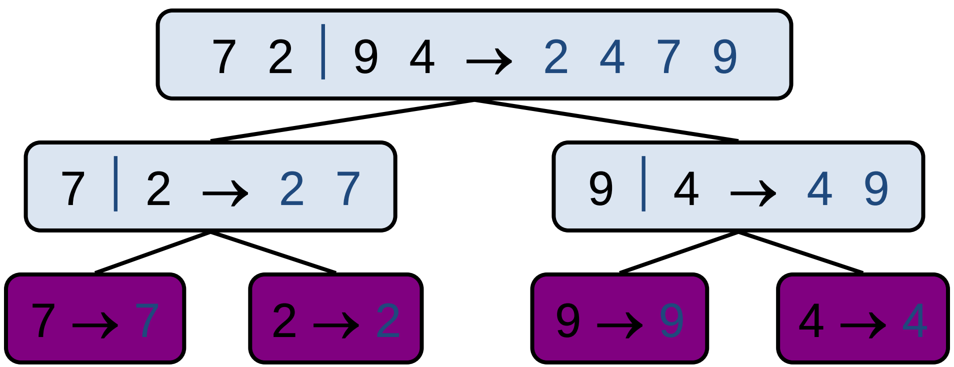
SORTING WITH DIVIDE AND CONQUER SCHEME

Presentation for use with the textbook Data Structures and Algorithms in Java, 6th edit ion, by M. T. Goodrich, R. Tamassia, and M. H. Goldwasser, Wiley, 2014

TYPES OF SORTING

- Sorting algorithms we have seen so far:
  - insertion-sort
  - selection-sort
  - heap-sort
- ***Divide-and-conquer based sorting***
  - **merge-sort**
  - **quick-sort**
- Linear time Sorting
  - **bucket-sort**
  - **radix-sort**

MERGE SORT



MERGE-SORT

- Merge-sort on an input sequence S with n elements consists of three steps:
  - *Divide*: If S has zero or one element, return S. *Otherwise* partition S into two sequences S_1 and S_2 of about $n/2$ elements each
  - *Conquer*: recursively sort S_1 and S_2
  - *Combine*: merge sorted S_1 and sorted S_2 into a unique sorted sequence

Algorithm ***mergeSort***(S)
  Input sequence S with n elements
  Output sequence S sorted according to C
  if $S.size() > 1$
    $(S_1, S_2) \leftarrow partition(S, n/2)$
    $mergeSort(S_1)$
    $mergeSort(S_2)$
    $S \leftarrow merge(S_1, S_2)$

DIVIDE-AND-CONQUER

- **Divide-and conquer** is a general algorithm design paradigm:
  - *Divide*: divide the input data S in two disjoint subsets S_1 and S_2
  - *Conquer*: solve the subproblems associated with S_1 and S_2
  - *Combine*: combine the solutions for S_1 and S_2 into a solution for S
- The base case for the recursion are subproblems of size 0 or 1

- **Merge-sort** is a sorting algorithm based on the divide-and-conquer paradigm
- Like heap-sort
  - It has $O(n \log n)$ running time
- Unlike heap-sort
  - It does not use an auxiliary priority queue
  - It accesses data in a sequential manner (suitable to sort data on a disk)

MERGING TWO SORTED SEQUENCES

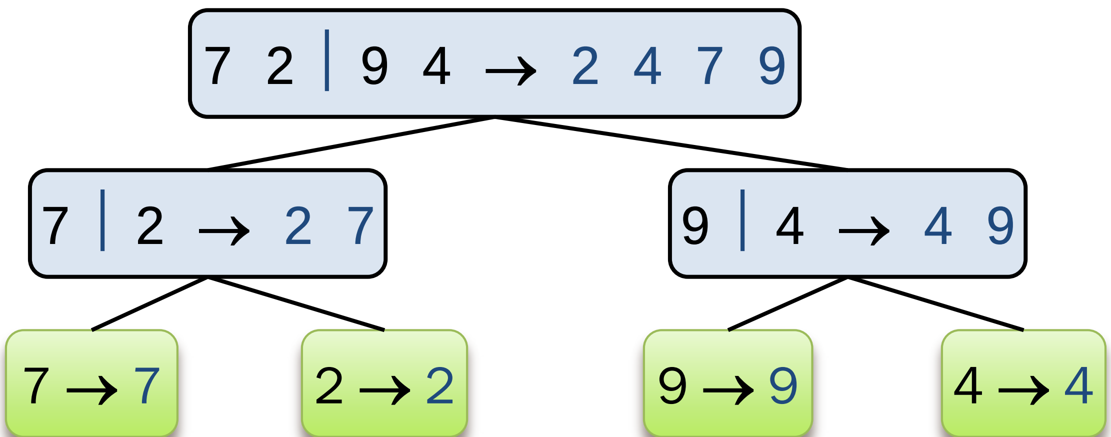
- The conquer step of merge-sort consists of merging two sorted sequences A and B into a sorted sequence S containing the union of the elements of A and B
- Merging two sorted sequences, each with $n/2$ elements and implemented by means of a doubly linked list, takes $O(n)$ time

Algorithm *merge*(*A*, *B*)
    Input sequences A and B with
        $n/2$ elements each
    Output sorted sequence of $A \cup B$

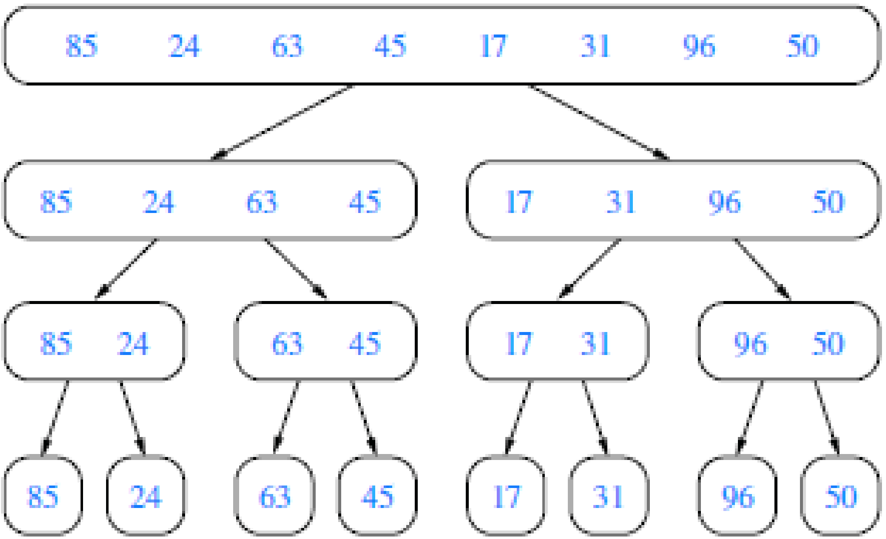
    $S \leftarrow$ empty sequence
    while $\neg A.isEmpty() \wedge \neg B.isEmpty()$
        if $A.first().element() < B.first().element()$
            $S.addLast(A.remove(A.first()))$
        else
            $S.addLast(B.remove(B.first()))$
    while $\neg A.isEmpty()$
        $S.addLast(A.remove(A.first()))$
    while $\neg B.isEmpty()$
        $S.addLast(B.remove(B.first()))$
    return S

MERGE-SORT TREE

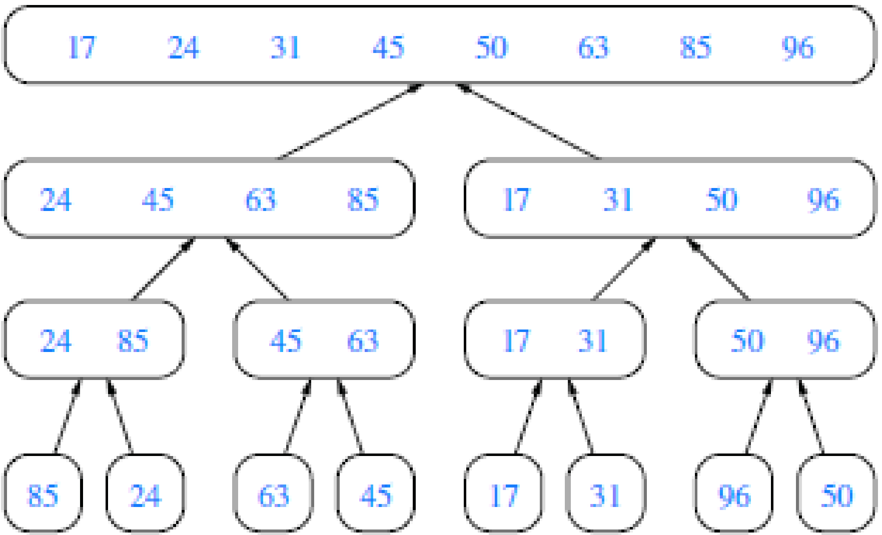
- An execution of merge-sort is depicted by a binary tree T, called the *merge-sort tree*
  - Each **node** represents a recursive call of merge-sort and stores
    - unsorted sequence before the execution and its partition
    - sorted sequence at the end of the execution
  - the **root** is the initial call
  - the **leaves** are calls on subsequences of size 0 or 1



EXAMPLE MERGE-SORT TREE *T*



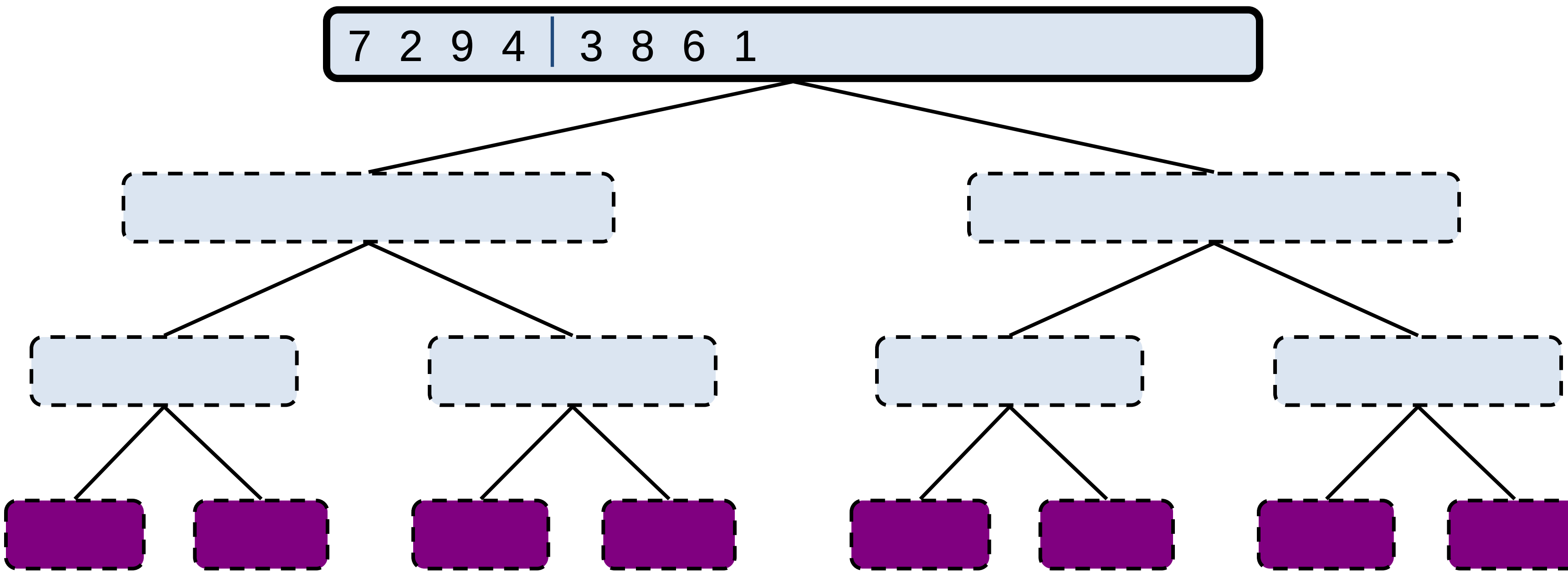
input sequences processed at each node of *T*



output sequences generated at each node of *T*.

