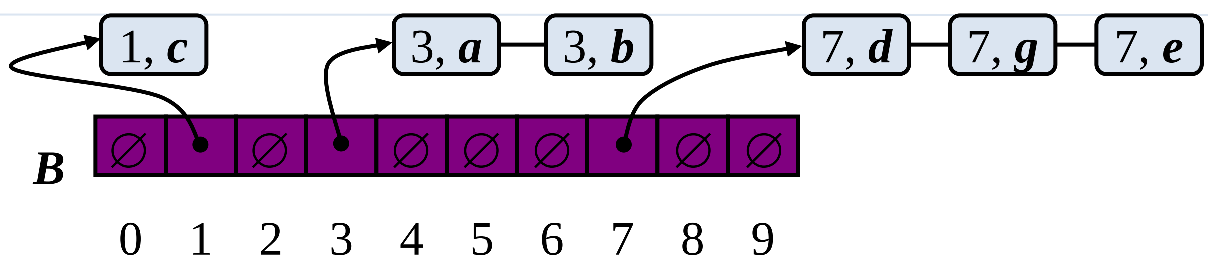
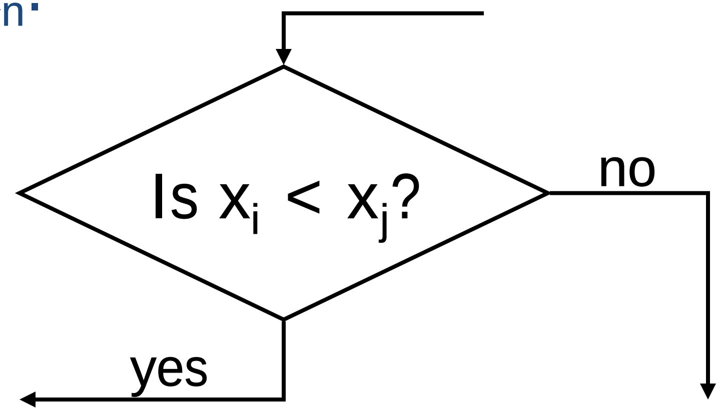
SORTING LOWER BOUND & BUCKET-SORT AND RADIX-SORT



Presentation for use with the textbook Data Structures and Algorithms in Java, 6th edition, by M. T. Goodrich, R. Tamassia, and M. H. Goldwasser, Wiley, 2014

