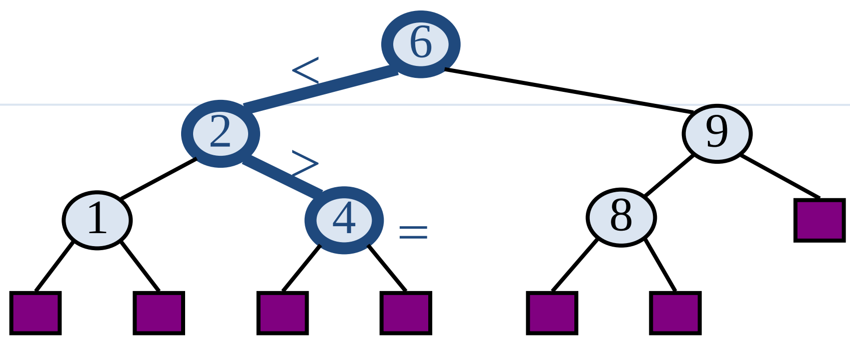
SEARCH TREES



Presentation for use with the textbook Data Structures and Algorithms in Java, 6th edition, by M. T. Goodrich, R. Tamassia, and M. H. Goldwasser, Wiley, 2014

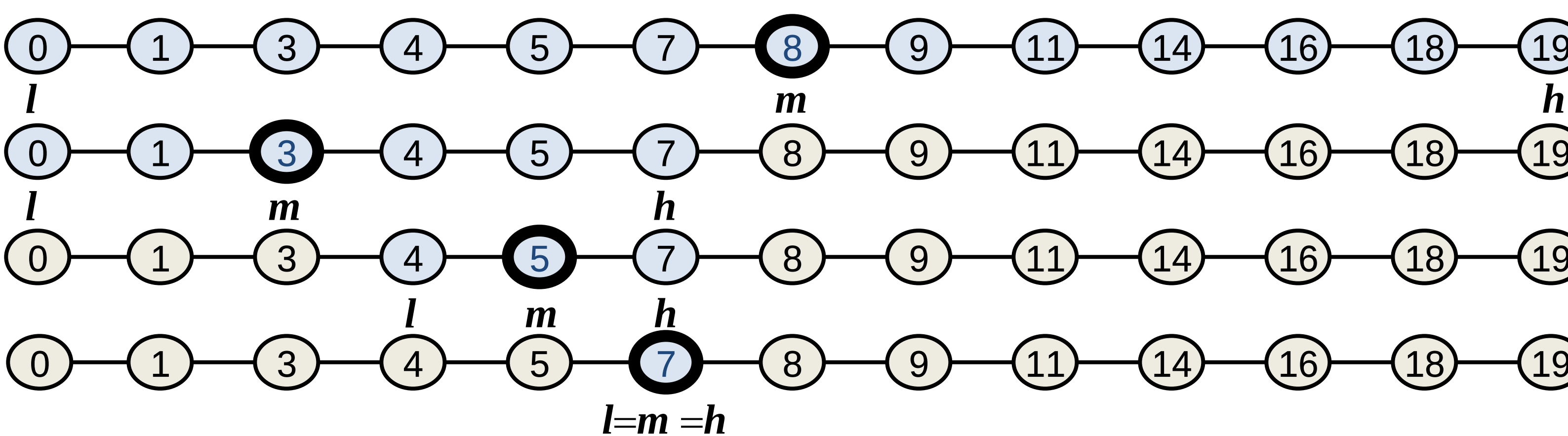
BINARY SEARCH TREES

ORDERED MAPS

- Keys are assumed to come from a total order.
- Items are stored in order by their keys
- This allows us to support nearest neighbor queries :
  - Item with largest key less than or equal to k
  - Item with smallest key greater than or equal to k

BINARY SEARCH

- Binary search can perform nearest neighbor queries on an ordered map that is implemented with an array, sorted by key
  - similar to the high-low children's game
  - at each step, the number of candidate items is halved
  - terminates after O(log n) steps
- Example: find(7)



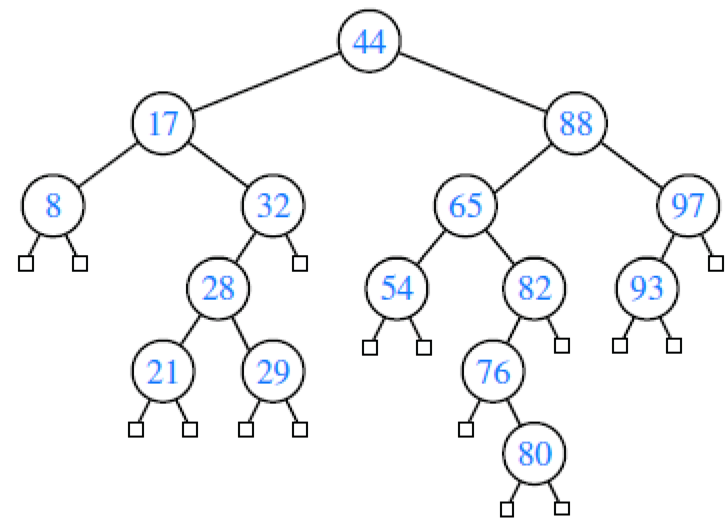
SEARCH TABLES

- A search table is an ordered map implemented by means of a sorted sequence
  - We store the items in an array-based sequence, sorted by key
  - We use an external comparator for the keys
- Performance:
  - Searches take $O(\log n)$ time, using binary search
  - Inserting a new item takes $O(n)$ time, since in the worst case we have to shift $n/2$ items to make room for the new item
  - Removing an item takes $O(n)$ time, since in the worst case we have to shift $n/2$ items to compact the items after the removal
- The lookup table is effective only for ordered maps of small size or for maps on which searches are the most common operations, while insertions and removals are rarely performed (e.g., credit card authorizations)

BINARY SEARCH TREES

- We define binary search tree as a proper binary tree storing keys (or key-value entries) at its internal nodes and satisfying the following property:
  - Let u, v, and w be three nodes such that u is in the left subtree of v and w is in the right subtree of v. We have $key(u) \le key(v) \le key(w)$
- External nodes do not store items
  - We use the leaves as "placeholders" (sentinels)
  - Represented as **null** references in practice,

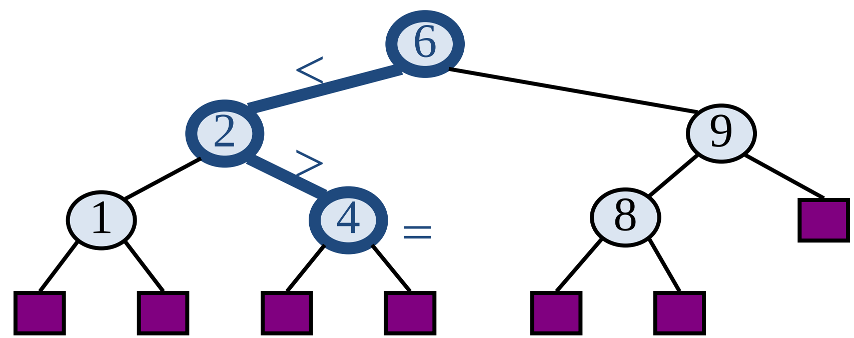
- An inorder traversal of a binary search trees visits the keys in increasing order



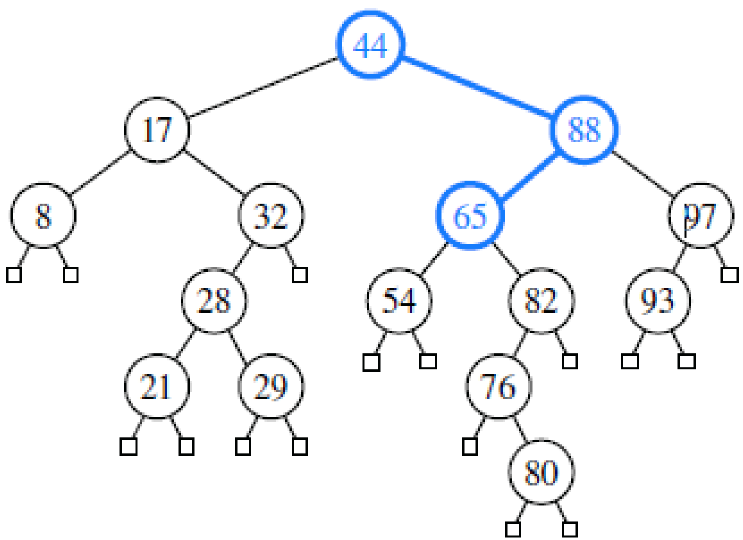
SEARCH

- To search for a key k, we trace a downward path starting at the root
- The next node visited depends on the comparison of k with the key of the current node
- If we reach a leaf, the key is not found
- Example: get(4):
  - Call TreeSearch(4,root)
- The algorithms for nearest neighbor queries are similar

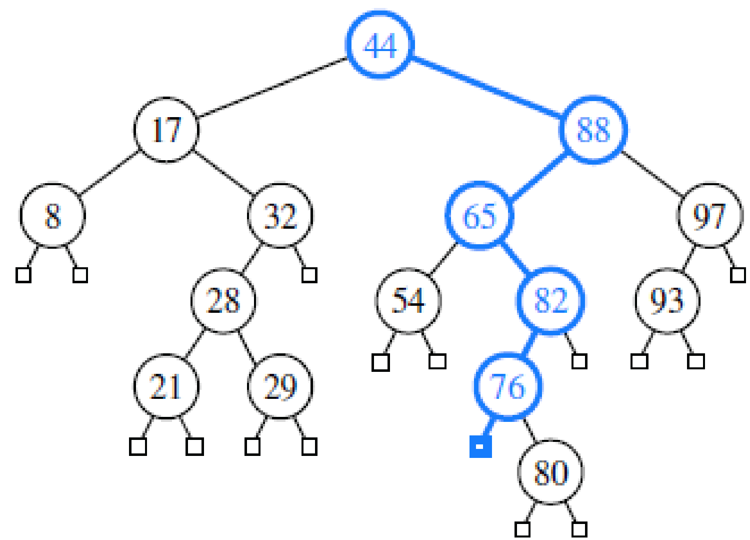
```
Algorithm TreeSearch(k, v)
    if T.isExternal (v)
        return v
    if k < key(v)
        return TreeSearch(k, left(v))
    else if k = key(v)
        return v
    else { k > key(v) }
        return TreeSearch(k, right(v))
```



ANOTHER EXAMPLE OF SEARCH



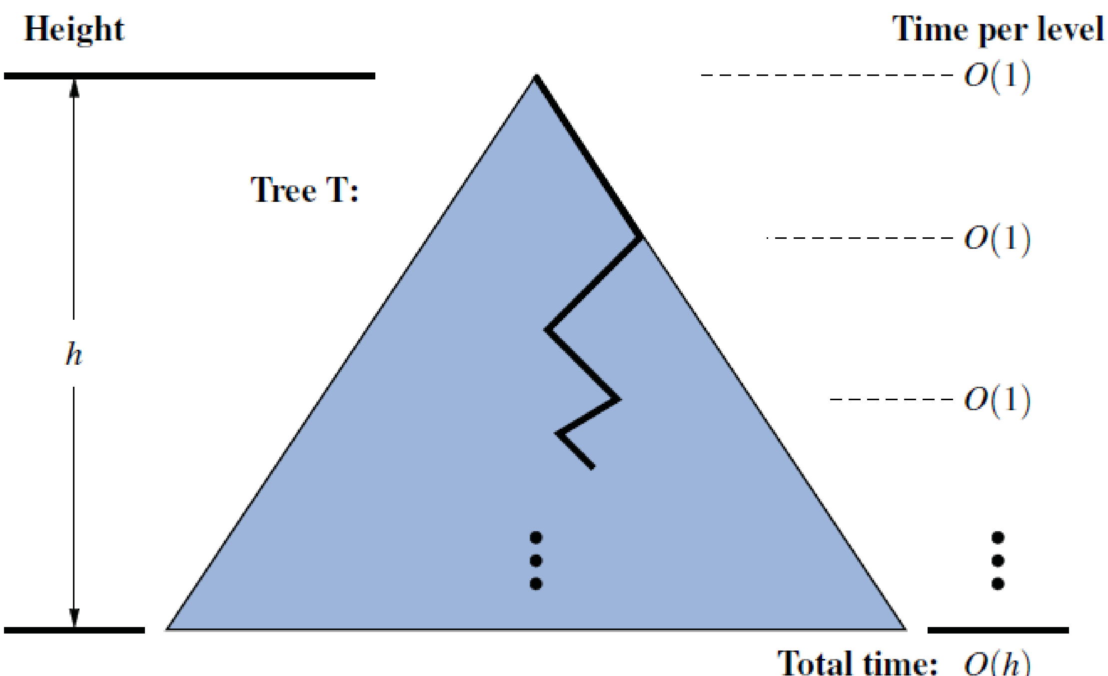
A successful search for key 65 in a binary search tree;



An unsuccessful search for key 68 that terminates at the leaf to the left of the key 76.