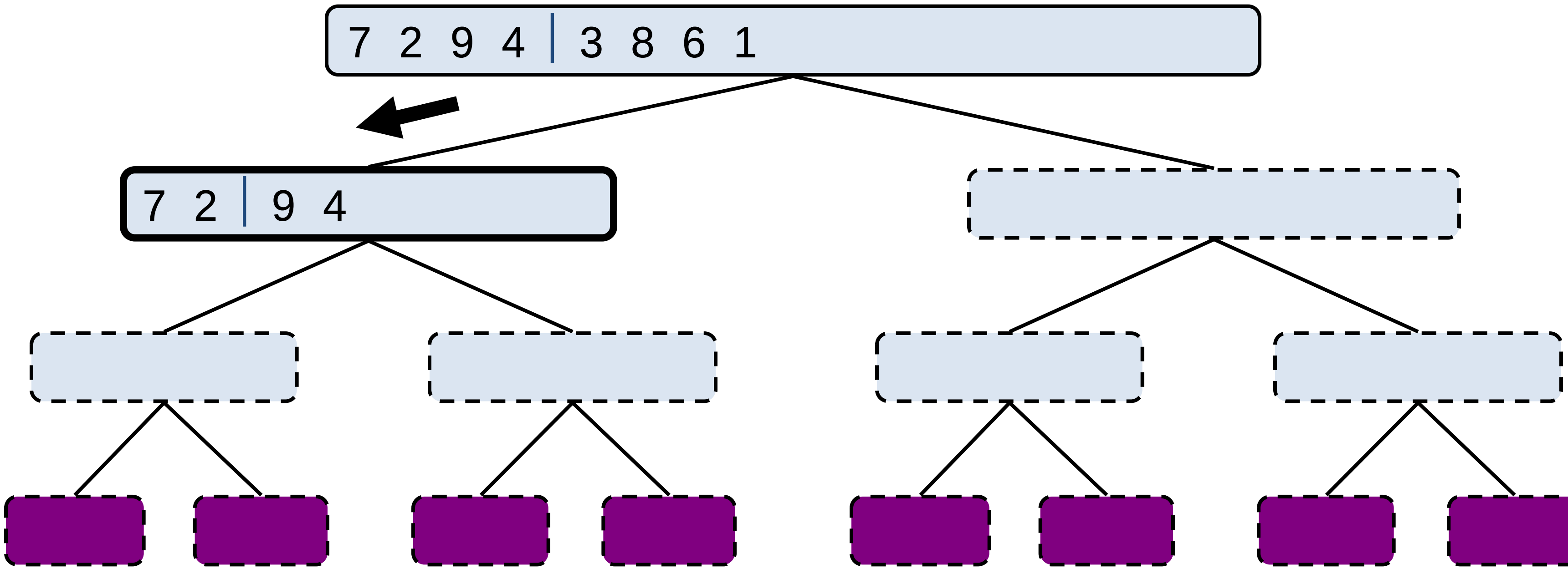
EXECUTION EXAMPLE

- Partition



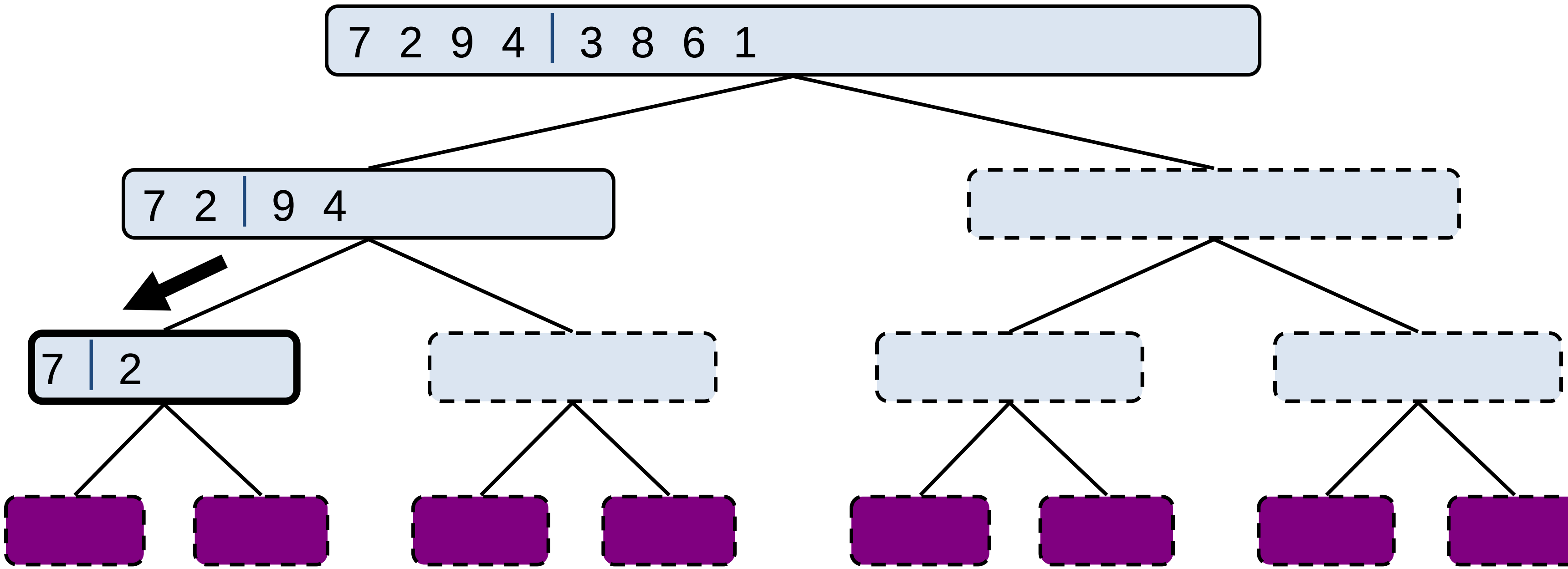
EXECUTION EXAMPLE (CONT.)

- Recursive call, partition



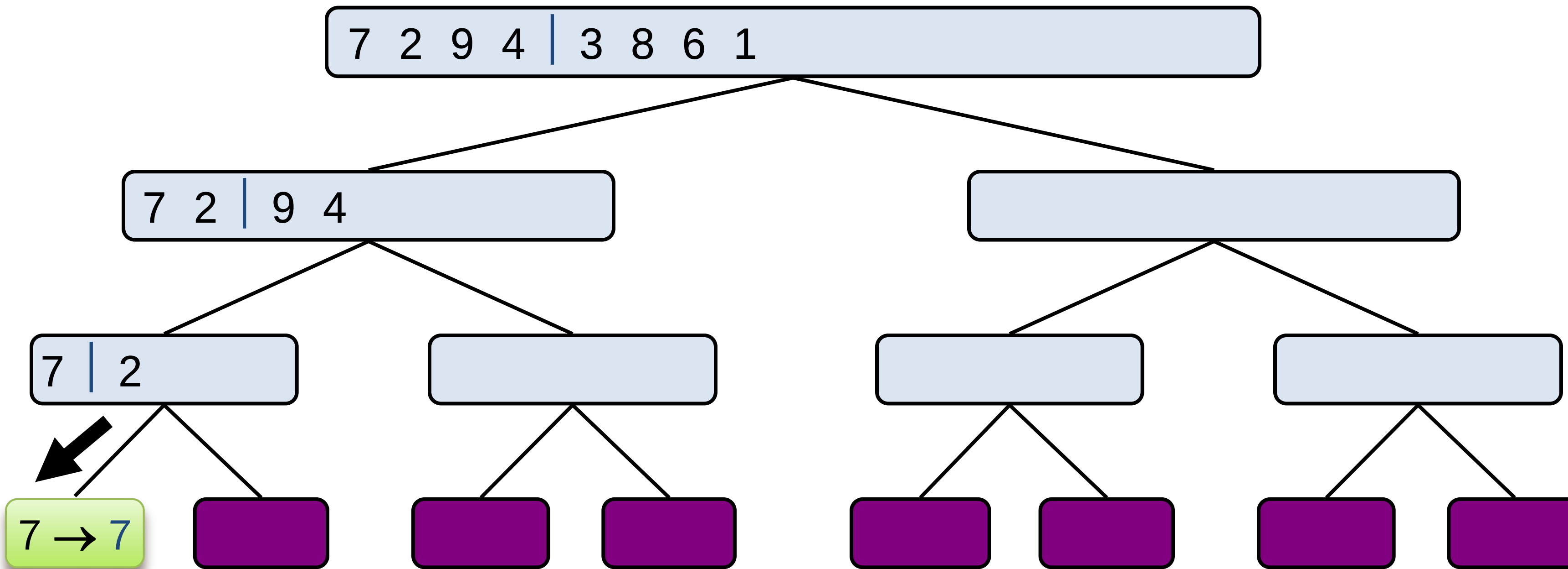
EXECUTION EXAMPLE (CONT.)

- Recursive call, partition



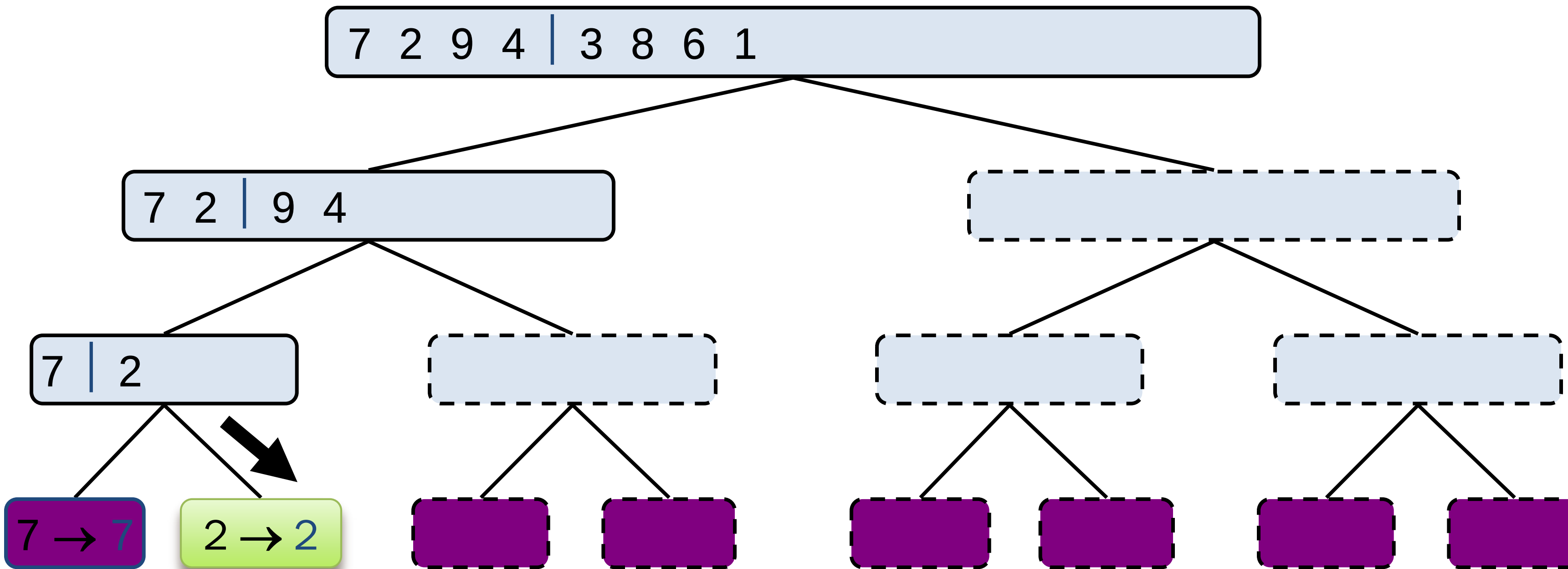
EXECUTION EXAMPLE (CONT.)