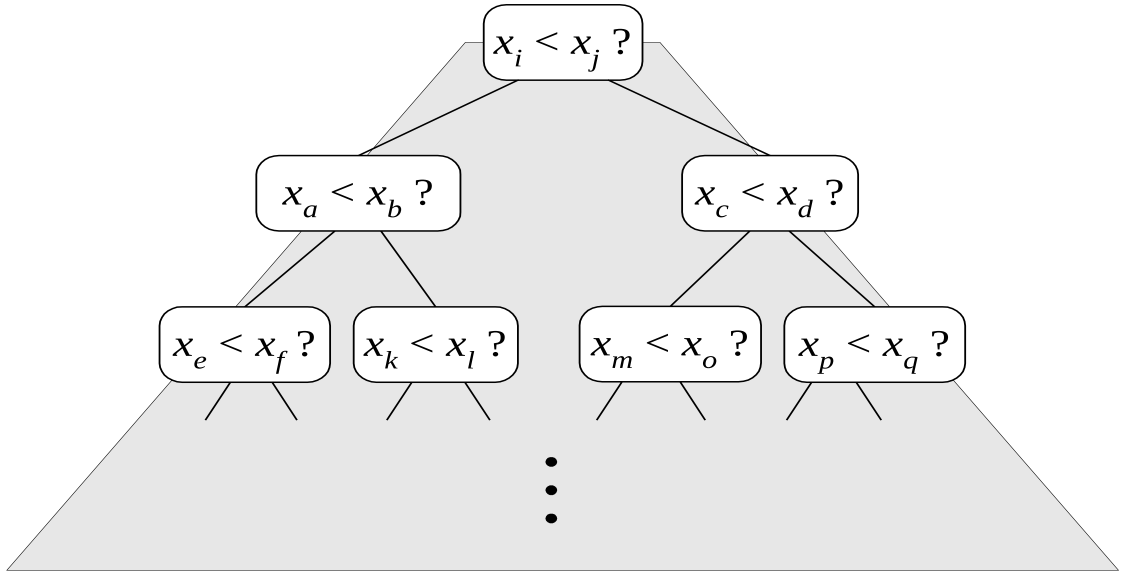
COMPARISON-BASED SORTING

- Many sorting algorithms are comparison based.
  - They sort by making comparisons between pairs of objects
  - Examples: selection-sort, insertion-sort, heap-sort, merge-sort, quick-sort, ...
- Let us therefore derive a lower bound on the running time of any algorithm that uses comparisons to sort n elements, $x_1, x_2, \ldots, x_n$.



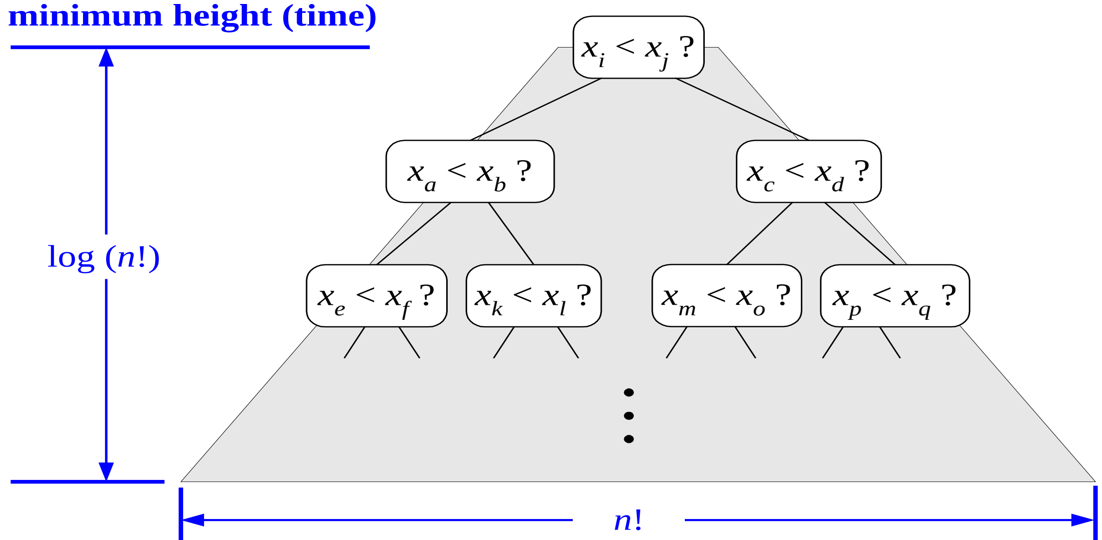
COUNTING COMPARISONS

- Let us just count comparisons then.
- Each possible run of the algorithm corresponds to a root-to-leaf path in a decision tree



DECISION TREE HEIGHT

- The height of the decision tree is a lower bound on the running time
- Every input permutation must lead to a separate leaf output
- If not, some input ...4...5... would have same output ordering as ...5...4..., which would be wrong
- Since there are n!=1·2 · ... ·n leaves, the height is at least log (n!)



THE LOWER BOUND

- Any comparison-based sorting algorithms takes at least log (n!) time
- Therefore, any such algorithm takes time at least

$$\log\,(n!) \geq \log\left(\frac{n}{2}\right)^{\frac{n}{2}} = (n/2)\log(n/2).$$

- That is, any comparison-based sorting algorithm must run in Ω(n log n) lower bound on its running time.