ANALYSIS OF BINARY TREE SEARCHING

- Algorithm TreeSearch is recursive and executes a constant number of primitive operations for each recursive call.



executes in time $O(h)$

We'll talk about various strategies to maintain an upper bound of $O(\log n)$ on the height soon

INSERTION

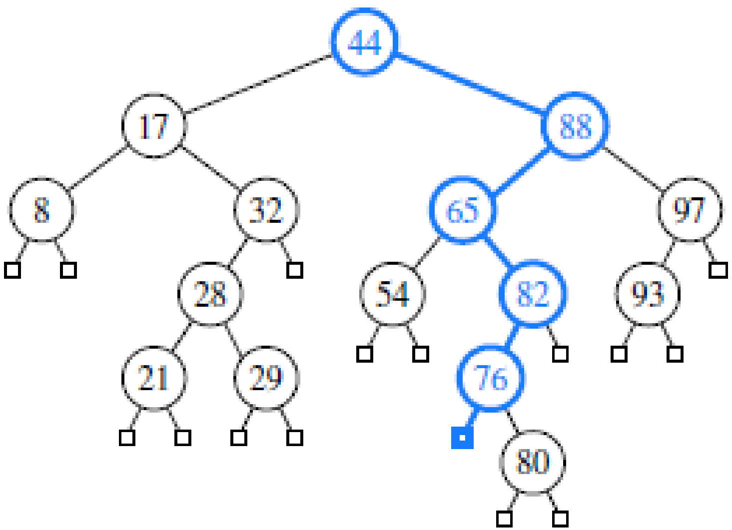
- To perform operation **put(k, o),** we search for key k (using TreeSearch)
- insertions, which always occur at a leaf).
- Assume a proper binary tree supports the following update operation
  - expandExternal(*p*, *e*): Stores entry *e* at the external position *p*, and expands *p* to be internal, having two new leaves as children.

Algorithm TreeInsert(k, v):
  Input: A search key k to be associated with value v
  p = TreeSearch(root(), k)
  if k == key(p) **then**
    Change p's value to (v)
  else
    expandExternal(p, (k, v))

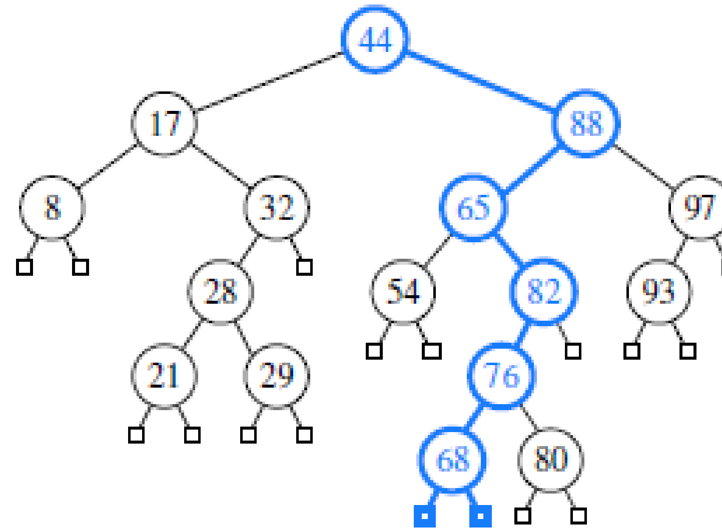
executes in time $O(h)$

EXAMPLE OF INSERT

Insertion of an entry with key 68 into the search tree



Finding the position to insert



the resulting tree

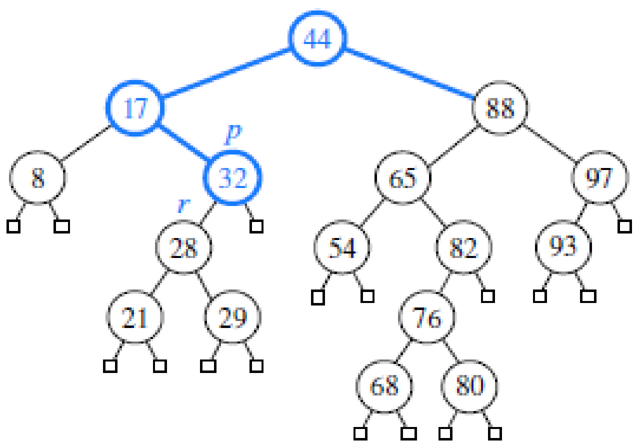
DELETION

- Deleting an entry from a binary search tree might happen anywhere in the tree
- To perform operation remove(k), we search for key k by calling TreeSearch(root( ), *k*) to find the position *p* storing an entry with key equal to *k* (if any).
  - If search returns an external node, then there is no entry to remove.
  - Otherwise,
    - at most one of the children of position *p* is internal,
    - Or position *p* has two internal children

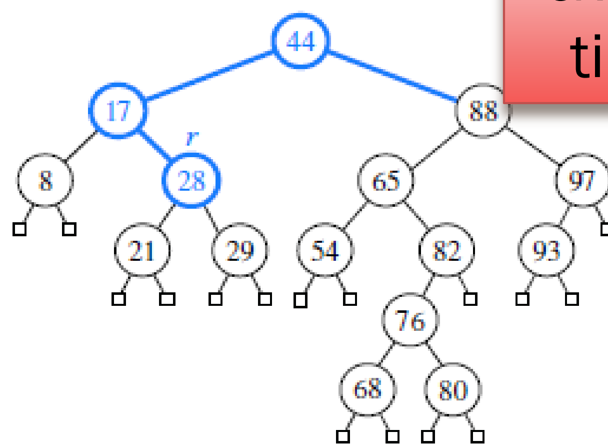
DELETION CONT.

- Deletion when at most one of the children of position *p* is internal.
  - Let position *r* be a child of *p* that is internal (or an arbitrary child, if both are leaves).
  - Remove *p* and the leaf that is *r*'s sibling, while promoting *r* upward to take the place of *p*.

executes in time *O*(*h*)



before the deletion of 32

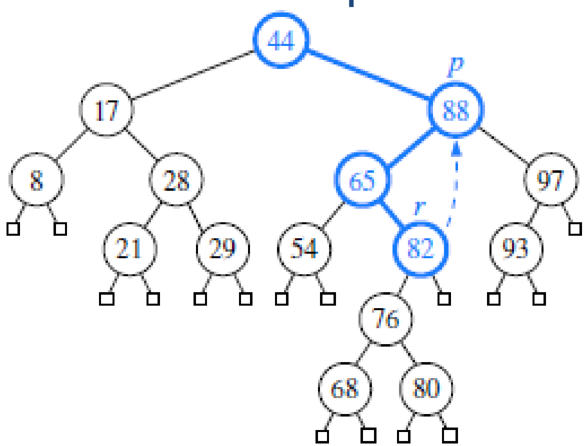


after the deletion of 32

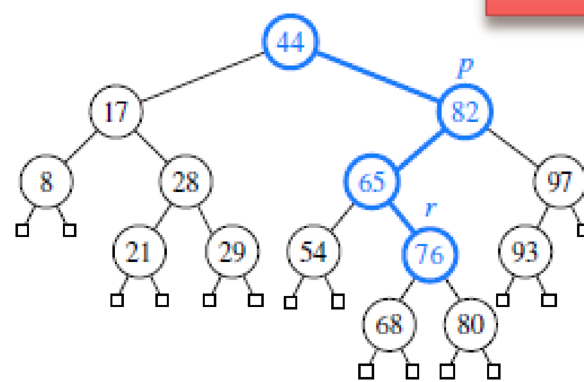
DELETION CONT.

- Deletion position p has two internal children
  - Locate position *r* containing the entry having the greatest key that is strictly less than that of position *p* (the rightmost internal position of the left subtree of position *p*)
  - Use *r*'s entry as a replacement for the one being deleted at position *p*.
  - Delete the node at position *r* from the tree.

executes in time $O(h)$



Before deleting 88



After deleting 88

JAVA IMPLEMENTATION 1

```
/** An implementation of a sorted map using a binary search tree. */
public class TreeMap<K,V> extends AbstractSortedMap<K,V> {
  // To represent the underlying tree structure, we use a specialized subclass of the
  // LinkedBinaryTree class that we name BalanceableBinaryTree (see Section 11.2).
  protected BalanceableBinaryTree<K,V> tree = new BalanceableBinaryTree<>( );

  /** Constructs an empty map using the natural ordering of keys. */
  public TreeMap( ) {
    super( );                                  // the AbstractSortedMap constructor
    tree.addRoot(null);                        // create a sentinel leaf as root
  }
  /** Constructs an empty map using the given comparator to order keys. */
  public TreeMap(Comparator<K> comp) {
    super(comp);                               // the AbstractSortedMap constructor
    tree.addRoot(null);                        // create a sentinel leaf as root
  }
  /** Returns the number of entries in the map. */
  public int size( ) {
    return (tree.size( ) − 1) / 2;             // only internal nodes have entries
  }
  /** Utility used when inserting a new entry at a leaf of the tree */
  private void expandExternal(Position<Entry<K,V>> p, Entry<K,V> entry) {
    tree.set(p, entry);                        // store new entry at p
    tree.addLeft(p, null);                     // add new sentinel leaves as children
    tree.addRight(p, null);
  }
```

JAVA IMPLEMENTATION 2

```
// Omitted from this code fragment, but included in the online version of the code,
// are a series of protected methods that provide notational shorthands to wrap
// operations on the underlying linked binary tree. For example, we support the
// protected syntax root() as shorthand for tree.root() with the following utility:
protected Position<Entry<K,V>> root( ) { return tree.root( ); }

/** Returns the position in p's subtree having given key (or else the terminal leaf).*/
private Position<Entry<K,V>> treeSearch(Position<Entry<K,V>> p, K key) {
  if (isExternal(p))
    return p;                               // key not found; return the final leaf
  int comp = compare(key, p.getElement( ));
  if (comp == 0)
    return p;                               // key found; return its position
  else if (comp < 0)
    return treeSearch(left(p), key);        // search left subtree
  else
    return treeSearch(right(p), key);       // search right subtree
}
```

JAVA IMPLEMENTATION 3

```
/** Returns the value associated with the specified key (or else null). */
public V get(K key) throws IllegalArgumentException {
  checkKey(key);                                  // may throw IllegalArgumentException
  Position<Entry<K,V>> p = treeSearch(root(), key);
  rebalanceAccess(p);                             // hook for balanced tree subclasses
  if (isExternal(p)) return null;                 // unsuccessful search
  return p.getElement().getValue();               // match found
}
/** Associates the given value with the given key, returning any overridden value.*/
public V put(K key, V value) throws IllegalArgumentException {
  checkKey(key);                                  // may throw IllegalArgumentException
  Entry<K,V> newEntry = new MapEntry<>(key, value);
  Position<Entry<K,V>> p = treeSearch(root(), key);
  if (isExternal(p)) {                            // key is new
    expandExternal(p, newEntry);
    rebalanceInsert(p);                           // hook for balanced tree subclasses
    return null;
  } else {                                        // replacing existing key
    V old = p.getElement().getValue();
    set(p, newEntry);
    rebalanceAccess(p);                           // hook for balanced tree subclasses
    return old;
  }
}
```

JAVA IMPLEMENTATION 4

```
/** Removes the entry having key k (if any) and returns its associated value. */
public V remove(K key) throws IllegalArgumentException {
  checkKey(key);                                    // may throw IllegalArgumentException
  Position<Entry<K,V>> p = treeSearch(root(), key);
  if (isExternal(p)) {                              // key not found
    rebalanceAccess(p);                             // hook for balanced tree subclasses
    return null;
  } else {
    V old = p.getElement().getValue();
    if (isInternal(left(p)) && isInternal(right(p))) { // both children are internal
      Position<Entry<K,V>> replacement = treeMax(left(p));
      set(p, replacement.getElement());
      p = replacement;
    } // now p has at most one child that is an internal node
    Position<Entry<K,V>> leaf = (isExternal(left(p)) ? left(p) : right(p));
    Position<Entry<K,V>> sib = sibling(leaf);
    remove(leaf);
    remove(p);                                      // sib is promoted in p's place
    rebalanceDelete(sib);                           // hook for balanced tree subclasses
    return old;
  }
}
```

JAVA IMPLEMENTATION 5

```
/** Returns the position with the maximum key in subtree rooted at Position p. */
protected Position<Entry<K,V>> treeMax(Position<Entry<K,V>> p) {
  Position<Entry<K,V>> walk = p;
  while (isInternal(walk))
    walk = right(walk);
  return parent(walk);                        // we want the parent of the leaf
}
/** Returns the entry having the greatest key (or null if map is empty). */
public Entry<K,V> lastEntry() {
  if (isEmpty()) return null;
  return treeMax(root()).getElement();
}
/** Returns the entry with greatest key less than or equal to given key (if any). */
public Entry<K,V> floorEntry(K key) throws IllegalArgumentException {
  checkKey(key);                              // may throw IllegalArgumentException
  Position<Entry<K,V>> p = treeSearch(root(), key);
  if (isInternal(p)) return p.getElement();   // exact match
  while (!isRoot(p)) {
    if (p == right(parent(p)))
      return parent(p).getElement();          // parent has next lesser key
    else
      p = parent(p);
  }
  return null;                                // no such floor exists
}
```