- Recursive call, base case



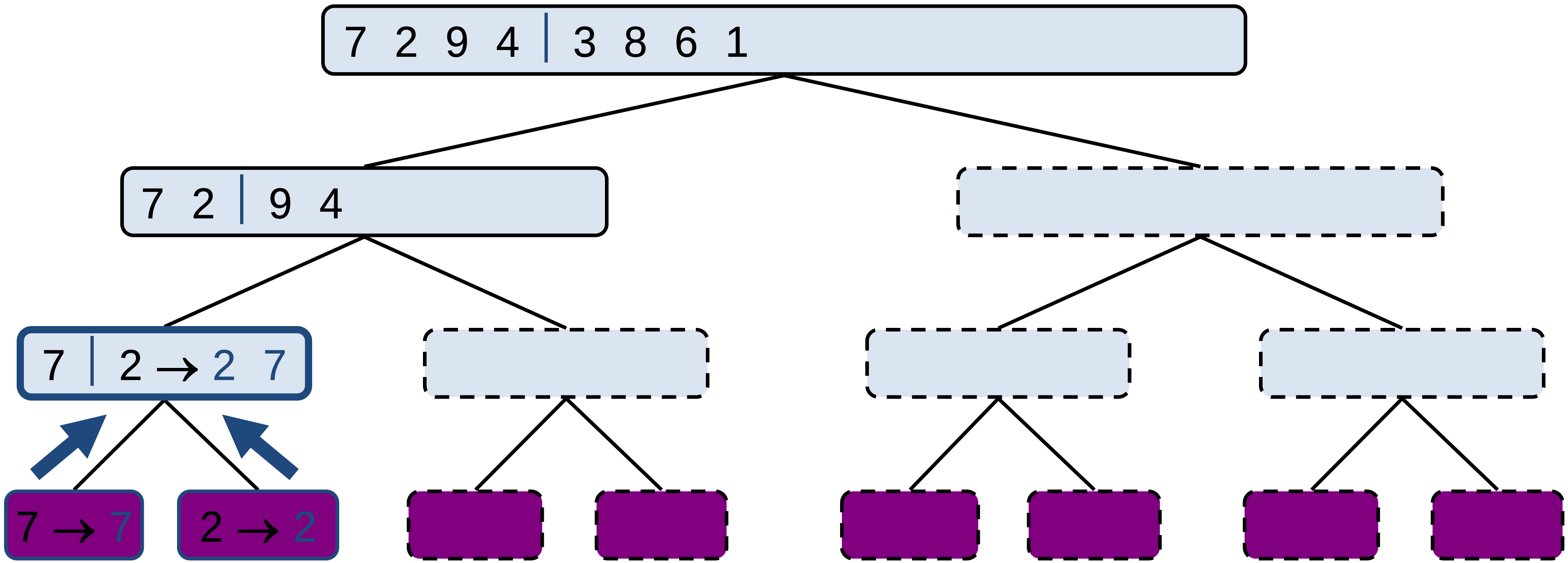
EXECUTION EXAMPLE (CONT.)

- Recursive call, base case



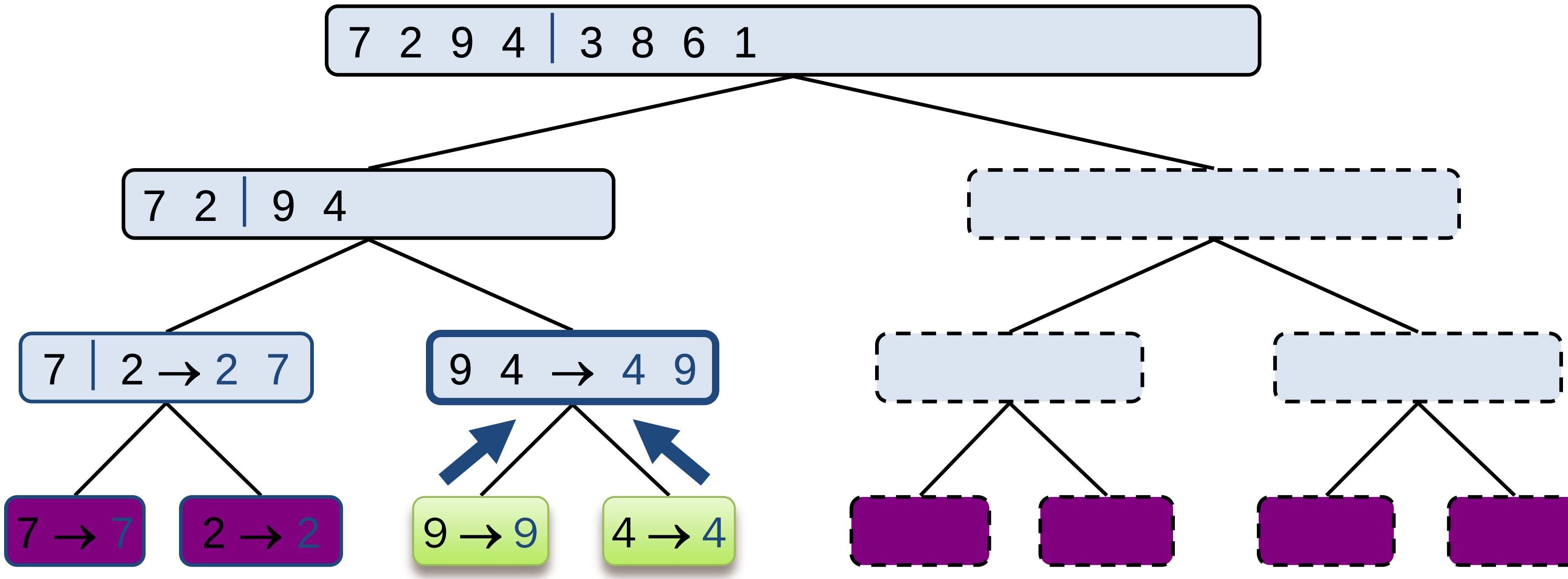
EXECUTION EXAMPLE (CONT.)

- Merge



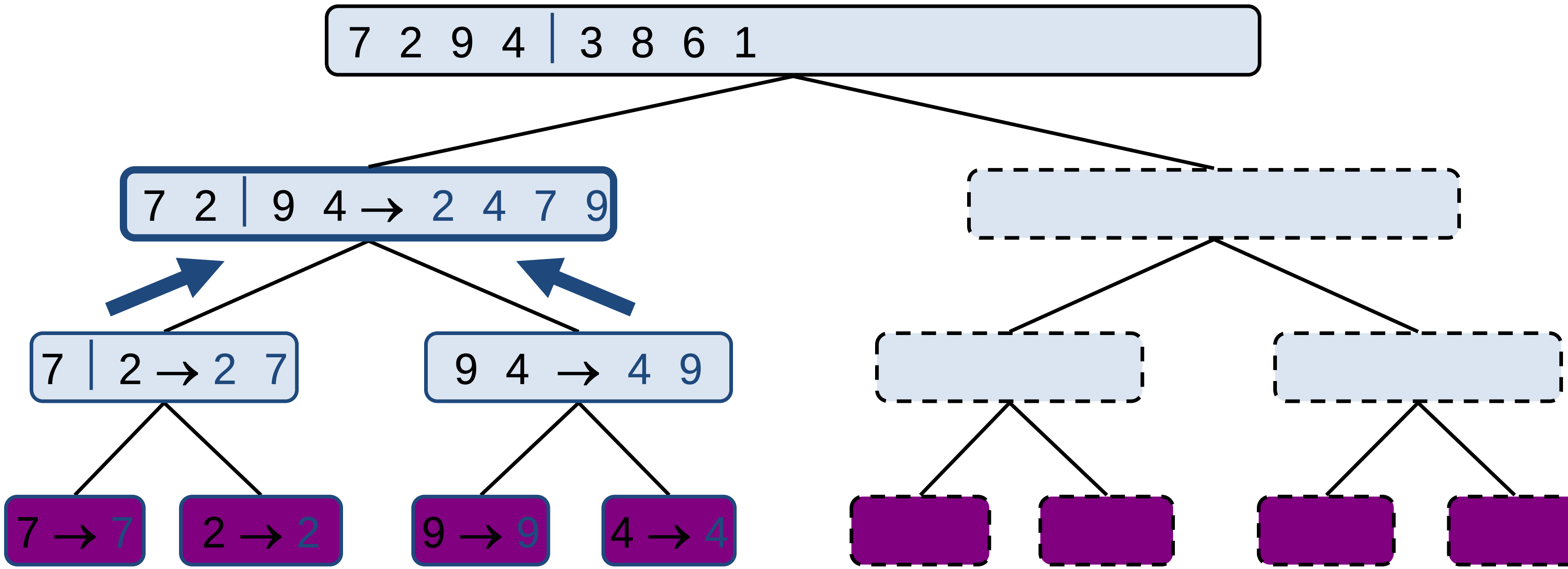
EXECUTION EXAMPLE (CONT.)

- Recursive call, ..., base case, merge



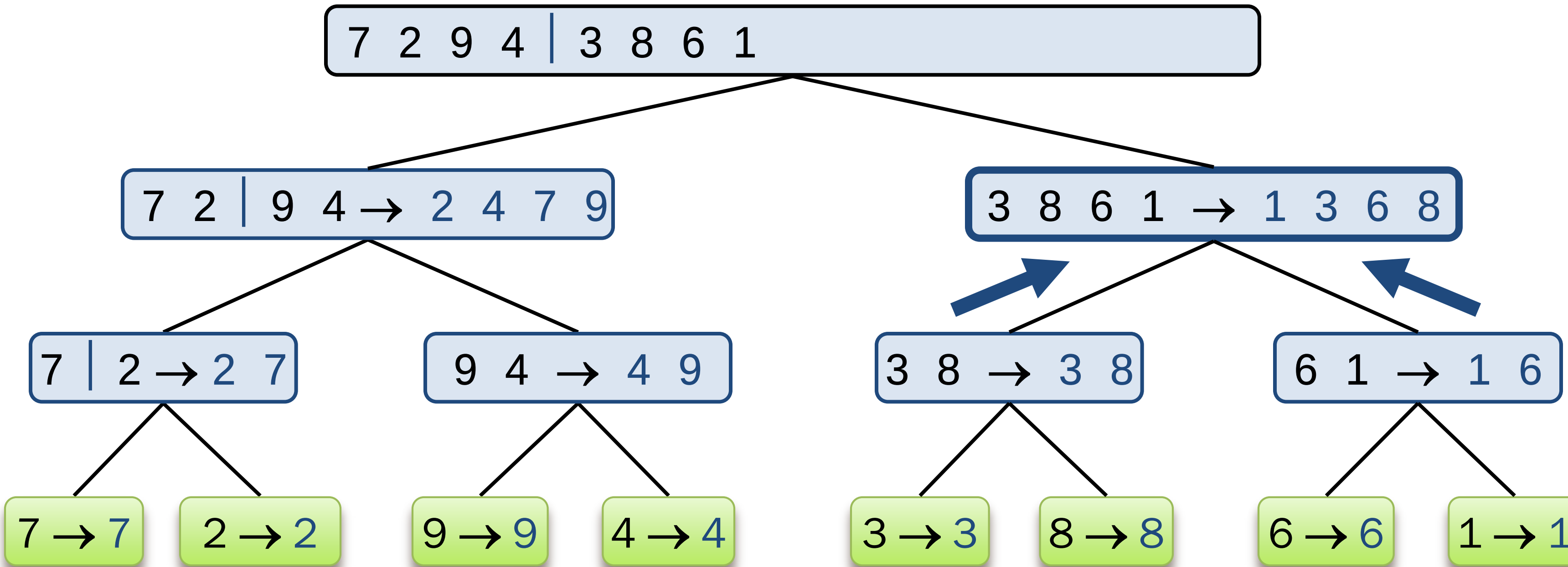
EXECUTION EXAMPLE (CONT.)