LINEAR TIME SORTING

- We showed that the lower bound of sorting with comparison is $\Omega(n \log n)$ time.
- Can we do better? Yes, with special assumptions about the input sequence to be sorted.
- We will consider the problem of sorting a sequence of entries, each a key-value pair, where the keys have a restricted type
  - Bucket-Sort
  - Radix-Sort

BUCKET-SORT

- Let be S be a sequence of n (key, element) entries with **integer keys** in the range [0, N-1], for some integer N≥2,
- **Bucket-sort** uses the keys as indices into an auxiliary array B of size N (buckets)

  Phase 1: Empty sequence S by moving each entry (k, o) into its bucket $B[k]$

  Phase 2: For $i = 0, \ldots, N - 1$, move the entries of bucket $B[i]$ to the end of sequence S
- Analysis:
  - Phase 1 takes $O(n)$ time
  - Phase 2 takes $O(n + N)$ time

  Bucket-sort takes $O(n + N)$ time

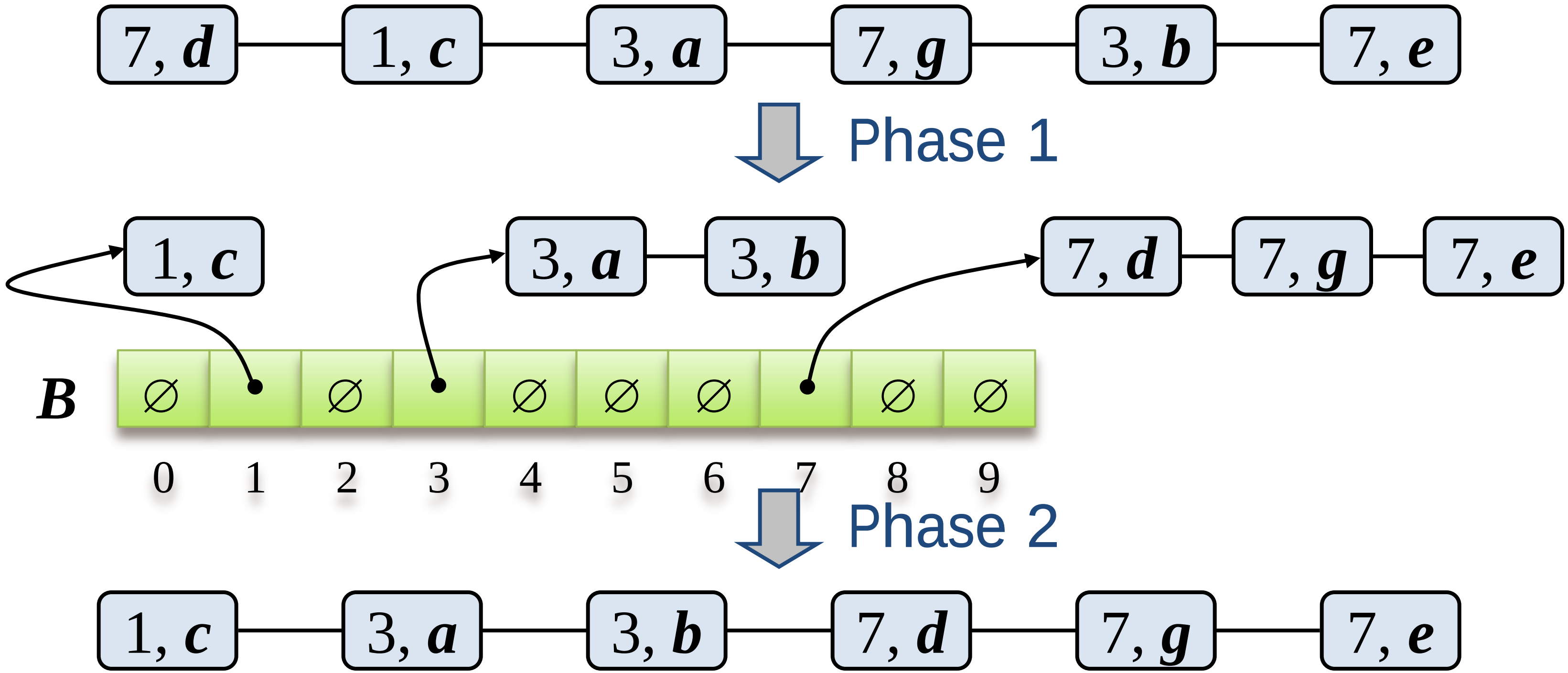
BUCKET-SORT ALGORITHM

Algorithm bucketSort(S):
Input: Sequence S of entries with integer keys in the range [0, N − 1]
Output: Sequence S sorted in nondecreasing order of the keys

let B be an array of N sequences, each of which is initially empty
for each entry e in S **do**
  k = the key of e
  remove e from S
  insert e at the end of bucket (sequence) B[*k*]
for *i* = 0 to N−1 **do**
  for each entry e in B[*i*] **do**
    remove e from B[*i*]
    insert e at the end of S