JAVA IMPLEMENTATION 6

```
/** Returns the entry with greatest key strictly less than given key (if any). */
public Entry<K,V> lowerEntry(K key) throws IllegalArgumentException {
  checkKey(key);                                    // may throw IllegalArgumentException
  Position<Entry<K,V>> p = treeSearch(root( ), key);
  if (isInternal(p) && isInternal(left(p)))
    return treeMax(left(p)).getElement( );          // this is the predecessor to p
  // otherwise, we had failed search, or match with no left child
  while (!isRoot(p)) {
    if (p == right(parent(p)))
      return parent(p).getElement( );               // parent has next lesser key
    else
      p = parent(p);
  }
  return null;                                      // no such lesser key exists
}
```

JAVA IMPLEMENTATION 7

```
/** Returns an iterable collection of all key-value entries of the map. */
public Iterable<Entry<K,V>> entrySet() {
  ArrayList<Entry<K,V>> buffer = new ArrayList<>(size());
  for (Position<Entry<K,V>> p : tree.inorder())
    if (isInternal(p)) buffer.add(p.getElement());
  return buffer;
}
/** Returns an iterable of entries with keys in range [fromKey, toKey). */
public Iterable<Entry<K,V>> subMap(K fromKey, K toKey) {
  ArrayList<Entry<K,V>> buffer = new ArrayList<>(size());
  if (compare(fromKey, toKey) < 0)           // ensure that fromKey < toKey
    subMapRecurse(fromKey, toKey, root(), buffer);
  return buffer;
}
private void subMapRecurse(K fromKey, K toKey, Position<Entry<K,V>> p,
                           ArrayList<Entry<K,V>> buffer) {
  if (isInternal(p))
    if (compare(p.getElement(), fromKey) < 0)
      // p's key is less than fromKey, so any relevant entries are to the right
      subMapRecurse(fromKey, toKey, right(p), buffer);
    else {
      subMapRecurse(fromKey, toKey, left(p), buffer); // first consider left subtree
      if (compare(p.getElement(), toKey) < 0) {        // p is within range
        buffer.add(p.getElement());                    // so add it to buffer, and consider
        subMapRecurse(fromKey, toKey, right(p), buffer); // right subtree as well
      }
    }
}
```

PERFORMANCE OF A BINARY SEARCH TREE

Method	Running Time
size, isEmpty	$O(1)$
get, put, remove	$O(h)$
firstEntry, lastEntry	$O(h)$
ceilingEntry, floorEntry, lowerEntry, higherEntry	$O(h)$
subMap	$O(s+h)$
entrySet, keySet, values	$O(n)$

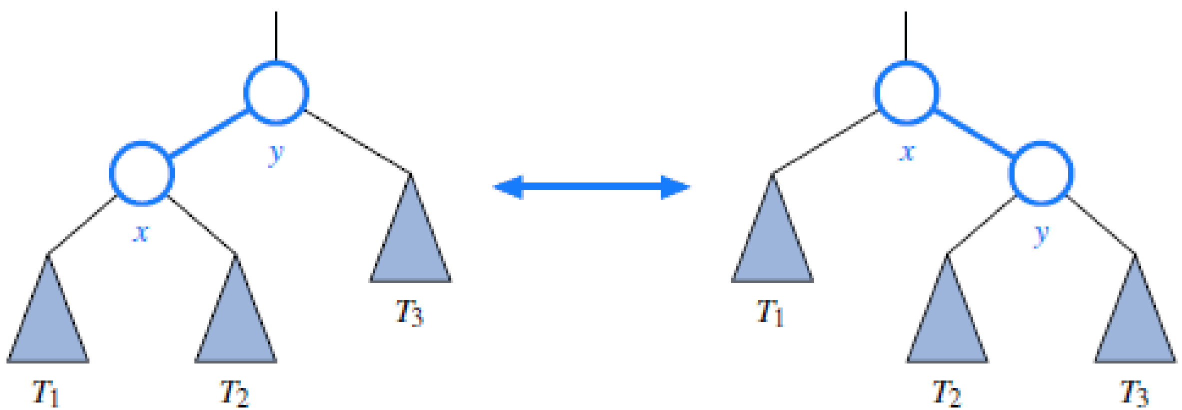
* subMap implementation can be shown to run in $O(s+h)$ worst-case bound for a call that reports s results

BALANCED SEARCH TREES

BALANCED SEARCH TREES

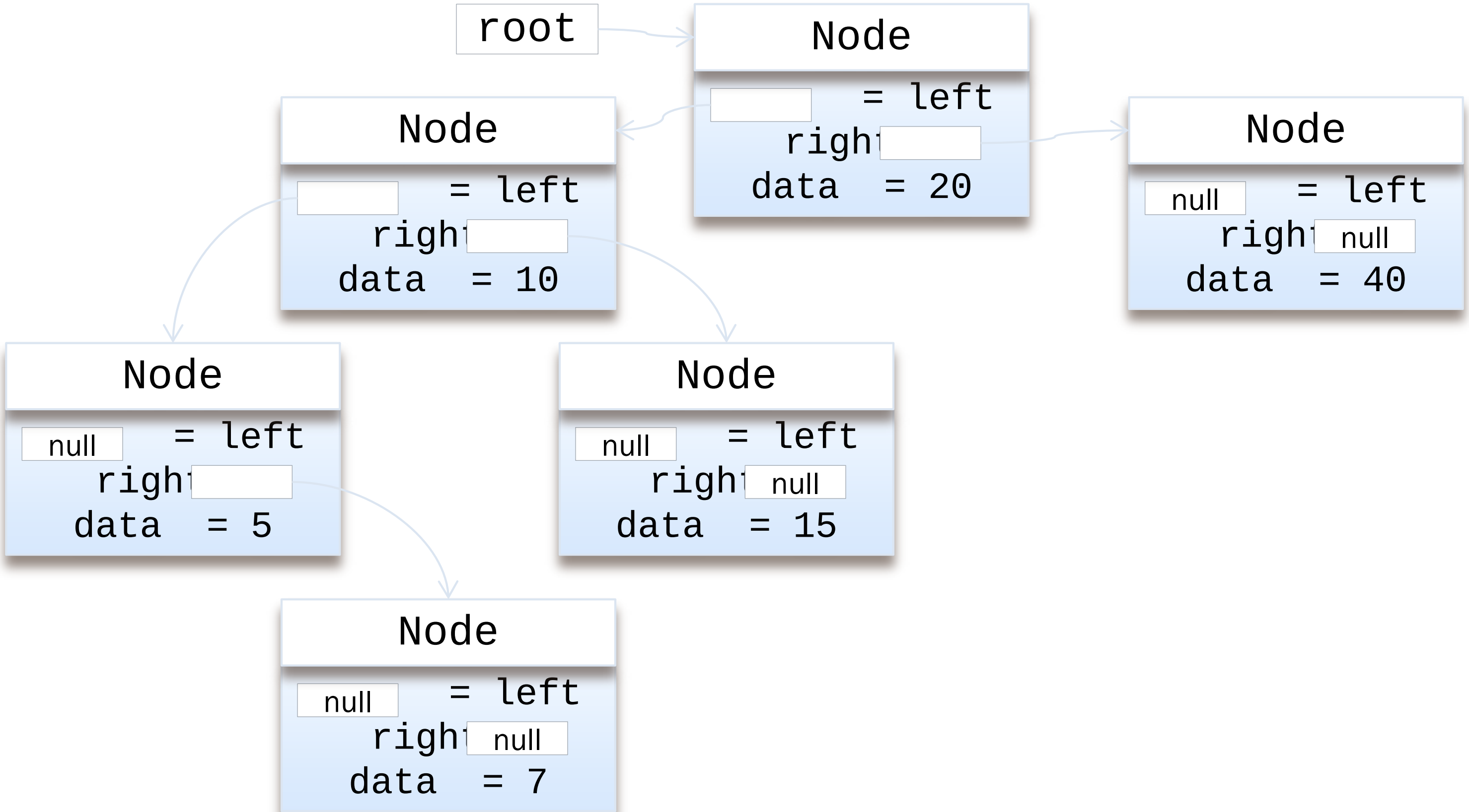
- Augmenting a standard binary search tree with occasional operations to reshape the tree and reduce its height
  - Examples> AVL trees, splay trees, and red-black trees
- The primary operation to rebalance a binary search tree is known as a *rotation*
  - allows the shape of a tree to be modified while maintaining the search-tree property.