- Merge



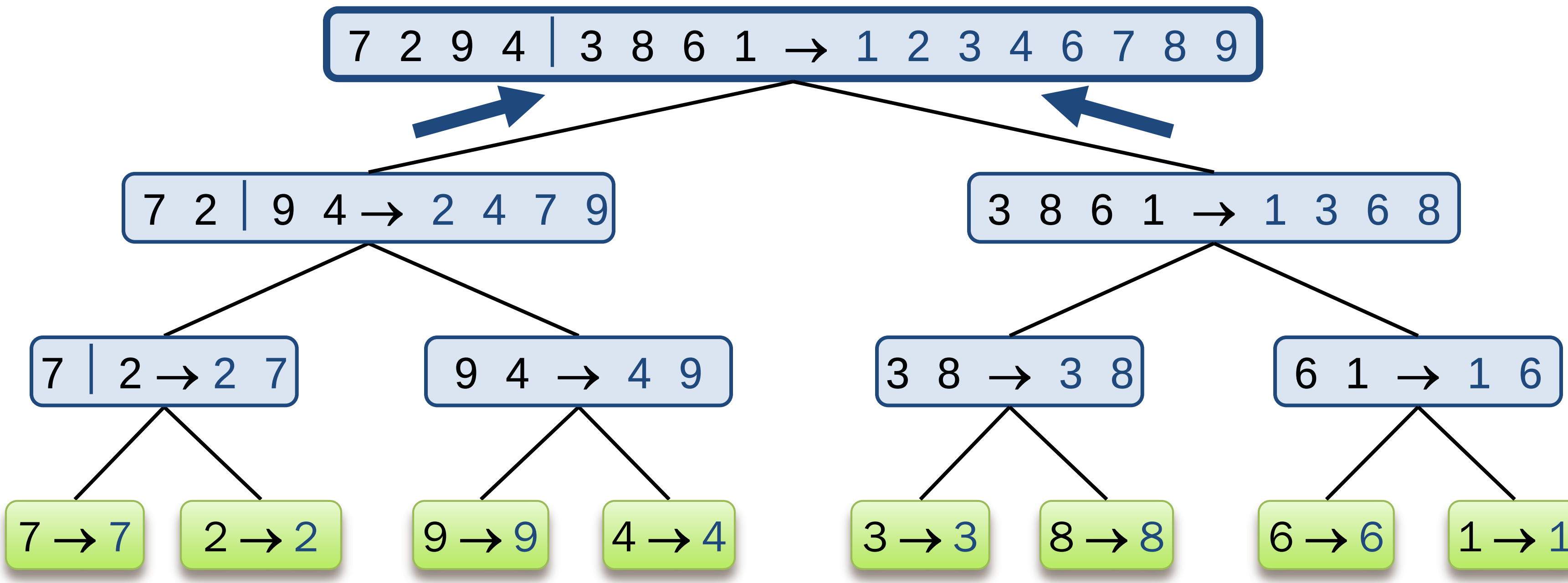
EXECUTION EXAMPLE (CONT.)

- Recursive call, ..., merge, merge



EXECUTION EXAMPLE (CONT.)

- Merge



ARRAY-BASED IMPLEMENTATION OF MERGE-SORT 1

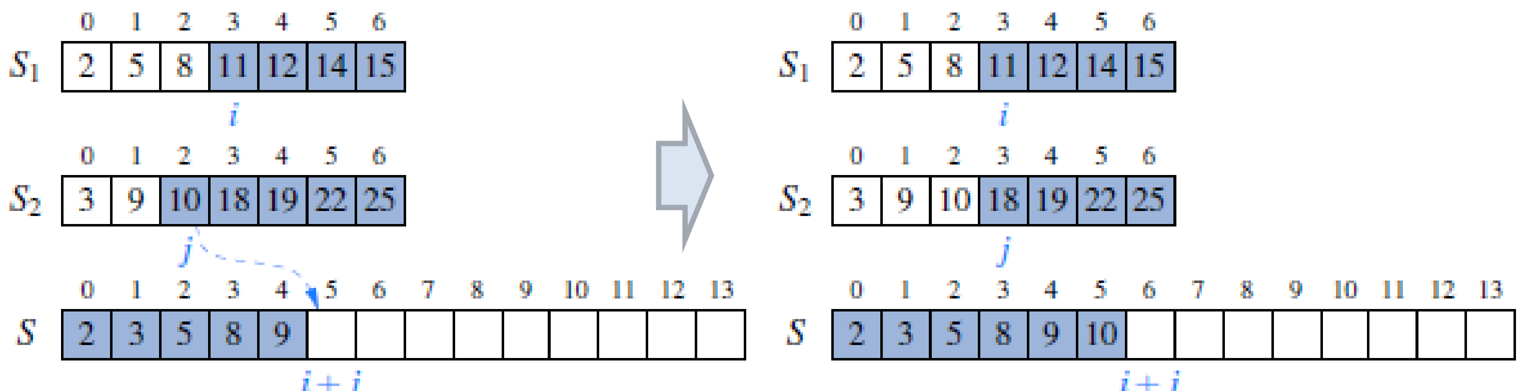
```
/** Merge-sort contents of array S. */
public static <K> void mergeSort(K[ ] S, Comparator<K> comp) {
  int n = S.length;
  if (n < 2) return;                                  // array is trivially sorted
  // divide
  int mid = n/2;
  K[ ] S1 = Arrays.copyOfRange(S, 0, mid);            // copy of first half
  K[ ] S2 = Arrays.copyOfRange(S, mid, n);            // copy of second half
  // conquer (with recursion)
  mergeSort(S1, comp);                                // sort copy of first half
  mergeSort(S2, comp);                                // sort copy of second half
  // merge results
  merge(S1, S2, S, comp);                 // merge sorted halves back into original
}
```

ARRAY-BASED IMPLEMENTATION OF MERGE-SORT 2

```
/** Merge contents of arrays S1 and S2 into properly sized array S. */
public static <K> void merge(K[ ] S1, K[ ] S2, K[ ] S, Comparator<K> comp) {
  int i = 0, j = 0;
  while (i + j < S.length) {
    if (j == S2.length || (i < S1.length && comp.compare(S1[i], S2[j]) < 0))
      S[i+j] = S1[i++];                    // copy ith element of S1 and increment i
    else
      S[i+j] = S2[j++];                    // copy jth element of S2 and increment j
  }
}
```

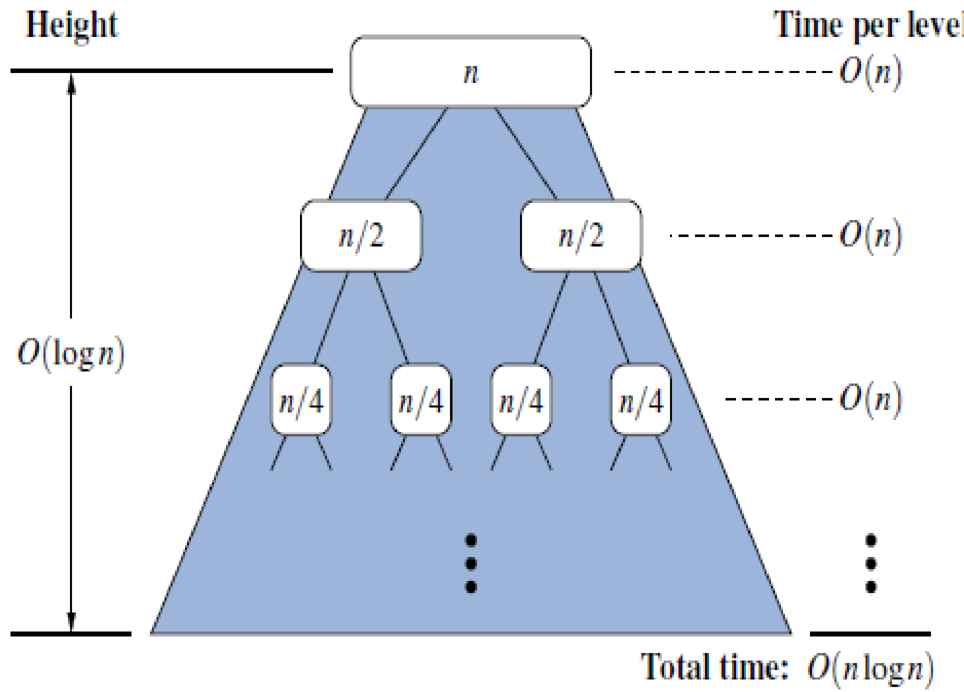
indices *i* & *j* represents the number of elements of *S*1 & S2 that have been copied to *S*

A step in the merge of two sorted arrays for which $S2[j] < S1[i]$.



ANALYSIS OF MERGE-SORT

- The height h of the merge-sort tree is $O(\log n)$
  - at each recursive call we divid in half the sequence,
- The overall work done at the nodes of depth i is $O(n)$
  - we partition and merge 2^i sequences of size $n/2^i$
  - we make 2^{i+1} recursive calls
- Thus, the total running time of merge-sort is $O(n\log n)$



EXTRA1. LINKED LIST IMPLEMENTATIONS OF MERGE-SORT 1

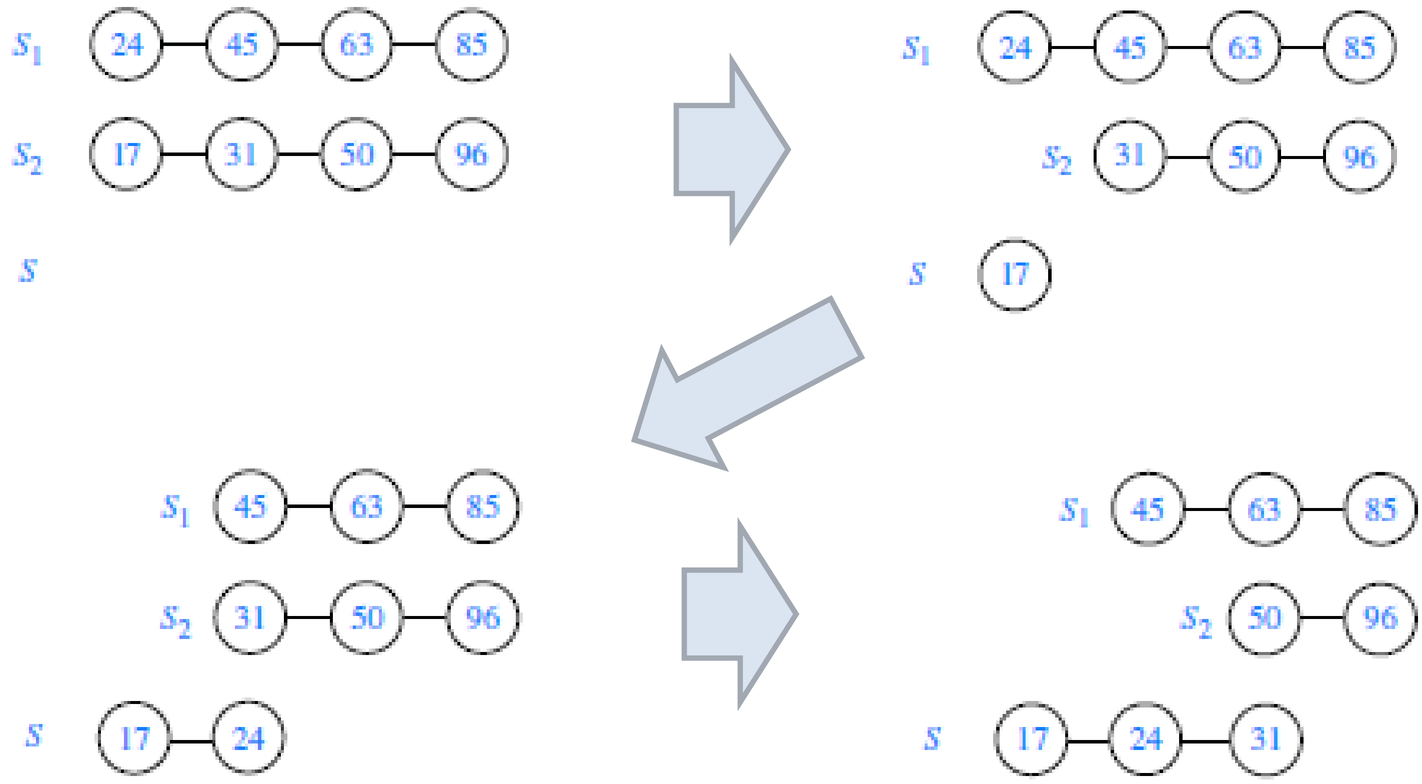
```
/** Merge-sort contents of queue. */
public static <K> void mergeSort(Queue<K> S, Comparator<K> comp) {
  int n = S.size();
  if (n < 2) return;                          // queue is trivially sorted
  // divide
  Queue<K> S1 = new LinkedQueue<>();          // (or any queue implementation)
  Queue<K> S2 = new LinkedQueue<>();
  while (S1.size() < n/2)
    S1.enqueue(S.dequeue());                  // move the first n/2 elements to S1
  while (!S.isEmpty())
    S2.enqueue(S.dequeue());                  // move remaining elements to S2
  // conquer (with recursion)
  mergeSort(S1, comp);                        // sort first half
  mergeSort(S2, comp);                        // sort second half
  // merge results
  merge(S1, S2, S, comp);                     // merge sorted halves back into original
}
```