EXAMPLE

- Key range [0, 9]



PROPERTIES AND EXTENSIONS

- Key-type Property
  - The keys are used as indices into an array and cannot be arbitrary objects
  - No external comparator
- Stable Sort Property
  - The relative order of any two items with the same key is preserved after the execution of the algorithm

- Extensions
  - Integer keys in the range $[a, b]$
    - Put entry (k, o) into bucket $B[k - a]$
  - String keys from a set D of possible strings, where D has constant size (e.g., names of the 50 U.S. states)
    - Sort D and compute the rank $r(k)$ of each string k of D in the sorted sequence
    - Put entry (k, o) into bucket $B[r(k)]$

LEXICOGRAPHIC ORDER

- A d-tuple is a sequence of d keys $(k_1, k_2, \ldots, k_d)$, where key k_i is said to be the i-th dimension of the tuple
- Example:
  - The Cartesian coordinates of a point in space are a 3-tuple
- The lexicographic order of two d-tuples is recursively defined as follows

$$(x_1, x_2, \ldots, x_d) < (y_1, y_2, \ldots, y_d)$$
$$\Leftrightarrow$$
$$x_1 < y_1 \lor x_1 = y_1 \land (x_2, \ldots, x_d) < (y_2, \ldots, y_d)$$

I.e., the tuples are compared by the first dimension, then by the second dimension, etc.

STABLE SORTING

- When sorting key-value pairs, an important issue is how equal keys are handled. Let $S = ((k_0,v_0), \ldots, (k_{n-1},v_{n-1}))$ be a sequence of such entries.
- We say that a sorting algorithm is ***stable*** if, for any two entries (k_i,v_i) and (k_j, v_j) of S such that $k_i = k_j$ and (k_i,v_i) precedes (k_j, v_j) in S before sorting (that is, $i < j$), entry (k_i,v_i) also precedes entry (k_j, v_j) after sorting.
- Stability is important for a sorting algorithm because applications may want to preserve the initial order of elements with the same key.
- Bucket-sort guarantees stability as long as we ensure that all sequences act as **queues**

LEXICOGRAPHIC-SORT

- Let C_i be the comparator that compares two tuples by their i-th dimension
- Let ***stableSort***(S, C) be a stable sorting algorithm that uses comparator C
- Lexicographic-sort sorts a sequence of d-tuples in lexicographic order by executing d times algorithm ***stableSort***, one per dimension
- Lexicographic-sort runs in $O(dT(n))$ time, where $T(n)$ is the running time of ***stableSort***

Algorithm ***lexicographicSort***(S)
 Input sequence S of d-tuples
 Output sequence S sorted in lexicographic order

 for $i \leftarrow d$ **downto** 1
  stableSort(S, C_i)

Example:

(7,4,6) (5,1,5) (2,4,6) (2, 1, 4) (3, 2, 4)

(2, 1, 4) (3, 2, 4) (5,1,5) (7,4,6) (2,4,6)

(2, 1, 4) (5,1,5) (3, 2, 4) (7,4,6) (2,4,6)

(2, 1, 4) (2,4,6) (3, 2, 4) (5,1,5) (7,4,6)

RADIX-SORT

- **Radix-sort** is a specialization of lexicographic-sort that uses bucket-sort as the stable sorting algorithm in each dimension
- Radix-sort is applicable to tuples where the keys in each dimension i are integers in the range $[0, N-1]$
- Radix-sort runs in time $O(d(n+N))$ where the d is the dimension of keys, n is the number of data, and keys range is [0…N-1]

Algorithm ***radixSort***(S, N)

Input sequence S of d-tuples such that $(0, \ldots, 0) \le (x_1, \ldots, x_d)$ and $(x_1, \ldots, x_d) \le (N-1, \ldots, N-1)$ for each tuple $(x_1, \ldots, x_d)$ in S

Output sequence S sorted in lexicographic order

for $i \leftarrow d$ **downto** 1

  bucketSort(S, N)