“rotate” a child to be above its parent

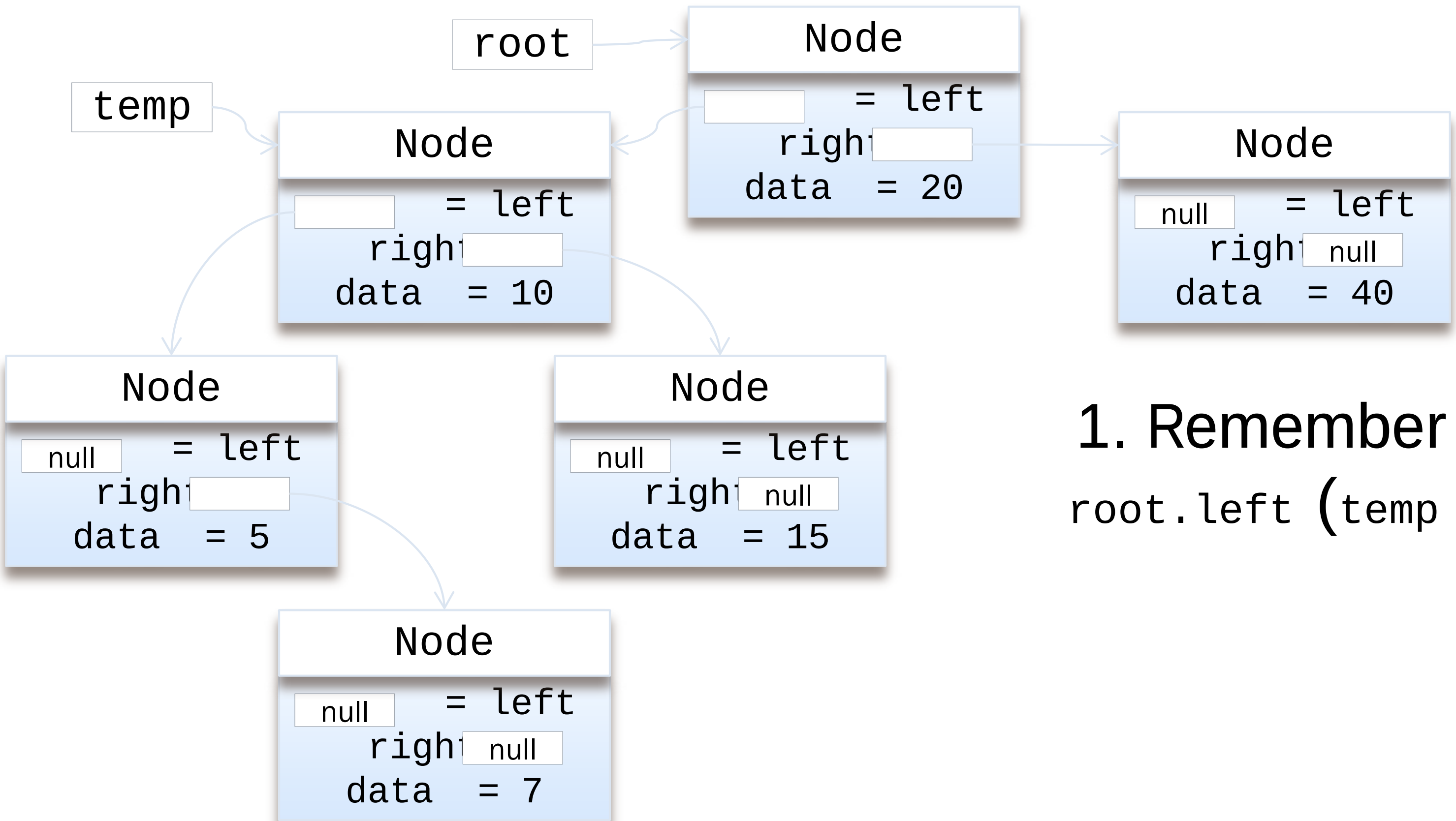


$O(1)$ time with a linked binary tree representation

ALGORITHM FOR ROTATION

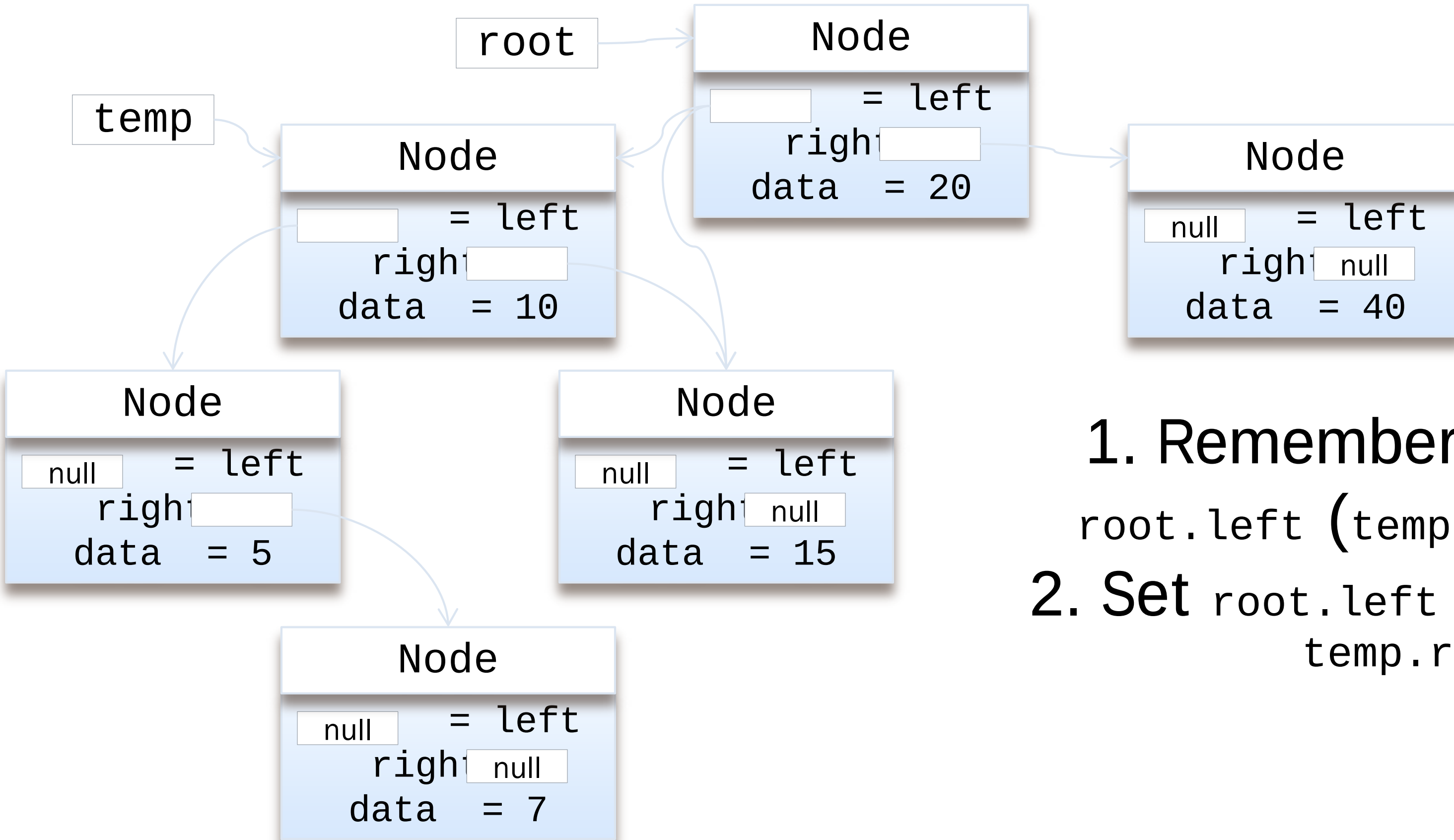


ALGORITHM FOR ROTATION (CONT.)



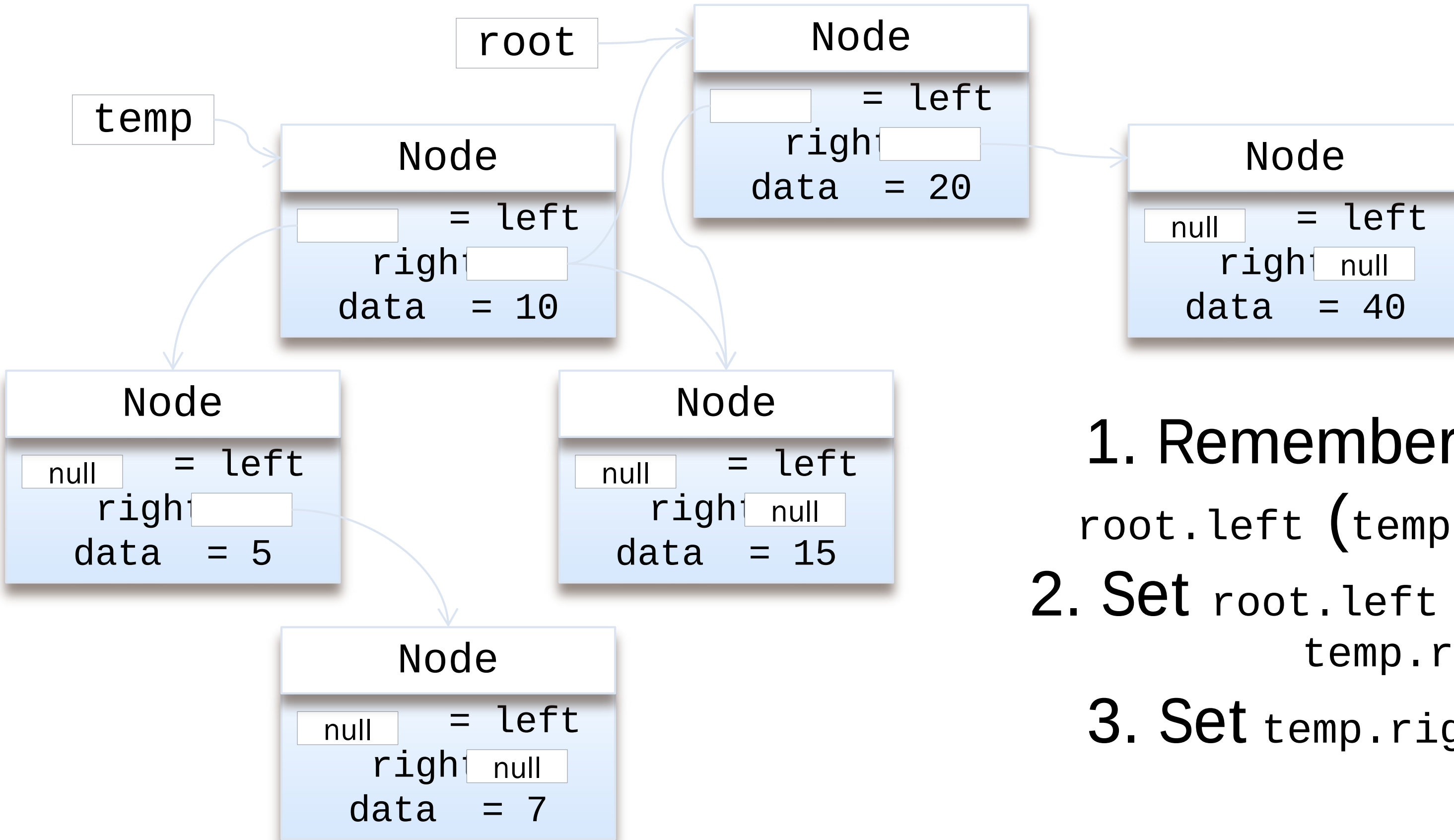
1. Remember value of `root.left` (`temp = root.left`)

ALGORITHM FOR ROTATION (CONT.)



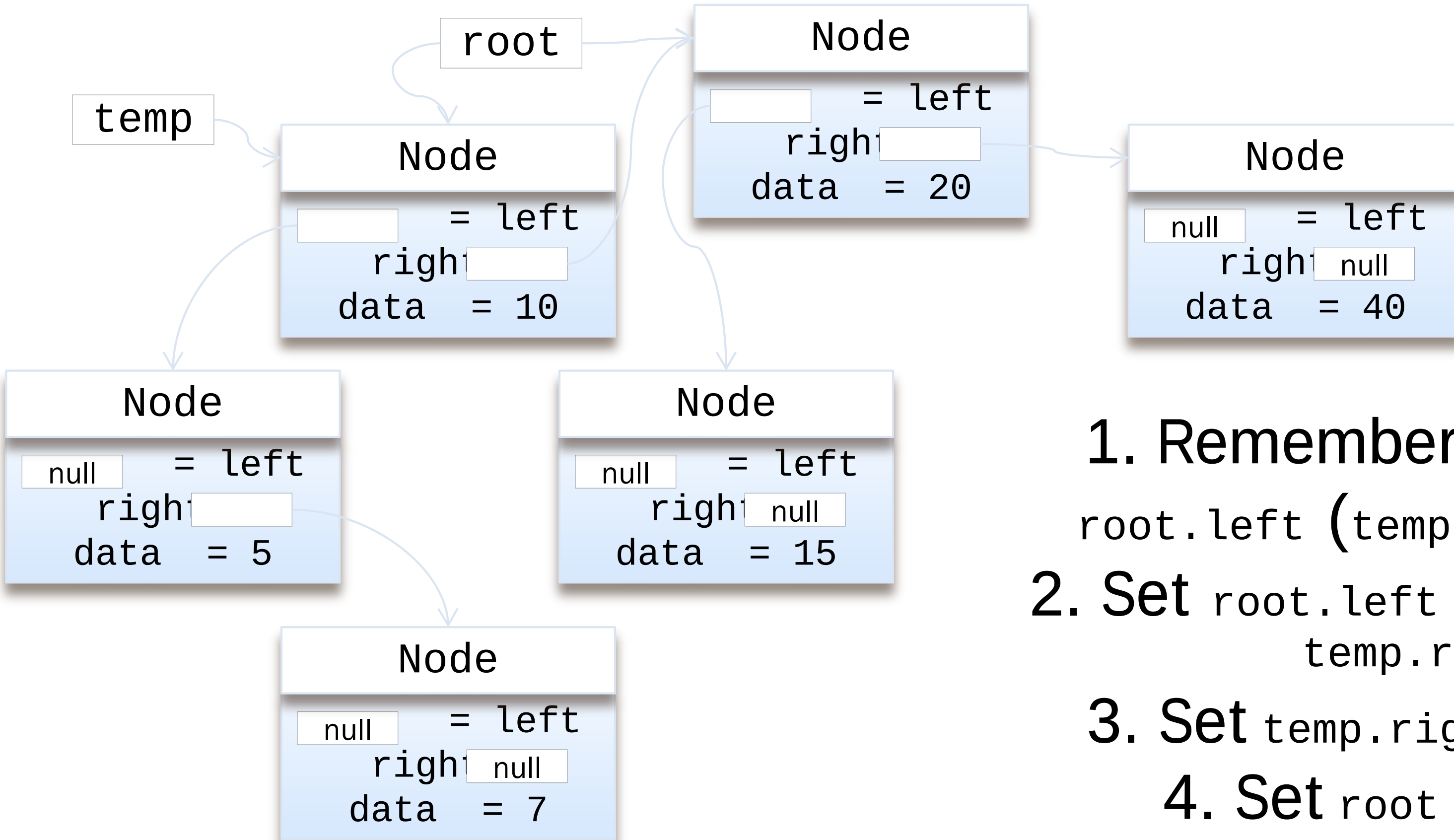
1. Remember value of `root.left` (`temp = root.left`)
2. Set `root.left` to value of `temp.right`

ALGORITHM FOR ROTATION (CONT.)



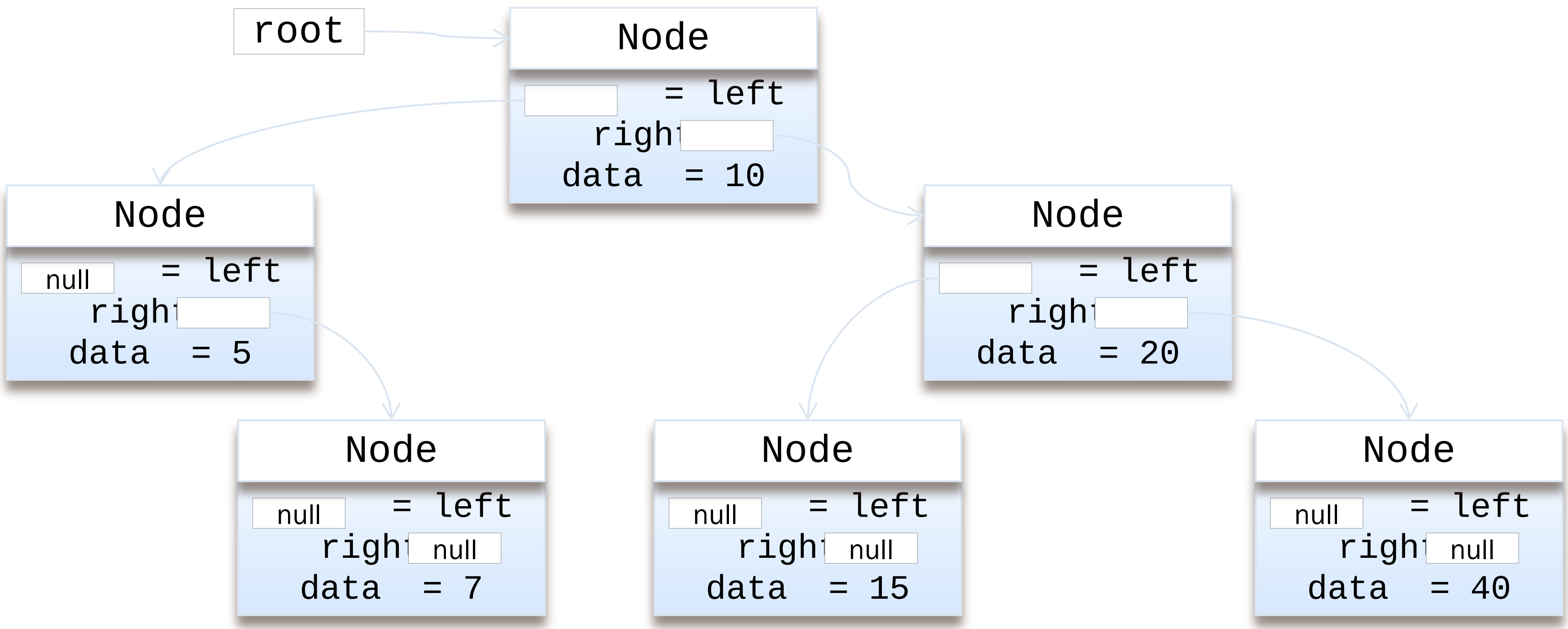
1. Remember value of `root.left` (`temp = root.left`)
2. Set `root.left` to value of `temp.right`
3. Set `temp.right` to `root`

ALGORITHM FOR ROTATION (CONT.)