LINKED LIST IMPLEMENTATIONS OF MERGE-SORT 2

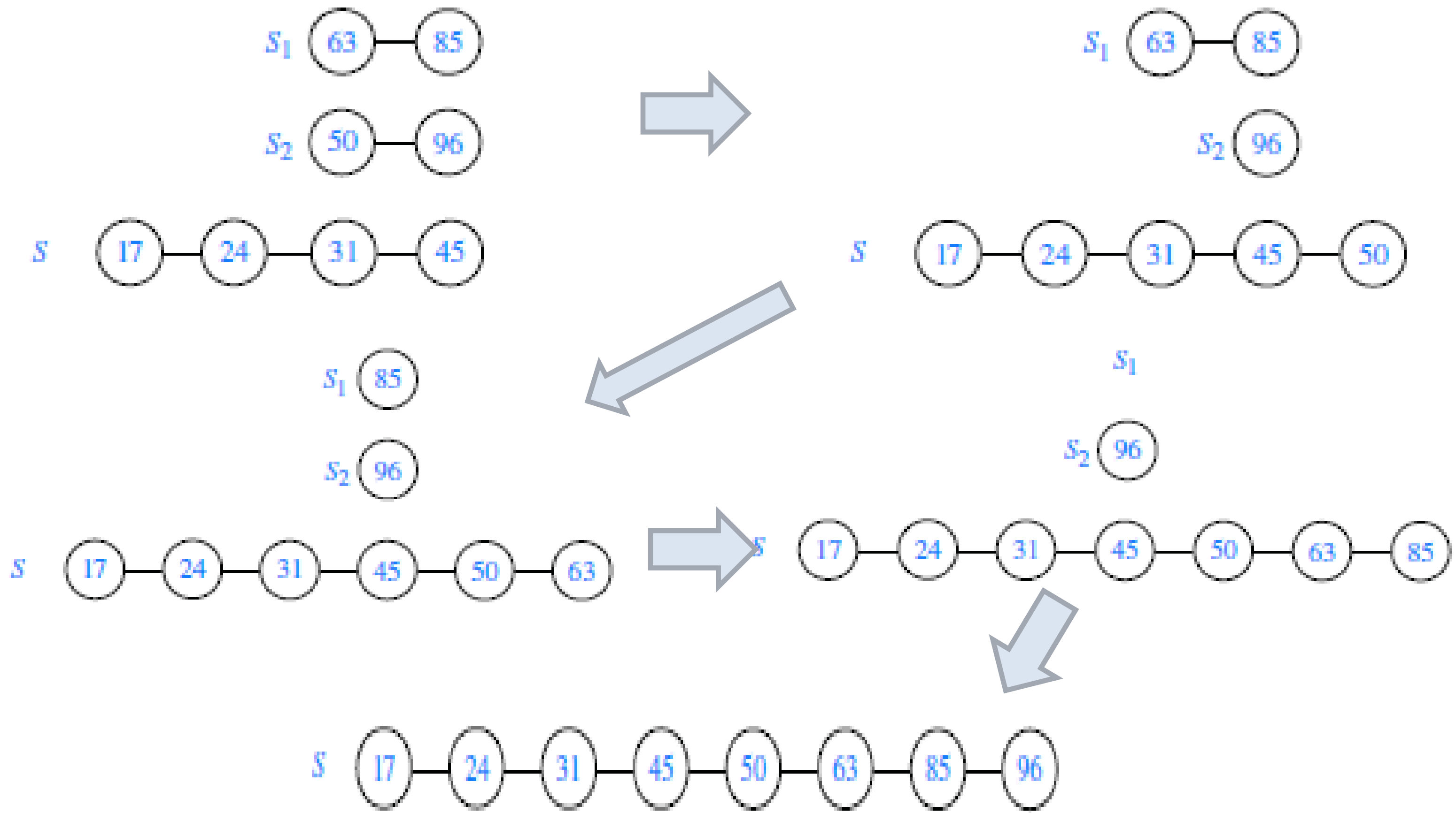
- Using basic queue as its container type

```
/** Merge contents of sorted queues S1 and S2 into empty queue S. */
public static <K> void merge(Queue<K> S1, Queue<K> S2, Queue<K> S,
                                                   Comparator<K> comp) {
  while (!S1.isEmpty() && !S2.isEmpty()) {
    if (comp.compare(S1.first(), S2.first()) < 0)
      S.enqueue(S1.dequeue());              // take next element from S1
    else
      S.enqueue(S2.dequeue());              // take next element from S2
  }
  while (!S1.isEmpty())
    S.enqueue(S1.dequeue());                // move any elements that remain in S1
  while (!S2.isEmpty())
    S.enqueue(S2.dequeue());                // move any elements that remain in S2
}
```

EXAMPLE MERGE IN LINKED-LIST IMPLEMENTATION



EXAMPLE MERGE IN LINKED-LIST IMPLEMENTATION 2



EXTRA2. A BOTTOM-UP (NONRECURSIVE) MERGE-SORT

- nonrecursive version of array-based merge-sort, which runs in $O(n\log n)$
- The main idea is to perform merge-sort bottom-up, performing the merges level by level going up the merge-sort tree.

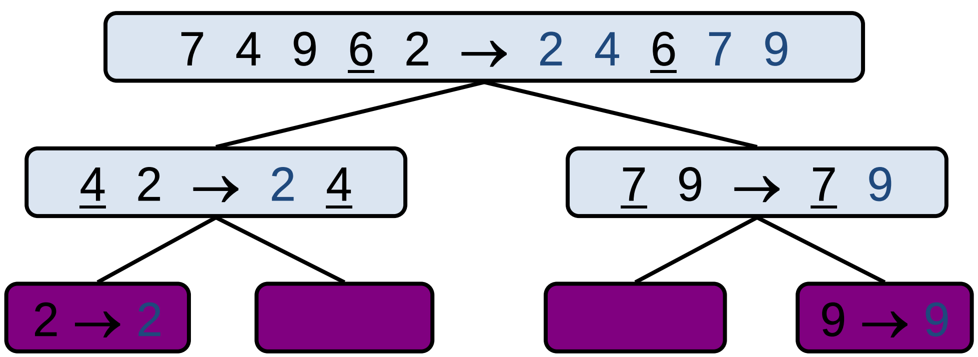
A BOTTOM-UP (NONRECURSIVE) MERGE-SORT

```
/** Merge-sort contents of data array. */
public static <K> void mergeSortBottomUp(K[ ] orig, Comparator<K> comp) {
  int n = orig.length;
  K[ ] src = orig;                        // alias for the original
  K[ ] dest = (K[ ]) new Object[n];       // make a new temporary array
  K[ ] temp;                              // reference used only for swapping
  for (int i=1; i < n; i *= 2) {          // each iteration sorts all runs of length i
    for (int j=0; j < n; j += 2*i)        // each pass merges two runs of length i
      merge(src, dest, comp, j, i);
    temp = src; src = dest; dest = temp;        // reverse roles of the arrays
  }
  if (orig != src)
    System.arraycopy(src, 0, orig, 0, n); // additional copy to get result to original
}
```

A BOTTOM-UP (NONRECURSIVE) MERGE-SORT

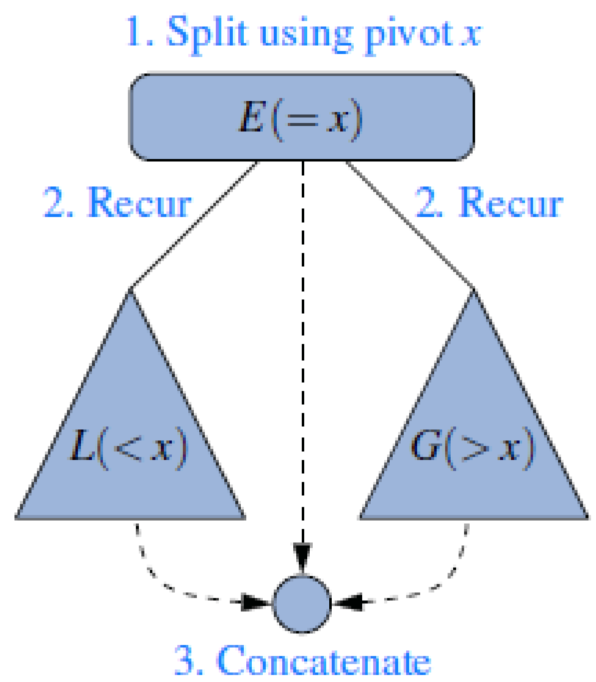
```
/** Merges in[start..start+inc-1] and in[start+inc..start+2*inc-1] into out. */
public static <K> void merge(K[ ] in, K[ ] out, Comparator<K> comp,
                                                    int start, int inc) {
  int end1 = Math.min(start + inc, in.length);          // boundary for run 1
  int end2 = Math.min(start + 2 * inc, in.length);      // boundary for run 2
  int x=start;                                          // index into run 1
  int y=start+inc;                                      // index into run 2
  int z=start;                                          // index into output
  while (x < end1 && y < end2)
    if (comp.compare(in[x], in[y]) < 0)
      out[z++] = in[x++];                               // take next from run 1
    else
      out[z++] = in[y++];                               // take next from run 2
  if (x < end1) System.arraycopy(in, x, out, z, end1 - x);      // copy rest of run 1
  else if (y < end2) System.arraycopy(in, y, out, z, end2 - y); // copy rest of run 2
}
```

QUICK-SORT



QUICK-SORT

- Quick-sort is a randomized sorting algorithm based on the divide-and-conquer paradigm:
  - *Divide*: pick a random element ***x*** (called pivot) and partition ***S*** into
    - ***L*** elements less than ***x***
    - ***E*** elements equal ***x***
    - ***G*** elements greater than ***x***
  - *Conquer:* Recursively sort ***L*** and ***G***
  - *Combine:* join ***L***, ***E*** and ***G***



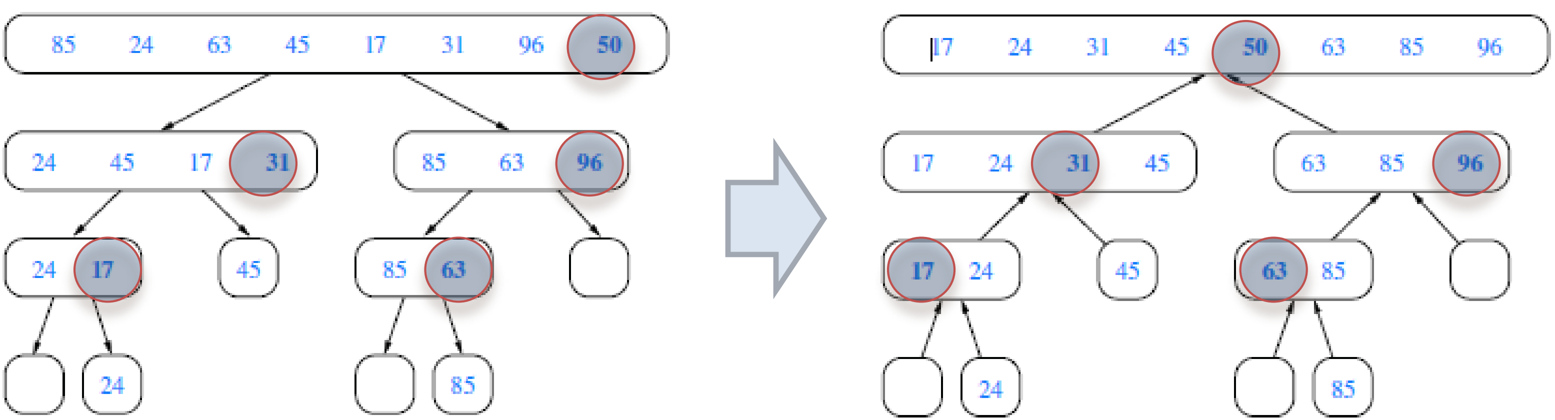
PARTITION

- We partition an input sequence as follows:
  - We remove, in turn, each element y from S and
  - We insert y into L, E or G, depending on the result of the comparison with the pivot x
- Each insertion and removal is at the beginning or at the end of a sequence, and hence takes $O(1)$ time
- Thus, the partition step of quick-sort takes $O(n)$ time

```
Algorithm partition(S, p)
    Input sequence S, position p of pivot
    Output subsequences L, E, G of the
        elements of S less than, equal to,
        or greater than the pivot, resp.
    L, E, G ← empty sequences
    x ← S.remove(p)
    while ¬S.isEmpty()
        y ← S.remove(S.first())
        if y < x
            L.addLast(y)
        else if y = x
            E.addLast(y)
        else { y > x }
            G.addLast(y)
    return L, E, G
```