RADIX-SORT FOR BINARY NUMBERS

- Consider a sequence of n b-bit integers

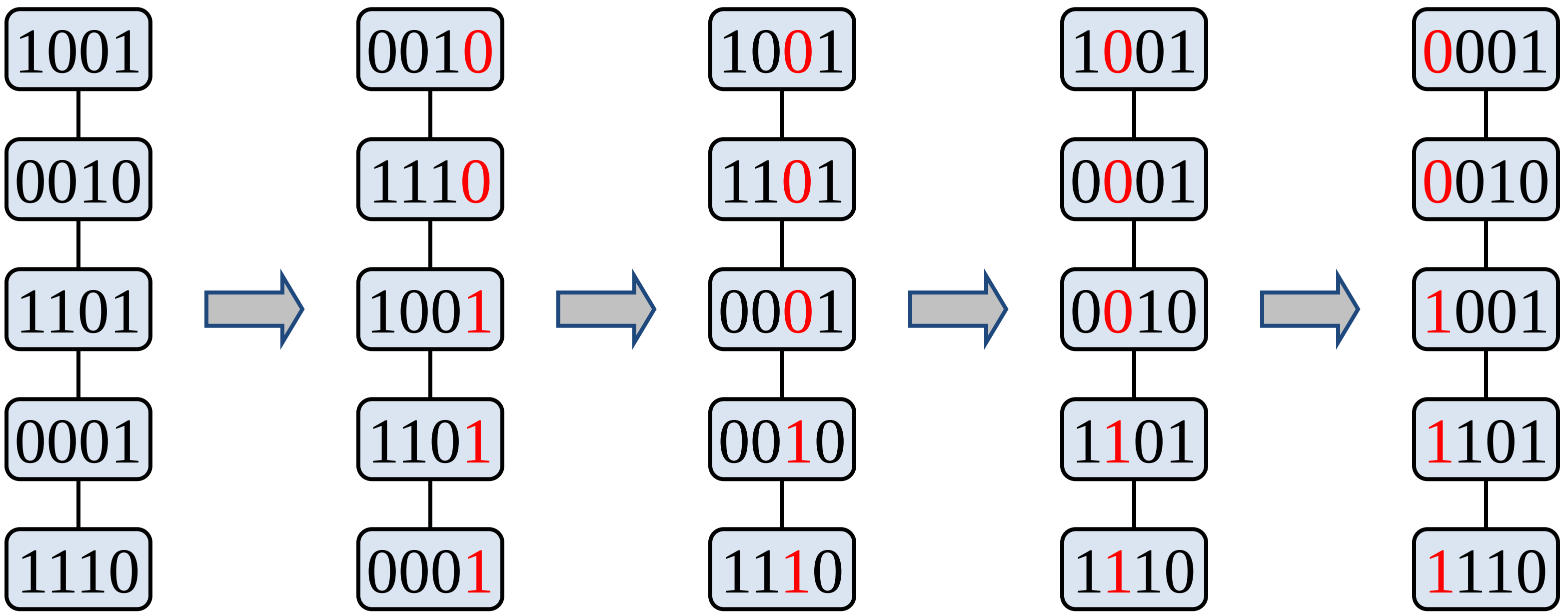
  $$x = x_{b-1} \dots x_1 x_0$$

- We represent each element as a b-tuple of integers in the range $[0, 1]$ and apply radix-sort with $N = 2$
- This application of the radix-sort algorithm runs in $O(bn)$ time
- For example, we can sort a sequence of 32-bit integers in linear time

Algorithm *binaryRadixSort*(S)
  Input sequence S of b-bit integers
  Output sequence S sorted
  replace each element x of S with the item $(0, x)$
  for $i \leftarrow 0$ **to** $b - 1$
    replace the key k of each item (k, x) of S with bit x_i of x
    bucketSort(S, 2)