1. Remember value of `root.left` (`temp = root.left`)
2. Set `root.left` to value of `temp.right`
3. Set `temp.right` to `root`
4. Set `root` to `temp`

ALGORITHM FOR ROTATION (CONT.)



TRINODE RESTRUCTURING.

- *Trinode restructuring* is a compound rotation operations with the goal to restructure the subtree rooted at *the grandparent z* in order to reduce the overall path length to current node *x* and its subtrees.

Algorithm restructure(x):

Input: A position x of a binary search tree T that has both a parent y and a grandparent z

Output: Tree T after a trinode restructuring (which corresponds to a single or double rotation) involving positions x, y, and z

1: Let (a, b, c) be a left-to-right (inorder) listing of the positions x, y, and z, and let (T_1, T_2, T_3, T_4) be a left-to-right (inorder) listing of the four subtrees of x, y, and z not rooted at x, y, or z.

2: Replace the subtree rooted at z with a new subtree rooted at b.

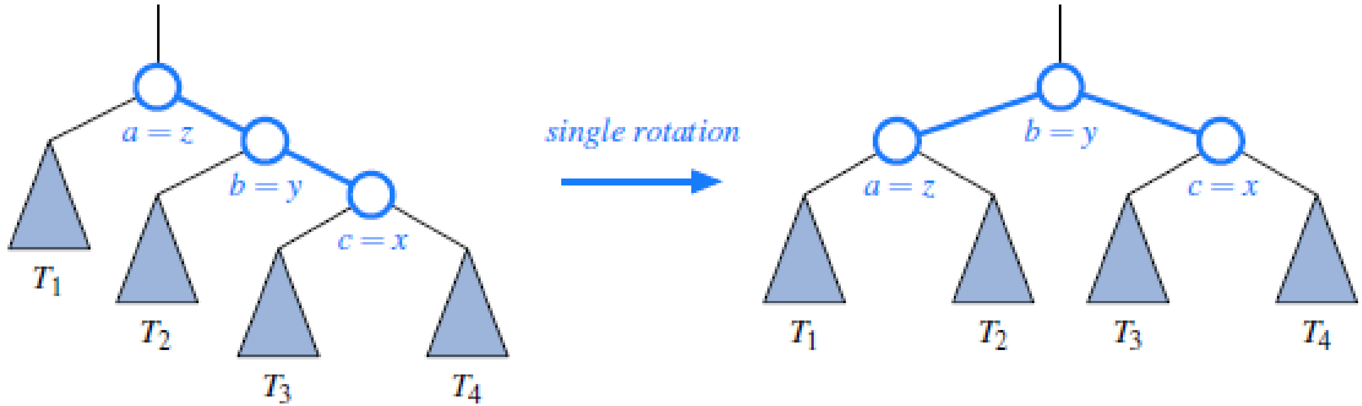
3: Let a be the left child of b and let T_1 and T_2 be the left and right subtrees of a, respectively.

4: Let c be the right child of b and let T_3 and T_4 be the left and right subtrees of c, respectively.

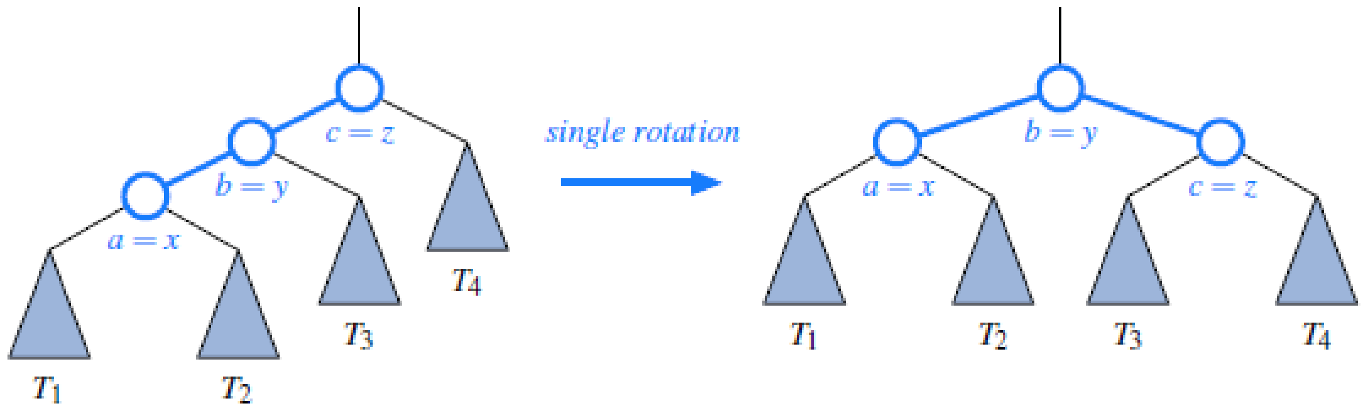
FOUR KINDS OF CRITICALLY UNBALANCED TREES

- Left-Left (parent balance is -2, left child balance is -1)
  - Rotate right around parent
- Left-Right (parent balance -2, left child balance +1)
  - Rotate left around child
  - Rotate right around parent
- Right-Right (parent balance +2, right child balance +1)
  - Rotate left around parent
- Right-Left (parent balance +2, right child balance -1)
  - Rotate right around child
  - Rotate left around parent

EXAMPLE OF A TRINODE RESTRUCTURING OPERATION 1

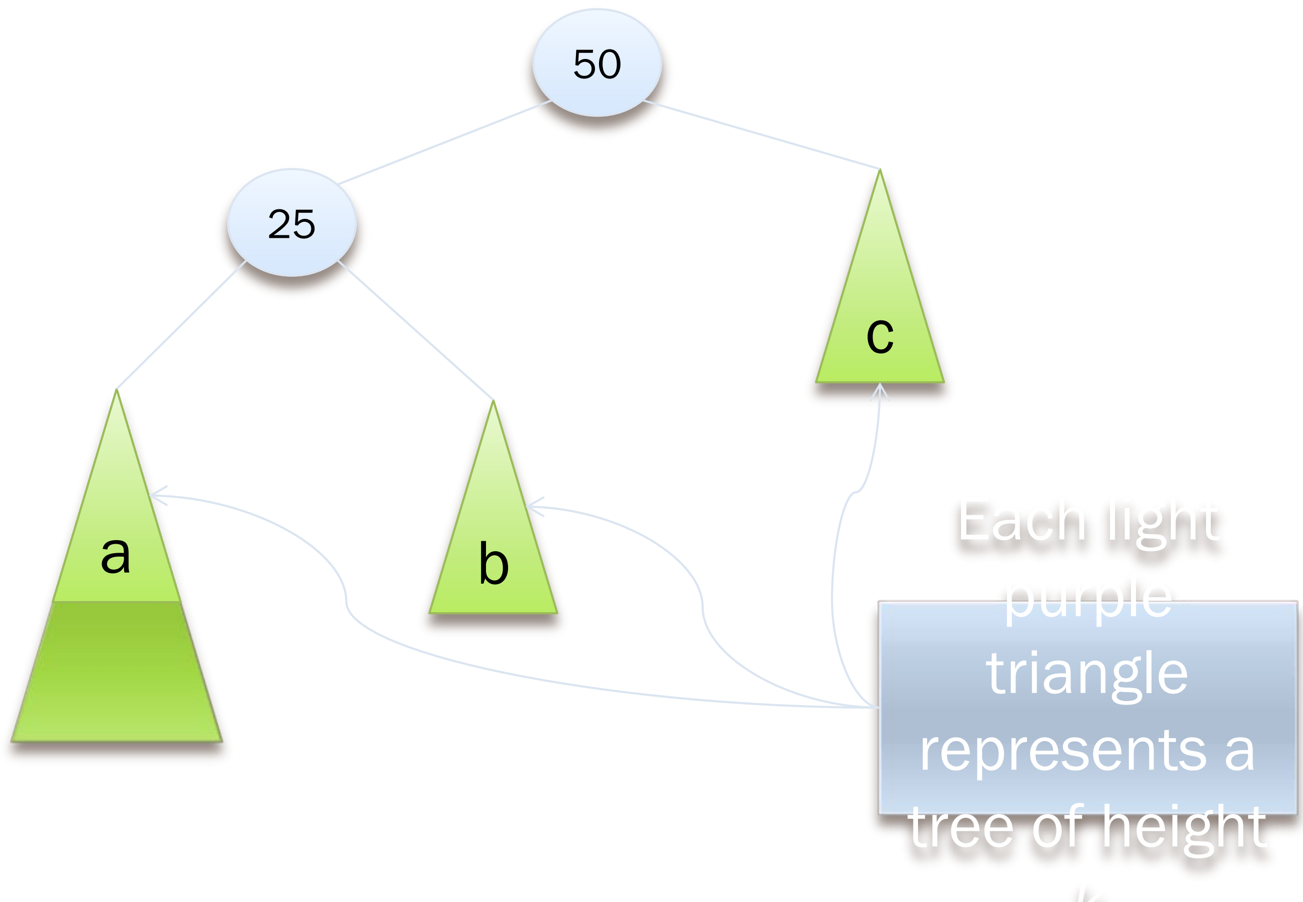


(a)

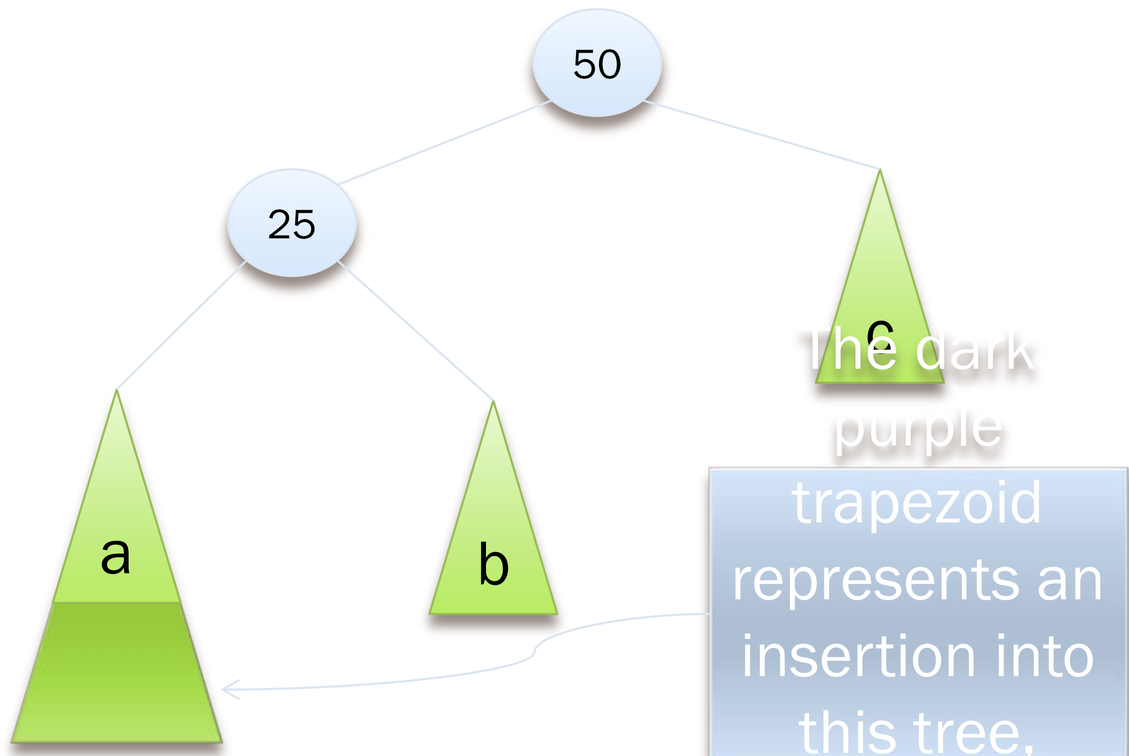


(b)