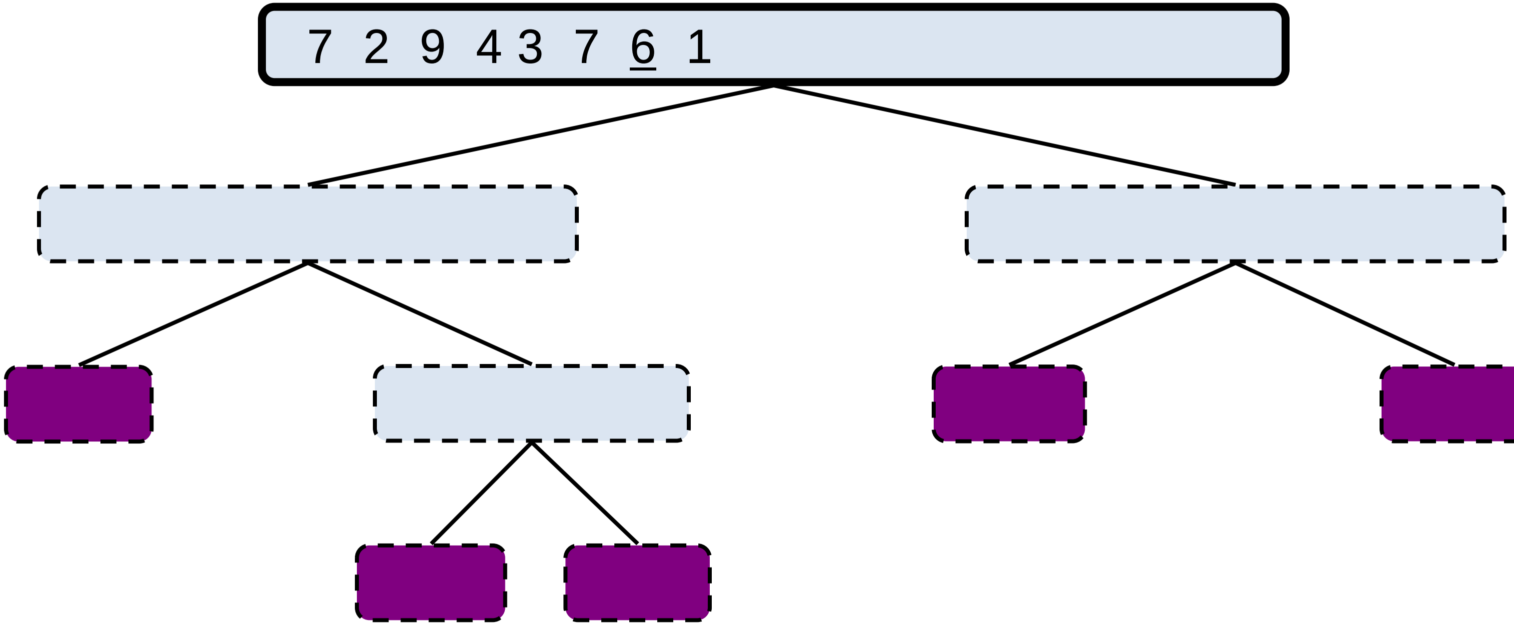
QUICK-SORT TREE

- An execution of quick-sort is depicted by a binary tree called ***quick-sort tree***.
  - Each **node** represents a recursive call of quick-sort and stores
    - Unsorted sequence before the execution and its pivot
    - Sorted sequence at the end of the execution
  - The **root** is the initial call
  - The **leaves** are calls on subsequences of size 0 or 1



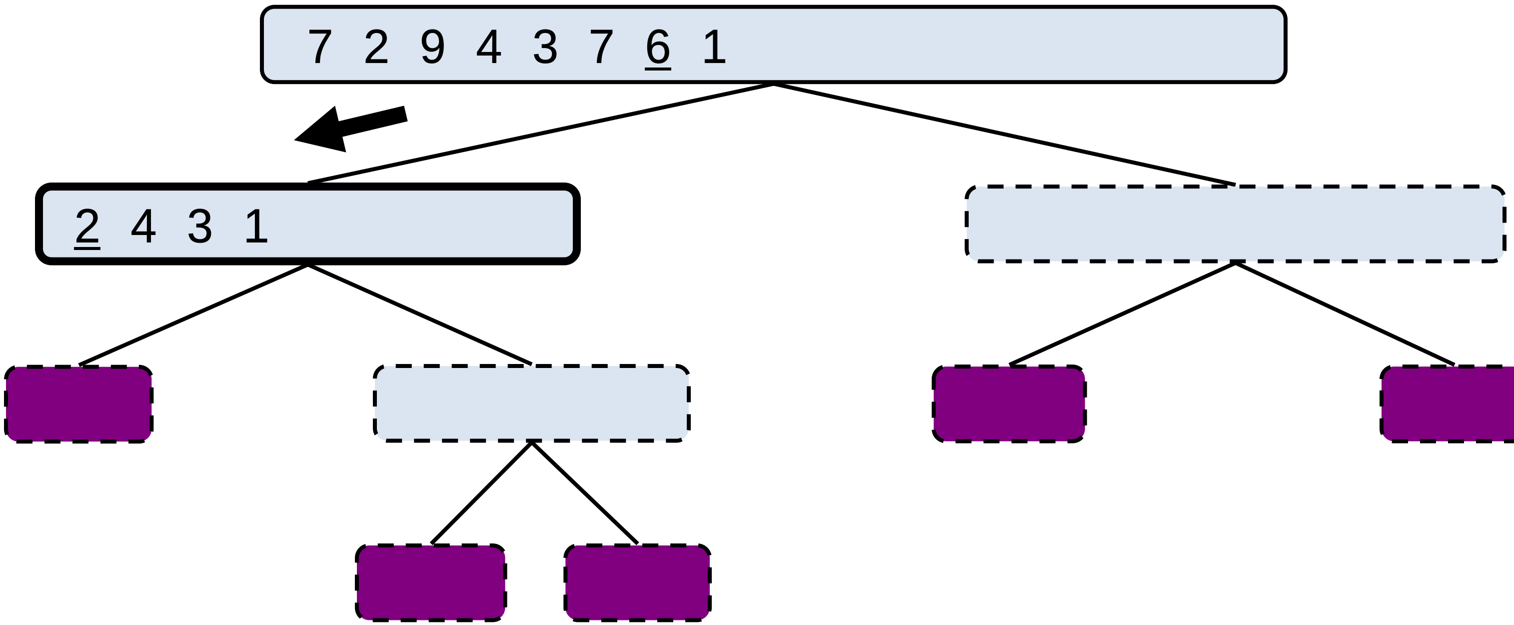
EXECUTION EXAMPLE

- Pivot selection



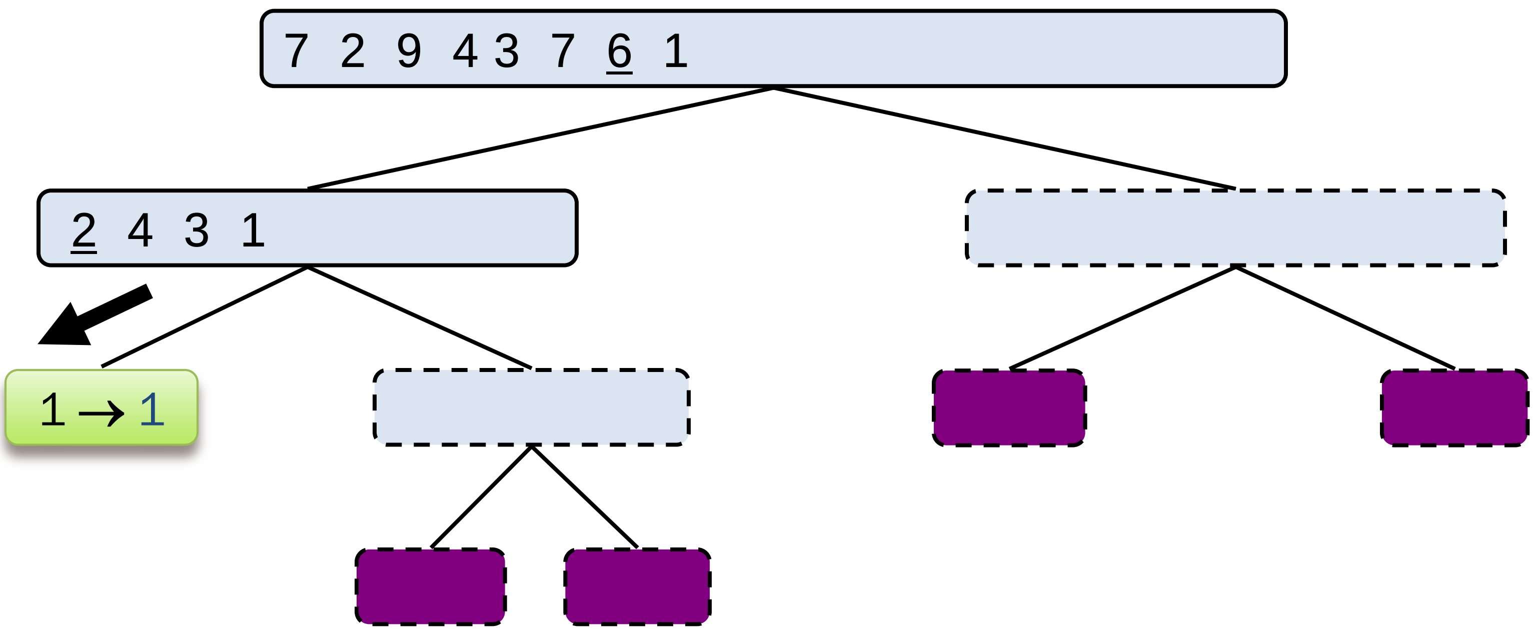
EXECUTION EXAMPLE (CONT.)

