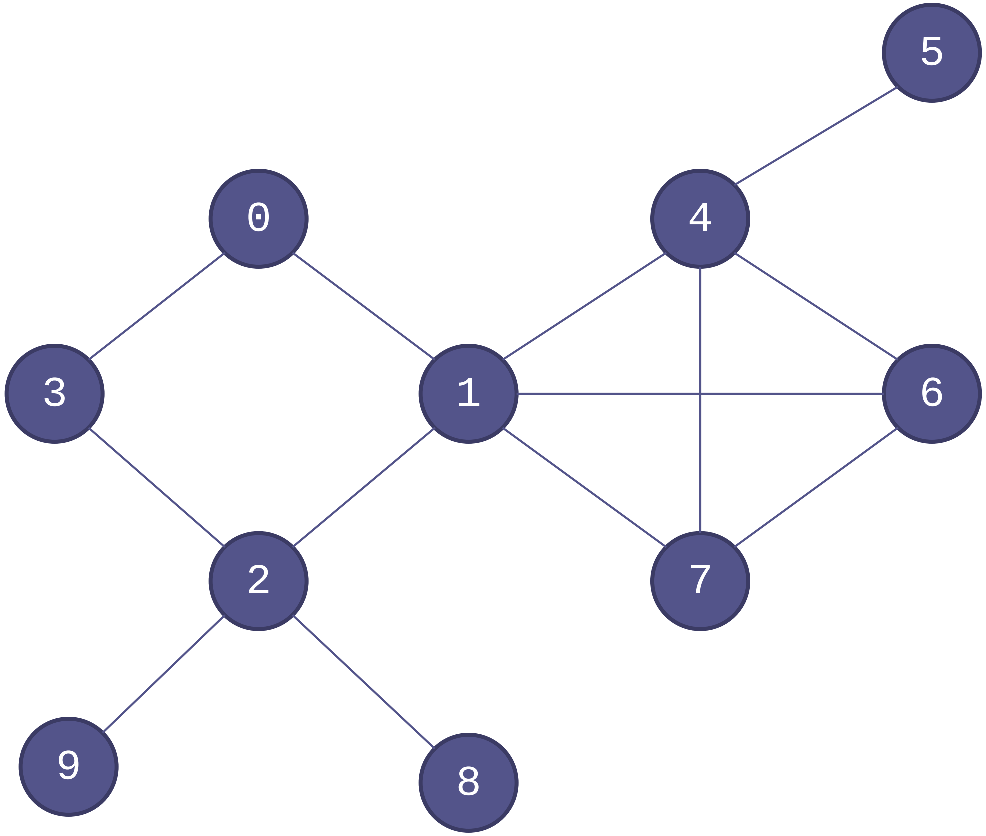


Example of a Breadth-First Search (cont.)

The queue is empty; all vertices have been visited

Queue:
empty

Visit sequence:
0, 1, 3, 2, 4, 6, 7, 8, 9, 5



PROPERTIES

Notation

G_s: connected component of s

Property 1

$BFS(G, s)$ visits all the vertices and edges of G_s

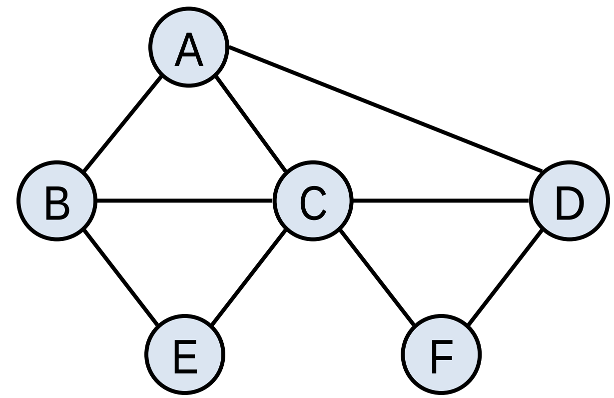
Property 2

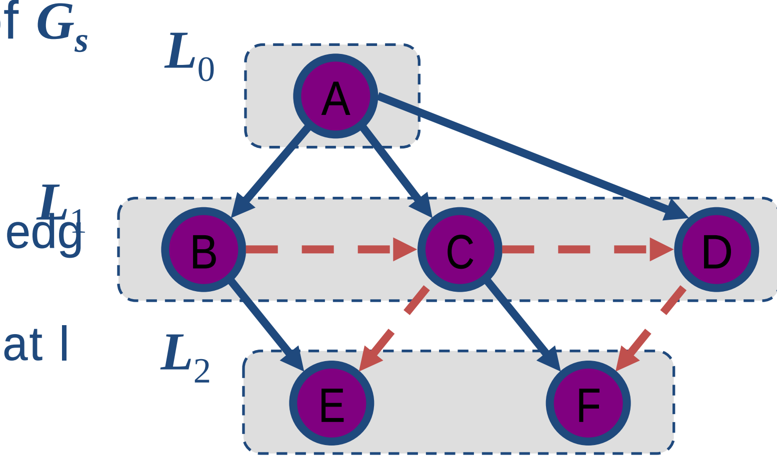
The discovery edges labeled by BF
$S(G, s)$ form a spanning tree T_s of G_s

Property 3

For each vertex v in L_i

- The path of T_s from s to v has i edg
  es
- Every path from s to v in G_s has at l
  east i edges





ANALYSIS

- Setting/getting a vertex/edge label takes $O(1)$ time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or CROSS
- Each vertex is inserted once into a sequence L_i
- Method incidentEdges is called once for each vertex
- BFS runs in $O(n + m)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$

APPLICATIONS

- Using the template method pattern, we can specialize the BFS traversal of a graph G to solve the following problems in $O(n + m)$ time
  - Compute the connected components of G
  - Compute a spanning forest of G
  - Find a simple cycle in G, or report that G is a forest
  - Given two vertices of G, find a path in G between them with the minimum number of edges, or report that no such path exists