EXAMPLE

- Sorting a sequence of 4-bit integers



SUMMARY OF SORTING ALGORITHMS

Algorithm	Time	Notes
selection-sort	$O(n^2)$	▪ in-place ▪ slow (good for small inputs)
insertion-sort	$O(n^2)$	▪ in-place ▪ slow (good for small inputs)
quick-sort	$O(n \log n)$ **expected**	▪ in-place, randomized ▪ fastest (good for large inputs)
heap-sort	$O(n \log n)$	▪ in-place ▪ fast (good for large inputs)
merge-sort	$O(n \log n)$	▪ sequential data access ▪ fast (good for huge inputs)
bucket-sort	$O(n+N)$	▪ integer keys of range [0 ... N]
radix-sort	$O(d(n+N))$	▪ d diinteger keys of range [0 ... N]

SELECTION PROBLEM

Presentation for use with the textbook Data Structures and Algorithms in Java, 6th edition, by M. T. Goodrich, R. Tamassia, and M. H. Goldwasser, Wiley, 2014

THE SELECTION PROBLEM

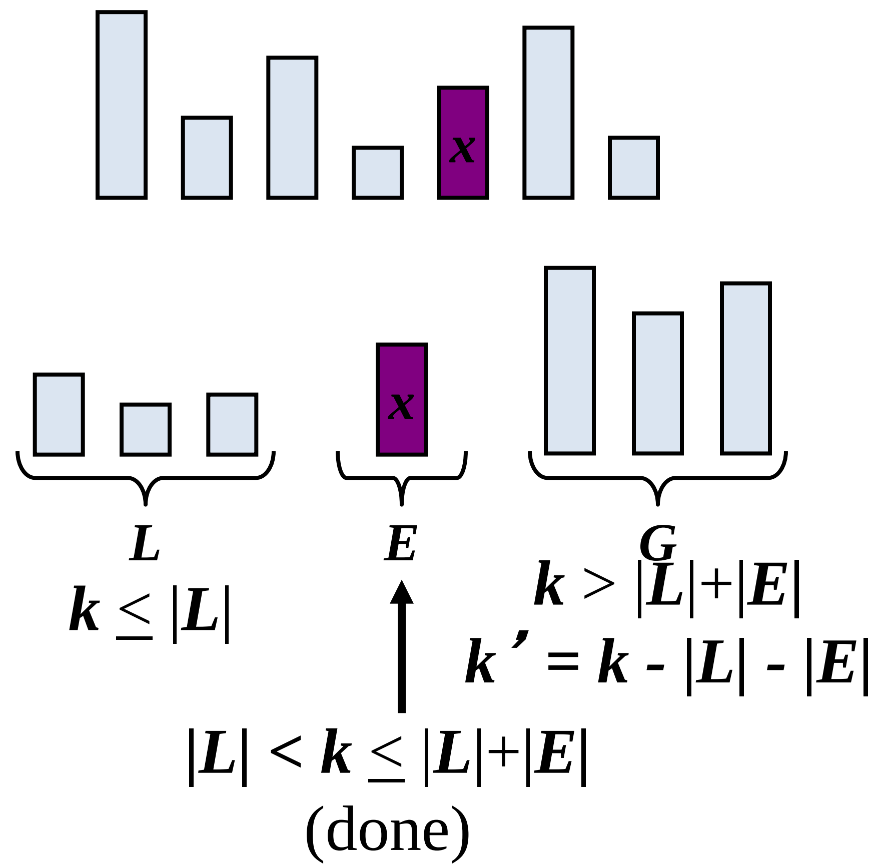
- Given an integer k and n elements $x_1, x_2, \ldots, x_n$, taken from a total order, find the k-th smallest element in this set.
- Of course, we can sort the set in O(n log n) time and then index the k-th element.

k=3 $\quad$ 7 4 9 $\underline{6}$ 2 $\rightarrow$ 2 4 $\underline{6}$ 7 9

- Can we solve the selection problem faster?