- Partition, recursive call, pivot selection



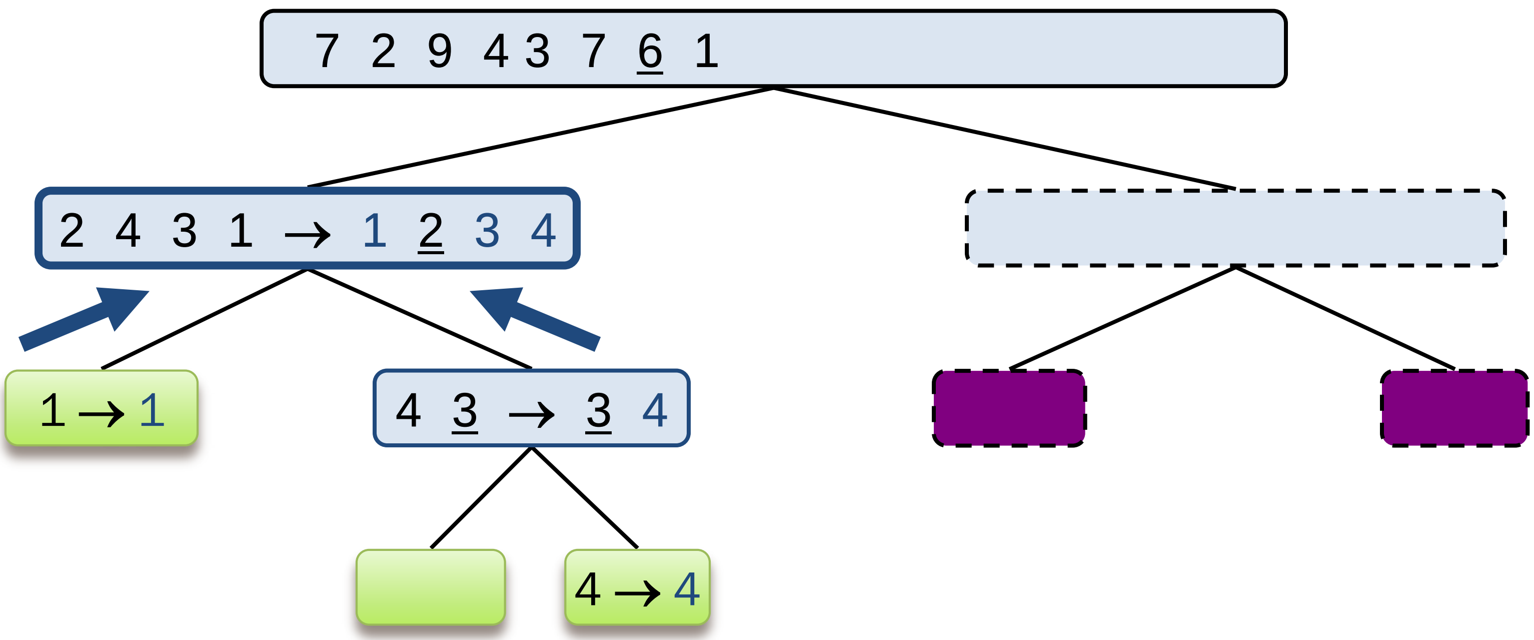
EXECUTION EXAMPLE (CONT.)

- Partition, recursive call, base case



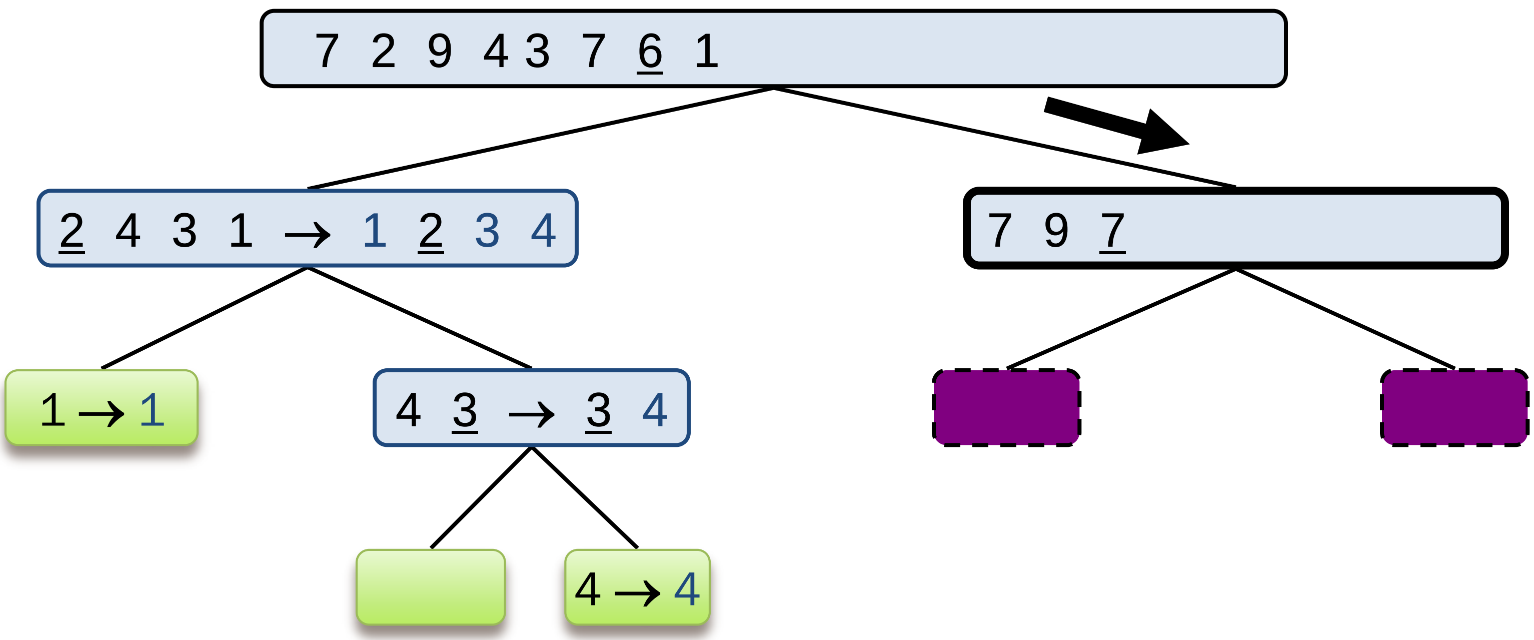
EXECUTION EXAMPLE (CONT.)

- Recursive call, ..., base case, join



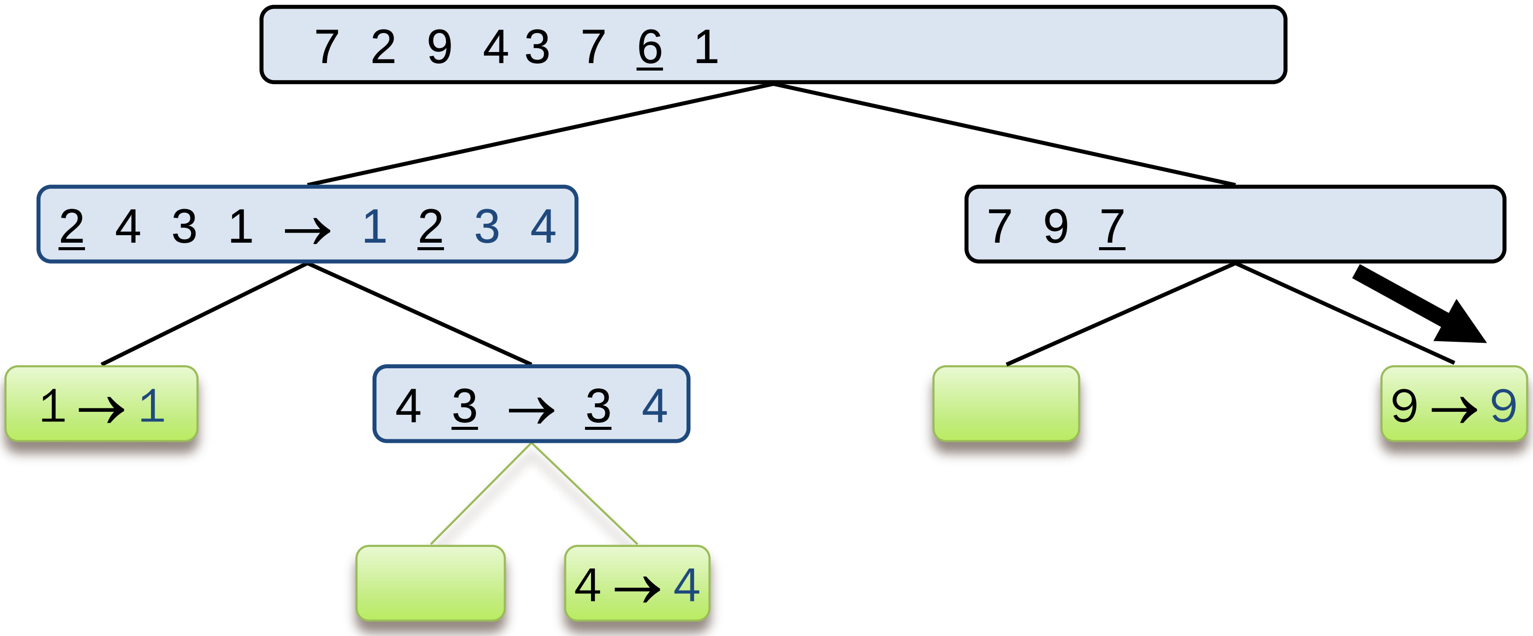
EXECUTION EXAMPLE (CONT.)

- Recursive call, pivot selection



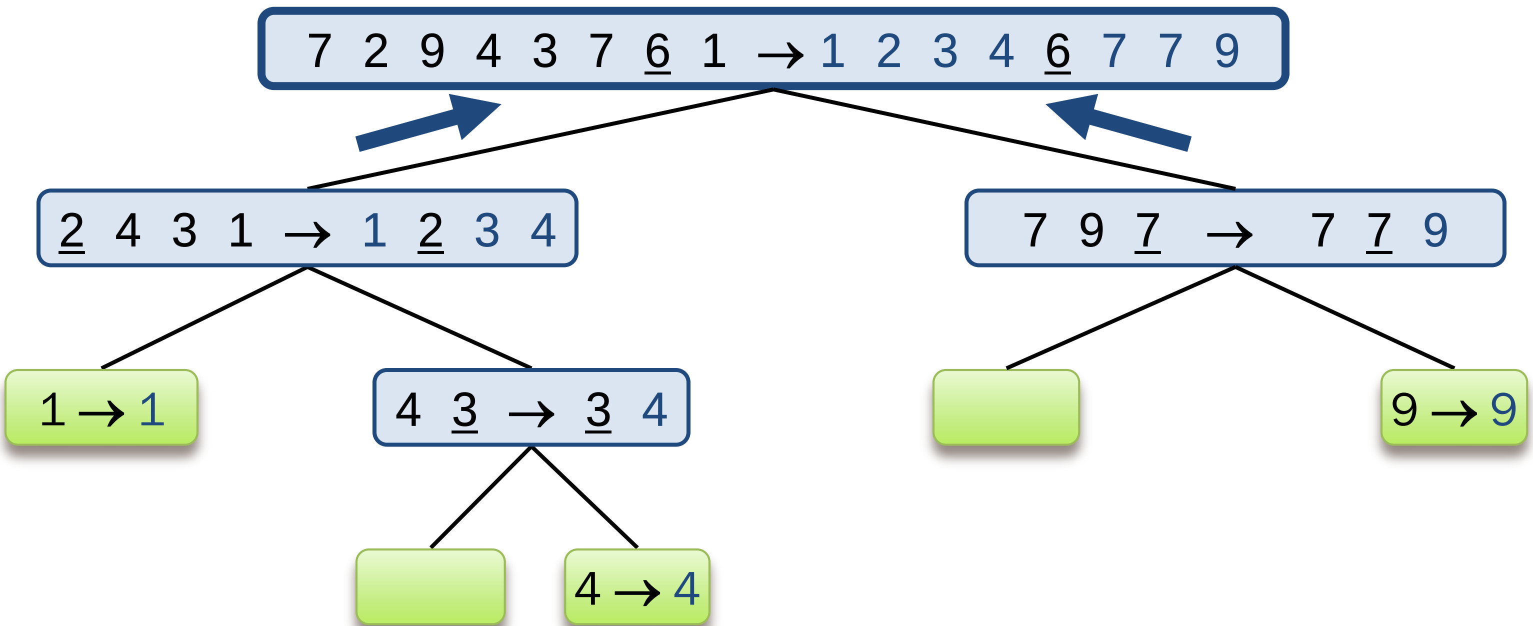
EXECUTION EXAMPLE (CONT.)

- Partition, ..., recursive call, base case



EXECUTION EXAMPLE (CONT.)