BALANCING A LEFT-LEFT TREE



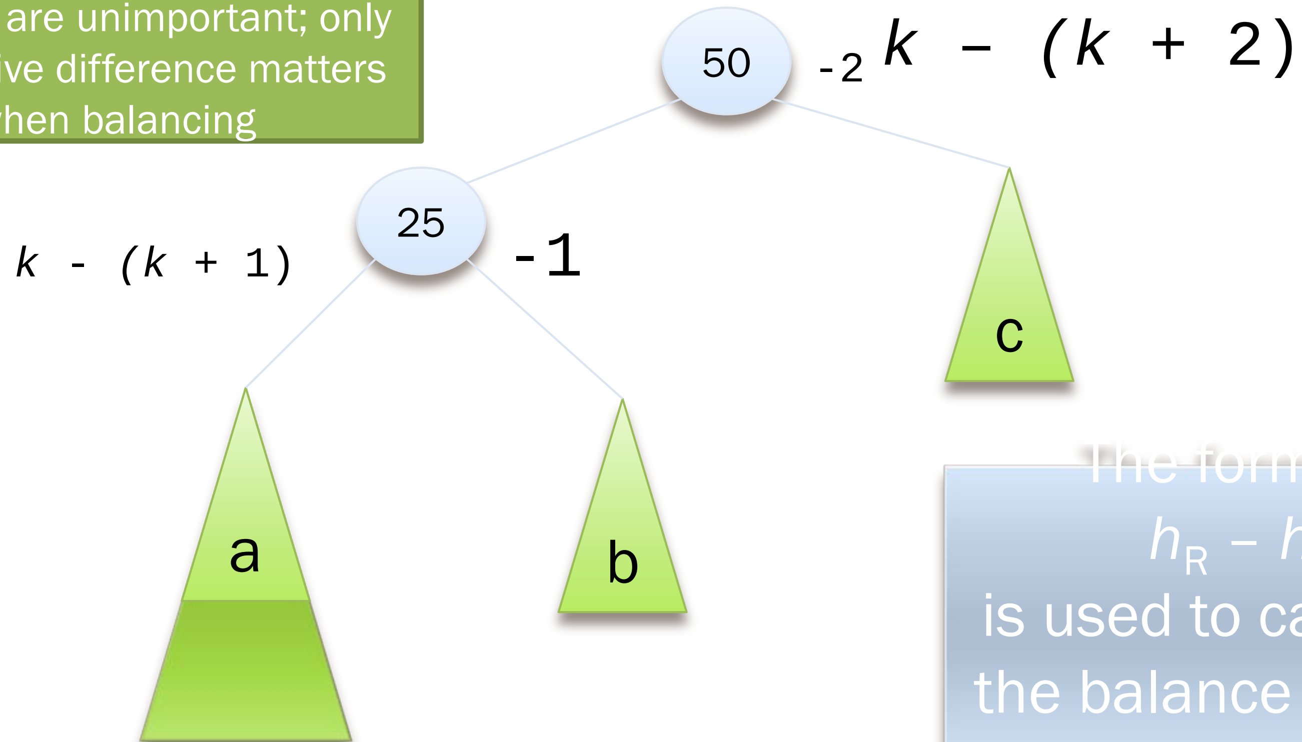
BALANCING A LEFT-LEFT TREE (CONT.)



The dark purple trapezoid represents an insertion into this tree, making its height $k + 1$

BALANCING A LEFT-LEFT TREE (CONT.)

The heights of the left and right subtrees are unimportant; only the relative difference matters when balancing

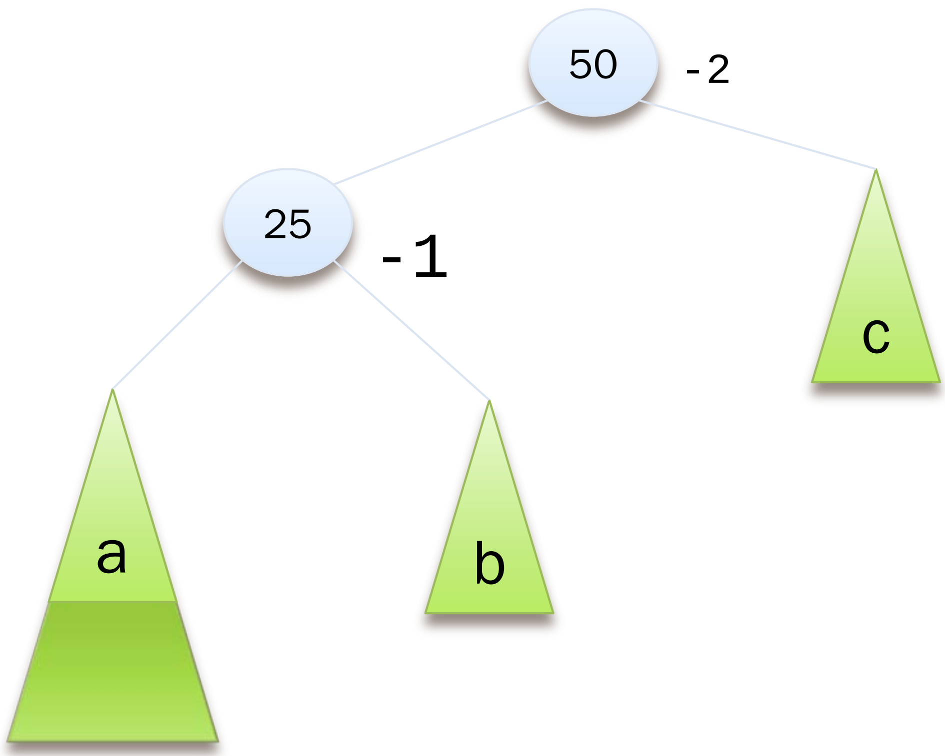


The formula

$h_R - h_L$

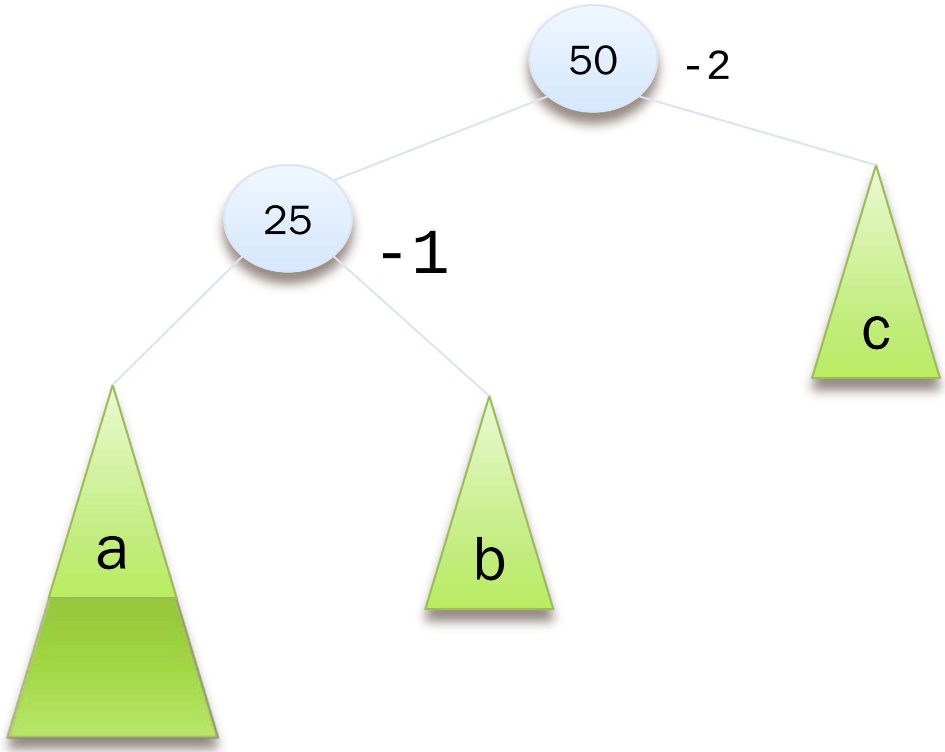
is used to calculate the balance of each node

BALANCING A LEFT-LEFT TREE (CONT.)



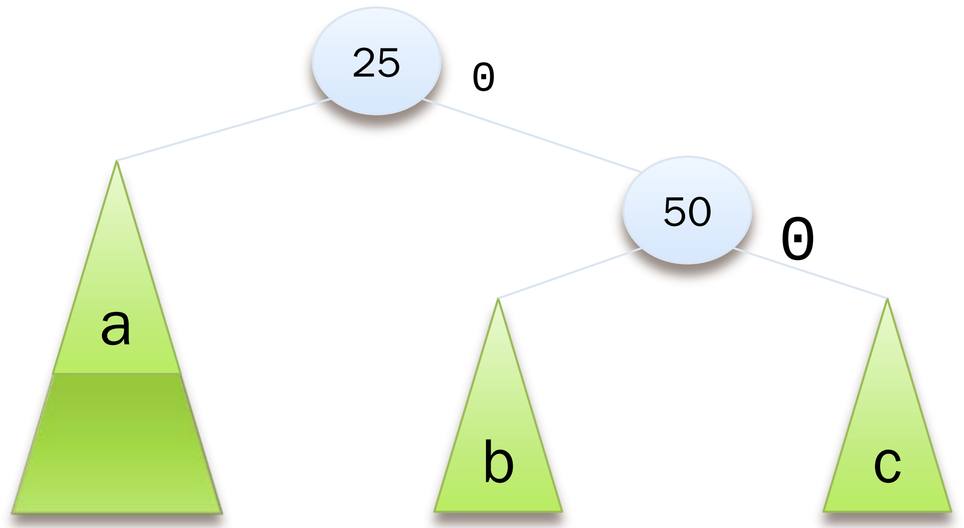
When the root and left subtree are both left-heavy, the tree is called a Left-Left tree

BALANCING A LEFT-LEFT TREE (CONT.)

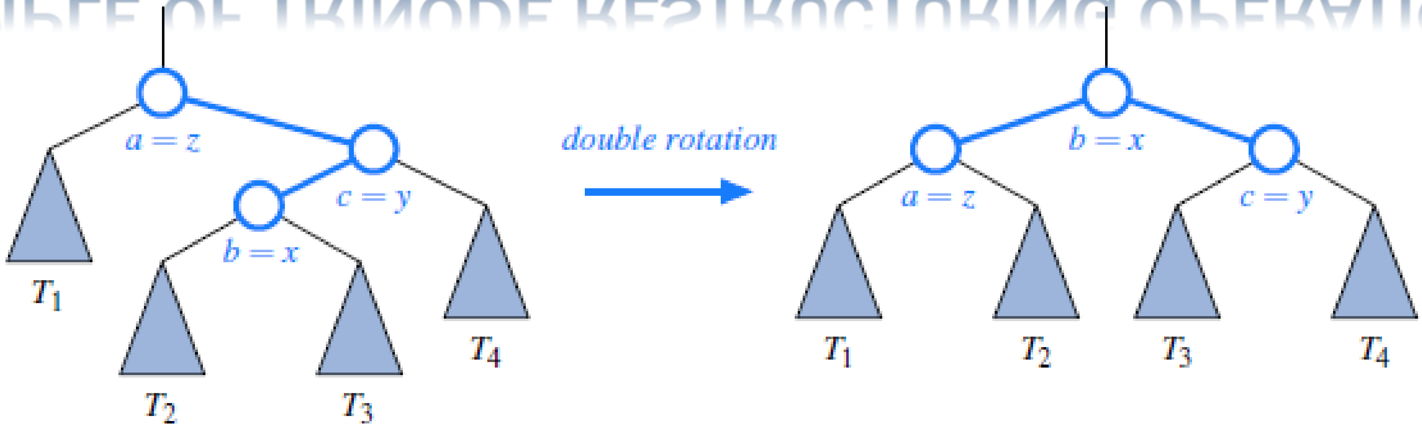


A Left-Left tree can be balanced by a rotation right

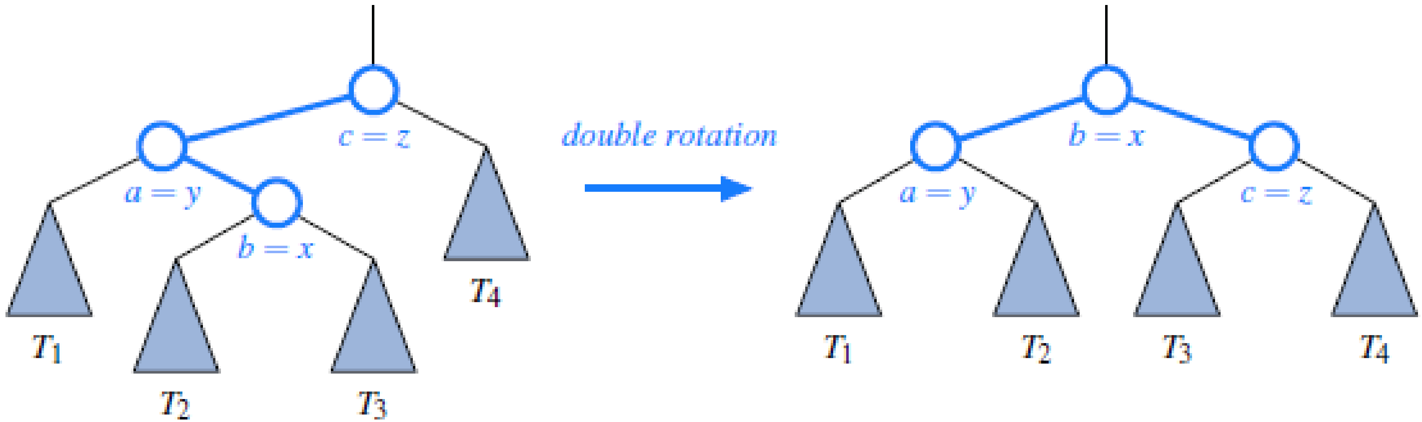
BALANCING A LEFT-LEFT TREE (CONT.)