PRUNE-AND-SEARCH

- Quick-select is a randomized selection algorithm based on the prune-and-search paradigm:
  - Prune: pick a random element x (called pivot) and partition S into
    - L: elements less than x
    - E: elements equal x
    - G: elements greater than x
  - Search: depending on k, either answer is in E, or we need to recur in either L or G



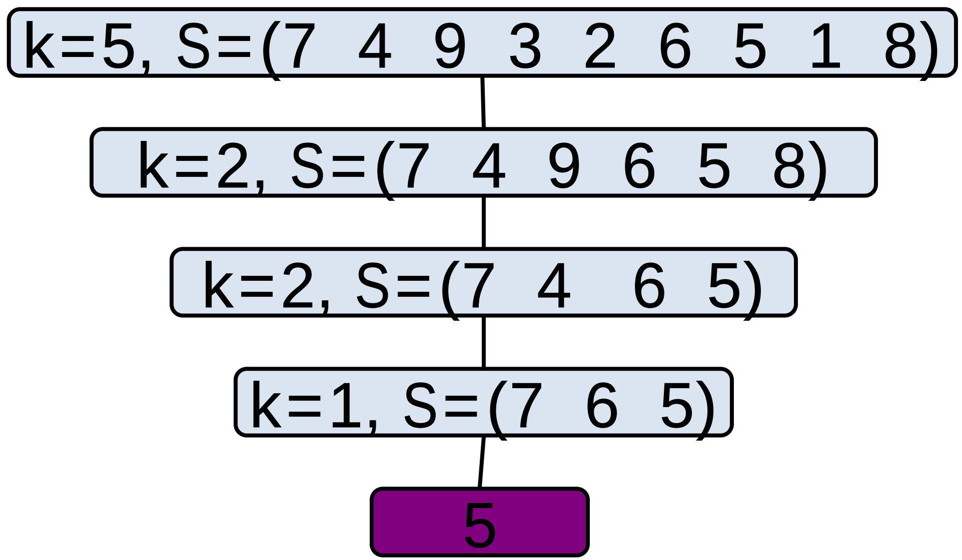
PARTITION

- We partition an input sequence as in the quick-select algorithm:
  - We remove, in turn, each element y from S and
  - We insert y into L, E or G, depending on the result of the comparison with the pivot x
- Each insertion and removal is at the beginning or at the end of a sequence, and hence takes $O(1)$ time
- Thus, the partition step of quick-select takes $O(n)$ time

```
Algorithm partition(S, p)
    Input sequence S, position p of pivot
    Output subsequences L, E, G of the
        elements of S less than, equal to,
        or greater than the pivot, resp.
    L, E, G ← empty sequences
    x ← S.remove(p)
    while ¬S.isEmpty()
        y ← S.remove(S.first())
        if y < x
            L.addLast(y)
        else if y = x
            E.addLast(y)
        else { y > x }
            G.addLast(y)
    return L, E, G
```

QUICK-SELECT VISUALIZATION

- An execution of quick-select can be visualized by a recursion path
  - Each node represents a recursive call of quick-select, and stores k and the remaining sequence

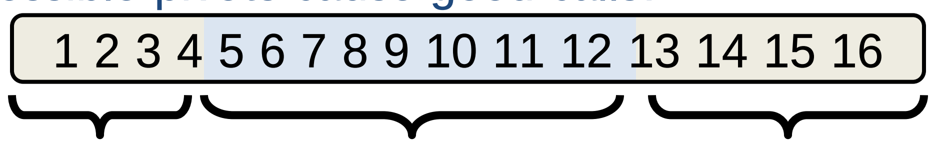


EXPECTED RUNNING TIME

- Consider a recursive call of quick-select on a sequence of size s
  - Good call: the sizes of L and G are each less than $3s/4$
  - Bad call: one of L and G has size greater than $3s/4$

Good call **Bad call**

- A call is good with probability $1/2$
  - 1/2 of the possible pivots cause good calls:



Bad pivots **Good pivots** **Bad pivots**

EXPECTED RUNNING TIME, PART 2

- Probabilistic Fact #1: The expected number of coin tosses required in ord er to get one head is two
- Probabilistic Fact #2: Expectation is a linear function:
  - $E(X + Y) = E(X) + E(Y)$
  - $E(cX) = cE(X)$
- Let T(n) denote the expected running time of quick-select.
- By Fact #2,
  - $T(n) \leq T(3n/4) + bn$*(expected # of calls before a good call)
- By Fact #1,
  - $T(n) \leq T(3n/4) + 2bn$
- That is, T(n) is a geometric series:
  - $T(n) \leq 2bn + 2b(3/4)n + 2b(3/4)^2 n + 2b(3/4)^3 n + \ldots$
- So T(n) is O(n).
- We can solve the selection problem in O(n) expected time.