- Join, join



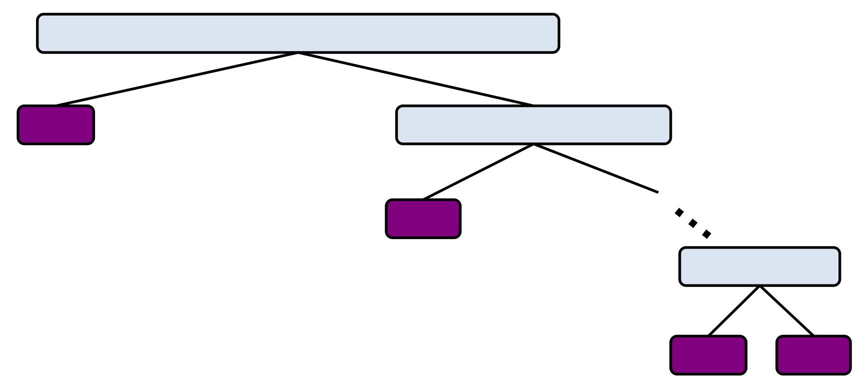
WORST-CASE RUNNING TIME

- The worst case for quick-sort occurs when the pivot is the unique minimum or maximum element
- One of L and G has size $n-1$ and the other has size 0
- The running time is proportional to the sum

$$n + (n-1) + \ldots + 2 + 1$$

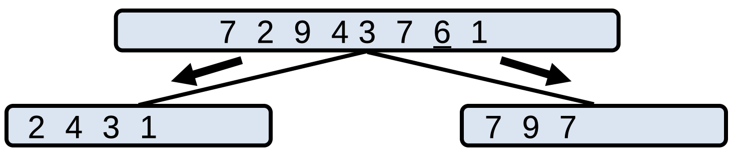
- Thus, the worst-case running time of quick-sort is $O(n^2)$

depth	time
0	n
1	$n-1$
…	…
$n-1$	1

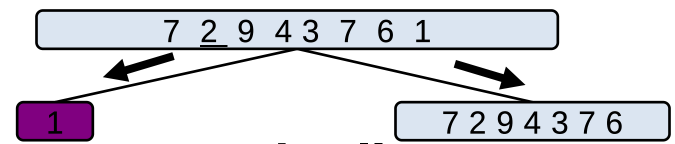


EXPECTED RUNNING TIME

- Consider a recursive call of quick-sort on a sequence of size s
  - **Good call:** the sizes of L and G are each less than $3s/4$
  - **Bad call:** one of L and G has size greater than $3s/4$

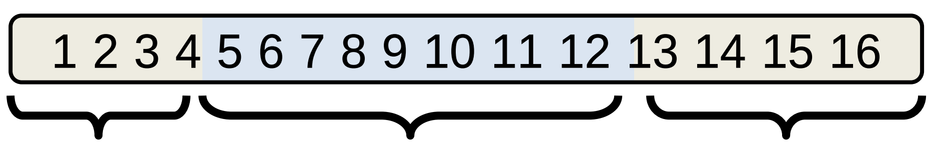


Good call



Bad call

- A call is good with probability $1/2$
  - 1/2 of the possible pivots cause good calls:



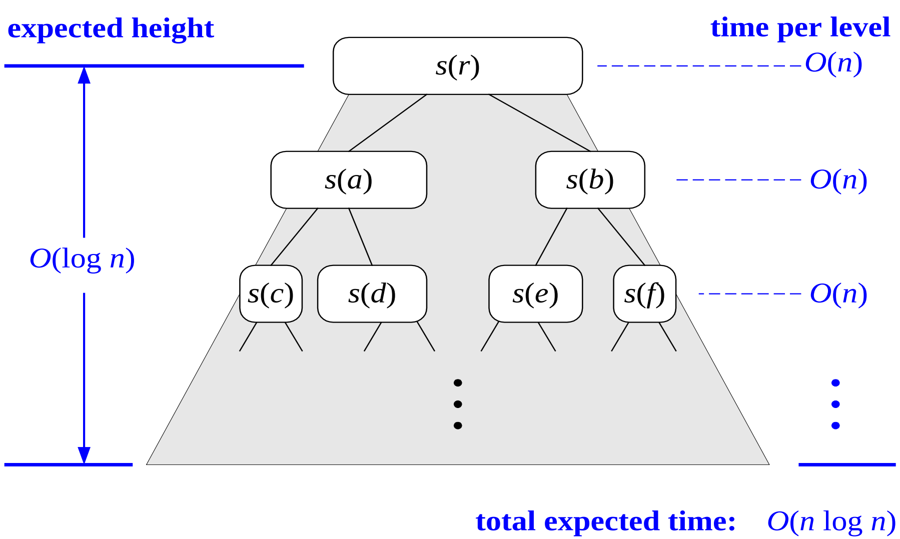
Bad pivots **Good pivots** **Bad pivots**

LINKED QUEUE BASED IMPLEMENTATION

```
/** Quick-sort contents of a queue. */
public static <K> void quickSort(Queue<K> S, Comparator<K> comp) {
  int n = S.size();
  if (n < 2) return;                              // queue is trivially sorted
  // divide
  K pivot = S.first();                            // using first as arbitrary pivot
  Queue<K> L = new LinkedQueue<>();
  Queue<K> E = new LinkedQueue<>();
  Queue<K> G = new LinkedQueue<>();
  while (!S.isEmpty()) {                          // divide original into L, E, and G
    K element = S.dequeue();
    int c = comp.compare(element, pivot);
    if (c < 0)                                    // element is less than pivot
      L.enqueue(element);
    else if (c == 0)                              // element is equal to pivot
      E.enqueue(element);
    else                                          // element is greater than pivot
      G.enqueue(element);
  }
  // conquer
  quickSort(L, comp);                             // sort elements less than pivot
  quickSort(G, comp);                             // sort elements greater than pivot
  // concatenate results
  while (!L.isEmpty())
    S.enqueue(L.dequeue());
  while (!E.isEmpty())
    S.enqueue(E.dequeue());
  while (!G.isEmpty())
    S.enqueue(G.dequeue());
}
```

EXPECTED RUNNING TIME, PART 2

- Probabilistic Fact: The expected number of coin tosses required in order to get k heads is $2k$
- For a node of depth i, we expect
  - $i/2$ ancestors are good calls
  - The size of the input sequence for the current call is at most $(3/4)^{i/2}n$
- Therefore, we have
  - For a node of depth 2log4/3n, the expected input size is one
  - The expected height of the quick-sort tree is O(log n)
- The amount or work done at the nodes of the same depth is O(n)
- Thus, the expected running time of quick-sort is O(n log n)



IN-PLACE QUICK-SORT

- Quick-sort can be implemented to run in-place
- In the partition step, we use replace operations to rearrange the elements of the input sequence such that
  - the elements less than the pivot have rank less than h
  - the elements equal to the pivot have rank between h and k
  - the elements greater than the pivot have rank greater than k
- The recursive calls consider
  - elements with rank less than h
  - elements with rank greater than k

Algorithm *inPlaceQuickSort*(S, l, r)
  Input sequence S, ranks l and r
  Output sequence S with the elements of rank between l and r rearranged in increasing order
  if $l \geq r$
    return
  $i \leftarrow$ a random integer between l and r
  $x \leftarrow S.elemAtRank(i)$
  $(h, k) \leftarrow inPlacePartition(x)$
  $inPlaceQuickSort(S, l, h - 1)$
  $inPlaceQuickSort(S, k + 1, r)$

IN-PLACE PARTITIONING

- Perform the partition using two indices to split S into L and E U G (a similar method can split E U G into E and G).

j k

3 2 5 1 0 7 3 5 9 2 7 9 8 9 7 **6** 9 (pivot = 6)

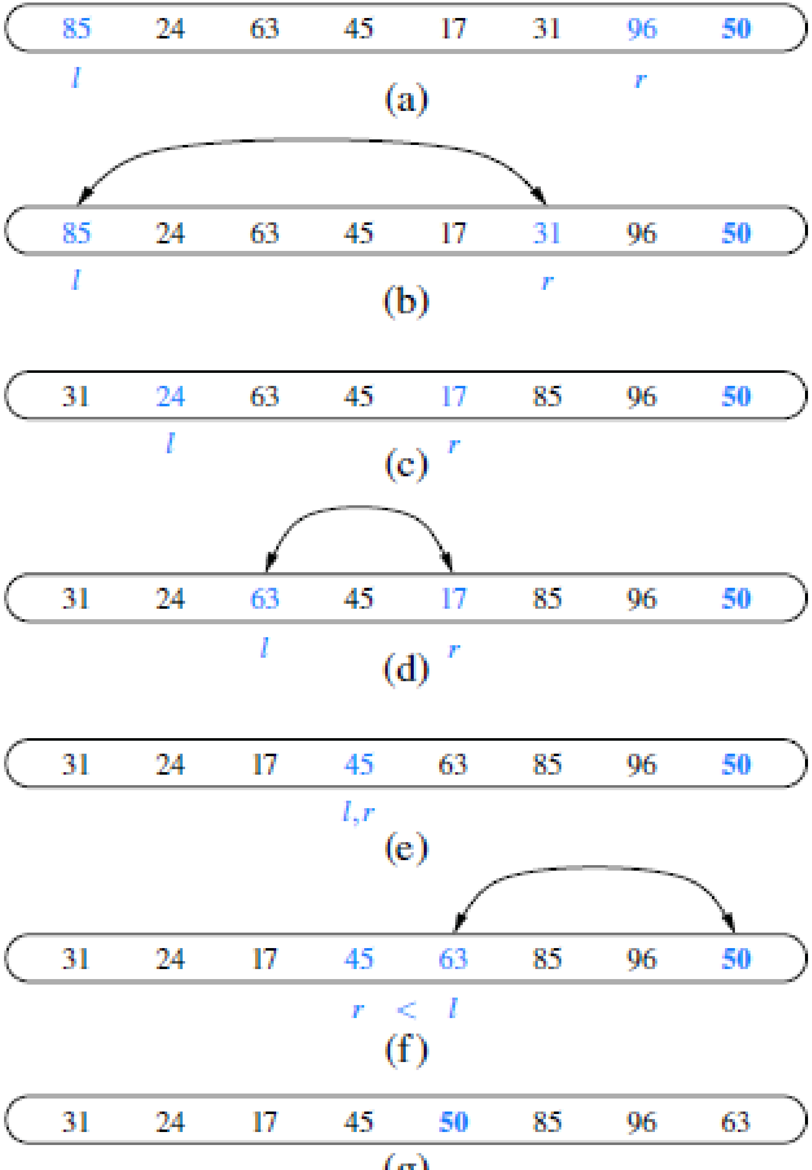
- Repeat until j and k cross:
  - Scan j to the right until finding an element ≥ x.
  - Scan k to the left until finding an element < x.
  - Swap elements at indices j and k

j k

3 2 5 1 0 $\boxed{7}$ 3 5 9 $\boxed{2}$ 7 9 8 9 7 **6** 9

JAVA IMPLEMENTATION

```
/** Sort the subarray S[a..b] inclusive. */
private static <K> void quickSortInPlace(K[ ] S, Comparator<K> comp,
                                                                  int a, int b) {
  if (a >= b) return;          // subarray is trivially sorted
  int left = a;
  int right = b−1;
  K pivot = S[b];
  K temp;                      // temp object used for swapping
  while (left <= right) {
    // scan until reaching value equal or larger than pivot (or right marker)
    while (left <= right && comp.compare(S[left], pivot) < 0) left++;
    // scan until reaching value equal or smaller than pivot (or left marker)
    while (left <= right && comp.compare(S[right], pivot) > 0) right−−;
    if (left <= right) {       // indices did not strictly cross
      // so swap values and shrink range
      temp = S[left]; S[left] = S[right]; S[right] = temp;
      left++; right−−;
    }
  }
  // put pivot into its final place (currently marked by left index)
  temp = S[left]; S[left] = S[b]; S[b] = temp;
  // make recursive calls
  quickSortInPlace(S, comp, a, left − 1);
  quickSortInPlace(S, comp, left + 1, b);
}
```



Divide step of in-place quick-sort, using index *l* as shorthand for identifier left, and index *r* as shorthand for identifier right.

- Index *l* scans the sequence from left to right, and
- index *r* scans the sequence from right to left.
- A swap is performed when *l* is at an element as large as the pivot and *r* is at an element as small as the pivot.
- A final swap with the pivot, in part (f), completes the divide step.

SUMMARY OF SORTING ALGORITHMS

Algorithm	Time	Notes
selection-sort	$O(n^2)$	▪ in-place ▪ slow (good for small inputs)
insertion-sort	$O(n^2)$	▪ in-place ▪ slow (good for small inputs)
quick-sort	$O(n \log n)$ expected	▪ in-place, randomized ▪ fastest (good for large inputs)
heap-sort	$O(n \log n)$	▪ in-place ▪ fast (good for large inputs)
merge-sort	$O(n \log n)$	▪ sequential data access ▪ fast (good for huge inputs)