EXAMPLE OF TRINODE RESTRUCTURING OPERATION 2

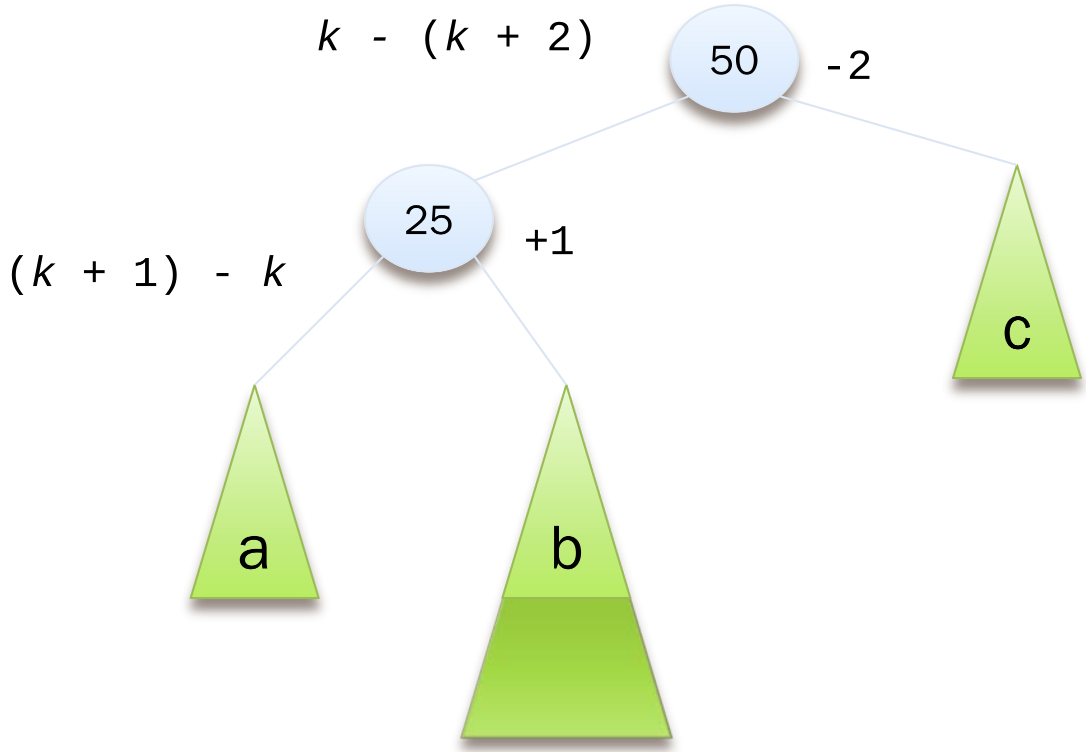


(c)

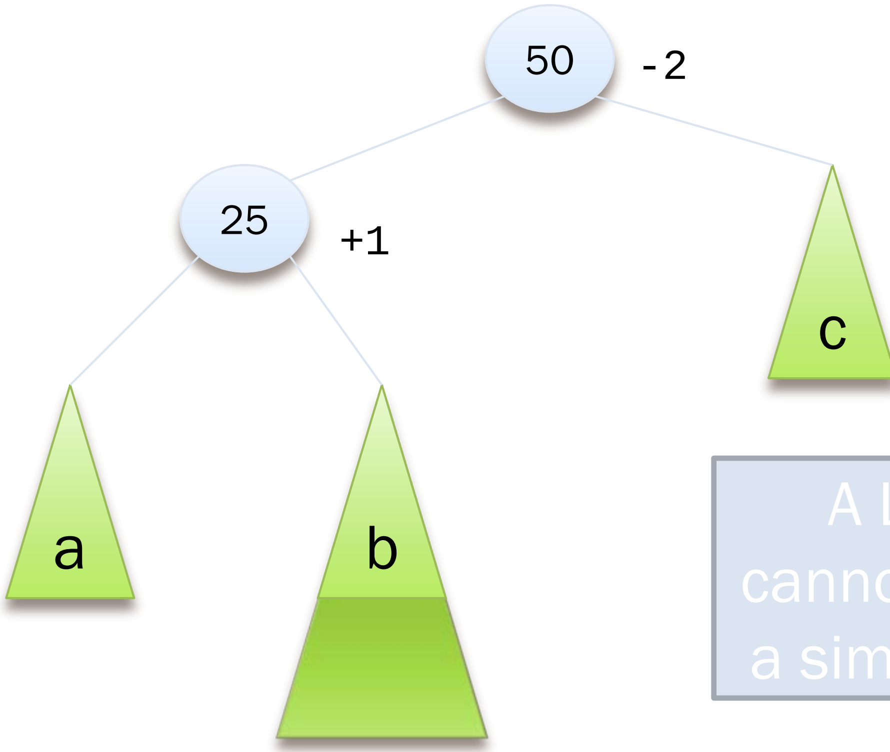


(d)

BALANCING A LEFT-RIGHT TREE

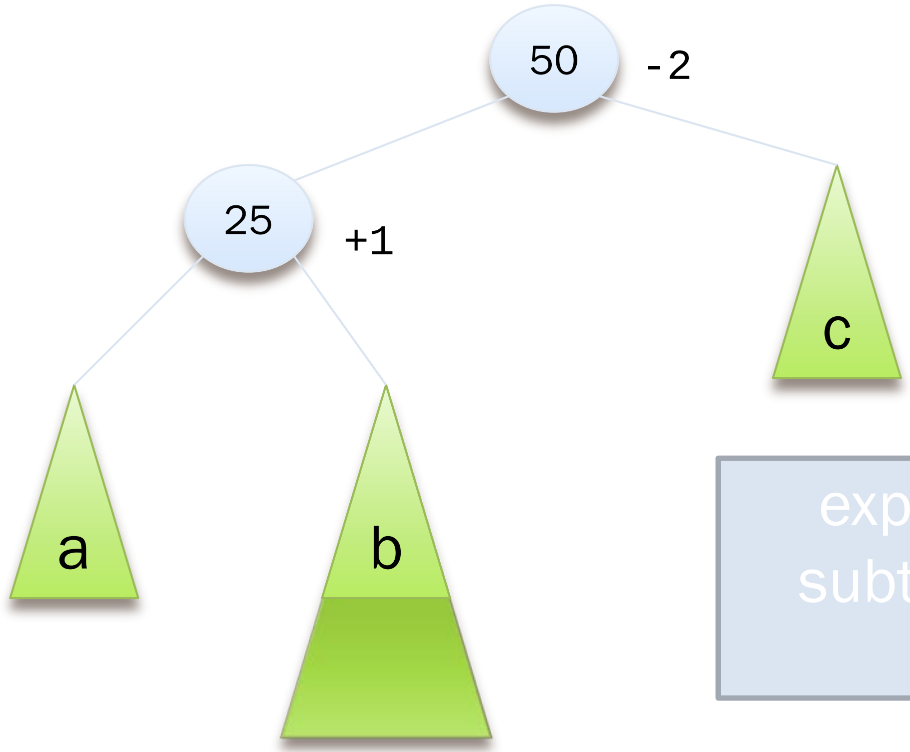


BALANCING A LEFT-RIGHT TREE (CONT.)



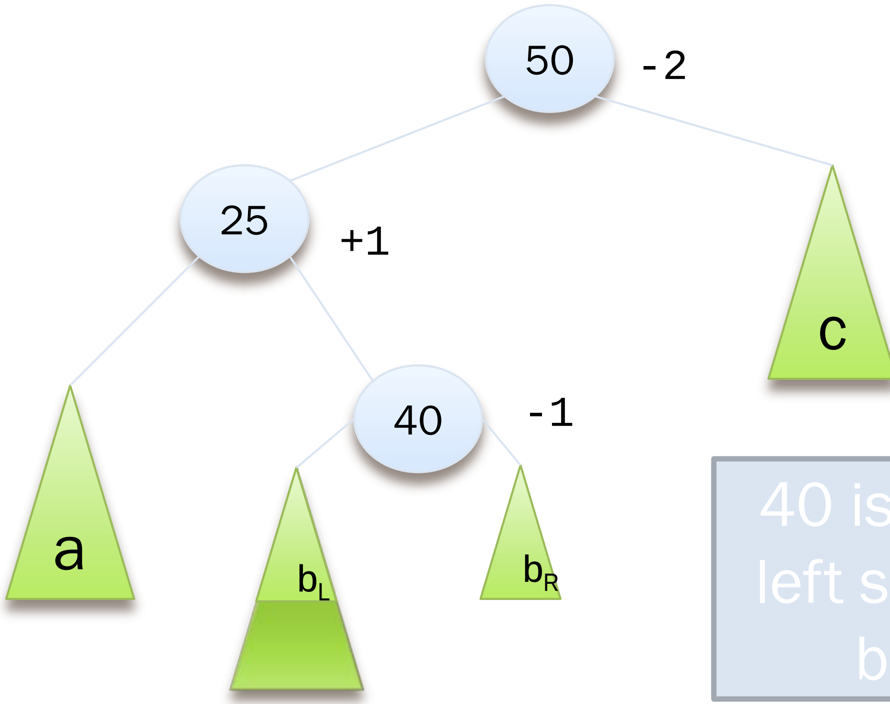
A Left-Right tree cannot be balanced by a simple rotation right

BALANCING A LEFT-RIGHT TREE (CONT.)



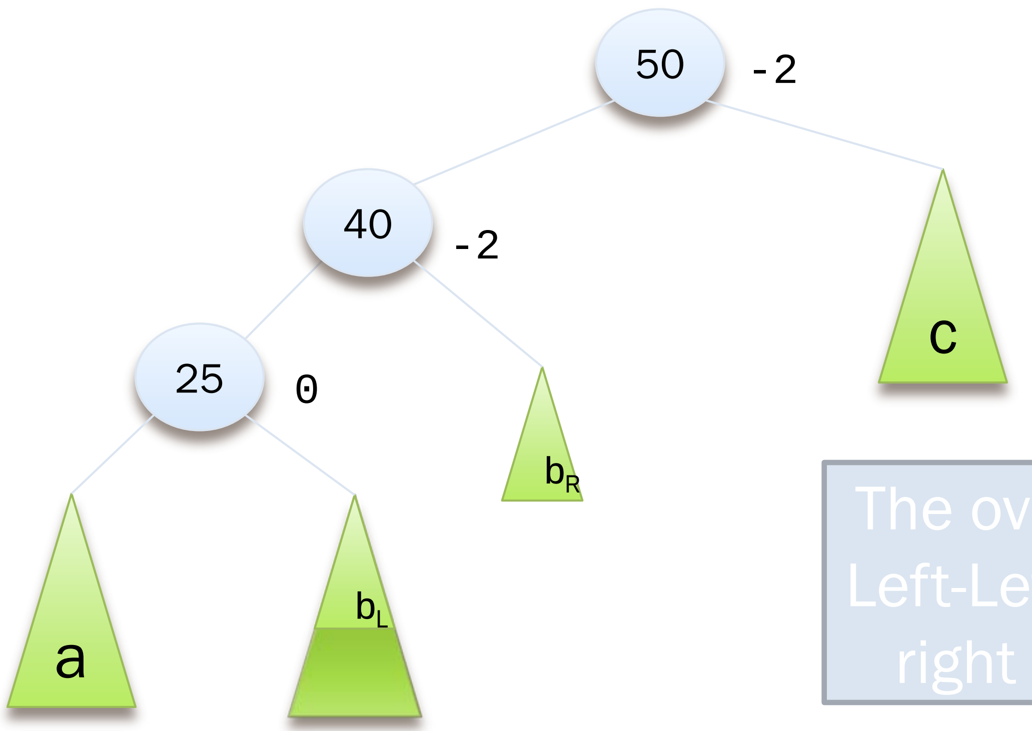
expanded into its subtrees b_L and b_R

BALANCING A LEFT-RIGHT TREE (CONT.)



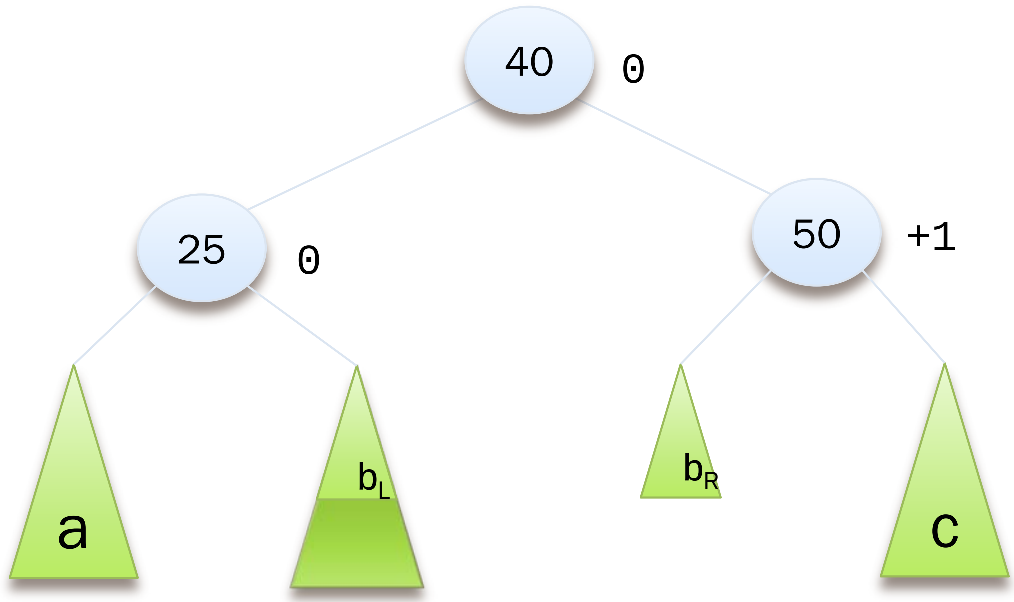
40 is left-heavy. The left substree can now be rotated left

BALANCING A LEFT-RIGHT TREE (CONT.)

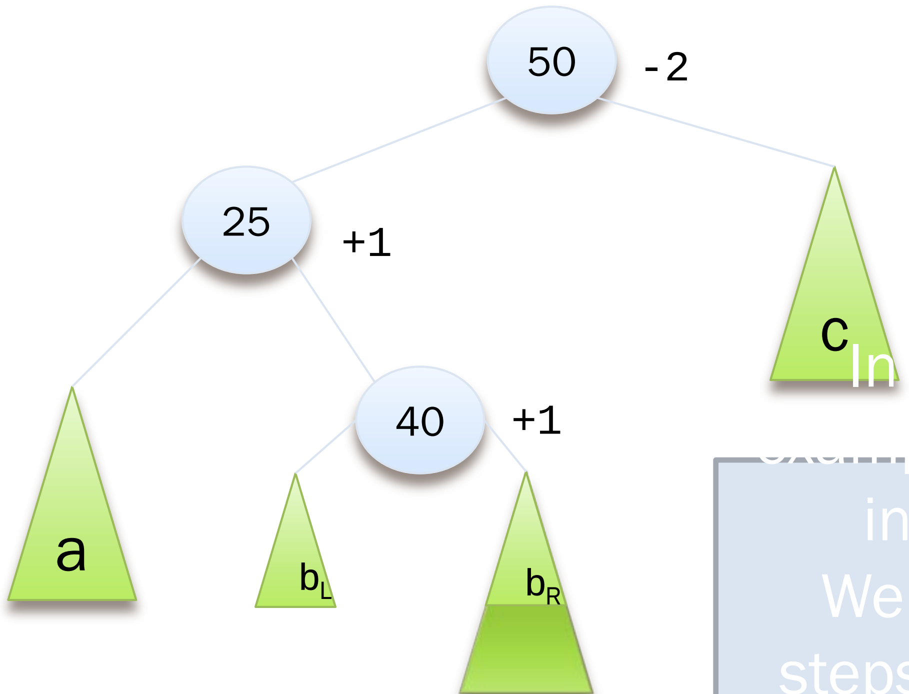


The overall tree is now Left-Left and a rotation right will balance it.

BALANCING A LEFT-RIGHT TREE (CONT.)



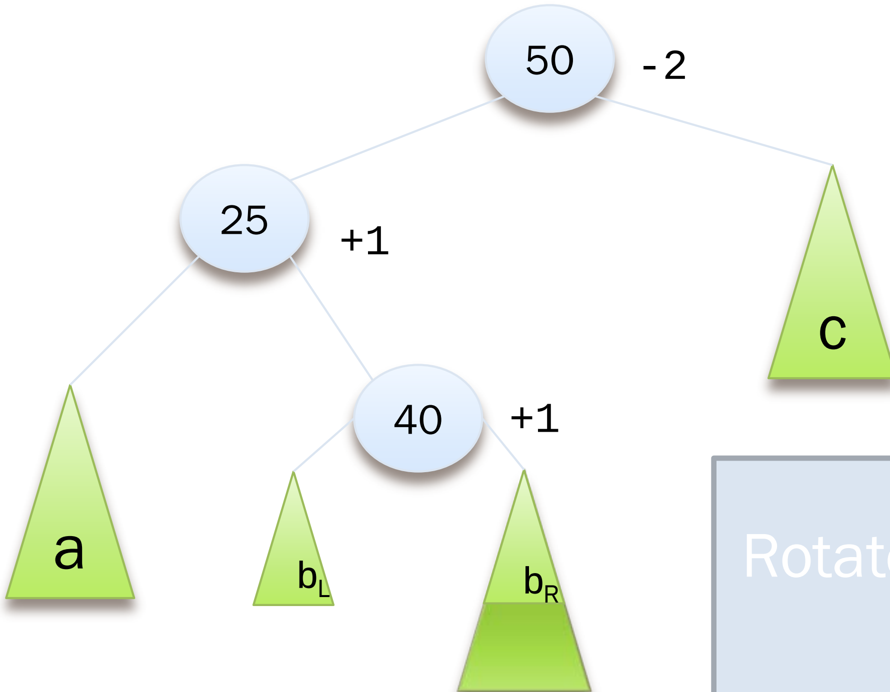
BALANCING A LEFT-RIGHT TREE (CONT.)



In

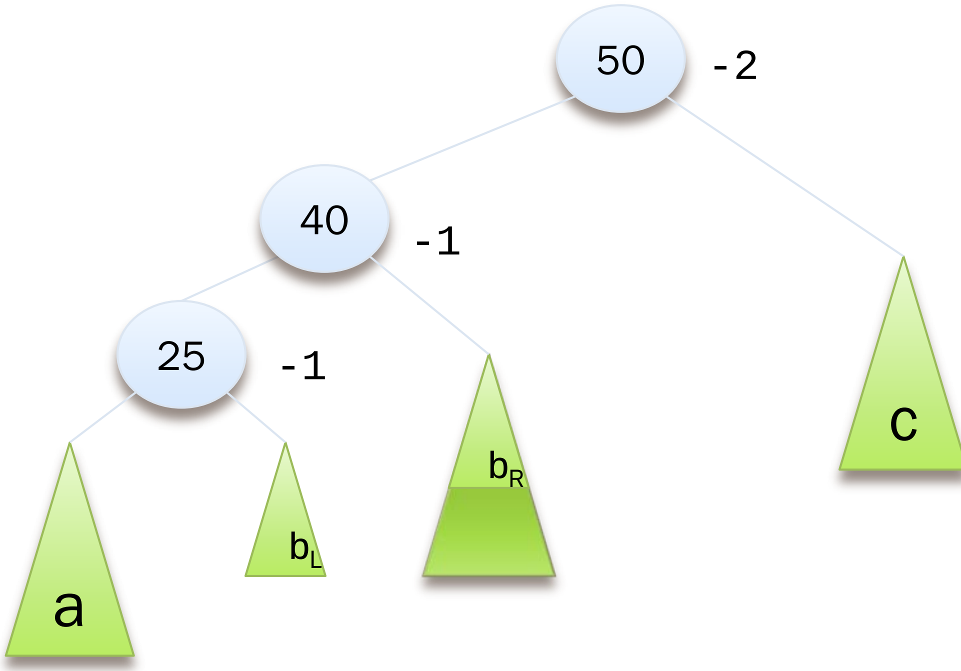
inserted in b_L.
We now show the
steps if an item was

BALANCING A LEFT-RIGHT TREE (CONT.)



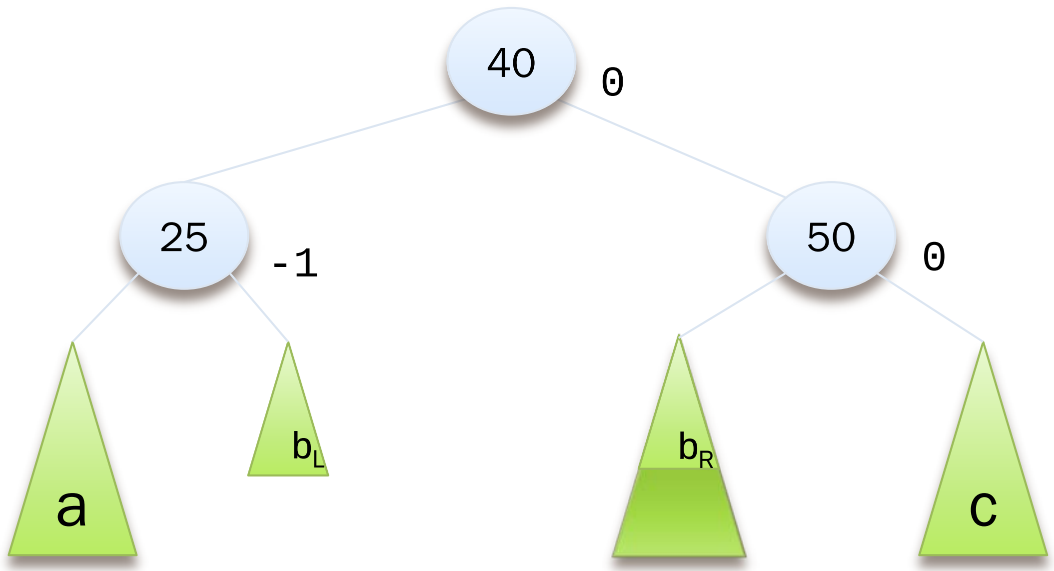
Rotate the left subtree left

BALANCING A LEFT-RIGHT TREE (CONT.)



Rotate the tree right

BALANCING A LEFT-RIGHT TREE (CONT.)



BALANCEABLE BINARYTREE CLASS

```
/** A specialized version of LinkedBinaryTree with support for balancing. */
protected static class BalanceableBinaryTree<K,V>
                     extends LinkedBinaryTree<Entry<K,V>> {
  //-------------- nested BSTNode class --------------
  // this extends the inherited LinkedBinaryTree.Node class
  protected static class BSTNode<E> extends Node<E> {
    int aux=0;
    BSTNode(E e, Node<E> parent, Node<E> leftChild, Node<E> rightChild) {
      super(e, parent, leftChild, rightChild);
    }
    public int getAux( ) { return aux; }
    public void setAux(int value) { aux = value; }
  } //--------- end of nested BSTNode class ---------

  // positional-based methods related to aux field
  public int getAux(Position<Entry<K,V>> p) {
    return ((BSTNode<Entry<K,V>>) p).getAux( );
  }
  public void setAux(Position<Entry<K,V>> p, int value) {
    ((BSTNode<Entry<K,V>>) p).setAux(value);
  }
  // Override node factory function to produce a BSTNode (rather than a Node)
  protected
  Node<Entry<K,V>> createNode(Entry<K,V> e, Node<Entry<K,V>> parent,
                         Node<Entry<K,V>> left, Node<Entry<K,V>> right) {
    return new BSTNode<>(e, parent, left, right);
  }
```

```
/** Relinks a parent node with its oriented child node. */
private void relink(Node<Entry<K,V>> parent, Node<Entry<K,V>> child,
                    boolean makeLeftChild) {
  child.setParent(parent);
  if (makeLeftChild)
    parent.setLeft(child);
  else
    parent.setRight(child);
}
```

```
/** Rotates Position p above its parent. */
public void rotate(Position<Entry<K,V>> p) {
  Node<Entry<K,V>> x = validate(p);
  Node<Entry<K,V>> y = x.getParent( );              // we assume this exists
  Node<Entry<K,V>> z = y.getParent( );              // grandparent (possibly null)
  if (z == null) {
    root = x;                                       // x becomes root of the tree
    x.setParent(null);
  } else
    relink(z, x, y == z.getLeft( ));                // x becomes direct child of z
  // now rotate x and y, including transfer of middle subtree
  if (x == y.getLeft( )) {
    relink(y, x.getRight( ), true);                 // x's right child becomes y's left
    relink(x, y, false);                            // y becomes x's right child
  } else {
    relink(y, x.getLeft( ), false);                 // x's left child becomes y's right
    relink(x, y, true);                             // y becomes left child of x
  }
}
```

```
  /** Performs a trinode restructuring of Position x with its parent/grandparent. */
  public Position<Entry<K,V>> restructure(Position<Entry<K,V>> x) {
    Position<Entry<K,V>> y = parent(x);
    Position<Entry<K,V>> z = parent(y);
    if ((x == right(y)) == (y == right(z))) {     // matching alignments
      rotate(y);                                  // single rotation (of y)
      return y;                                   // y is new subtree root
    } else {                                      // opposite alignments
      rotate(x);                                  // double rotation (of x)
      rotate(x);
      return x;                                   // x is new subtree root
    }
  }
}
```