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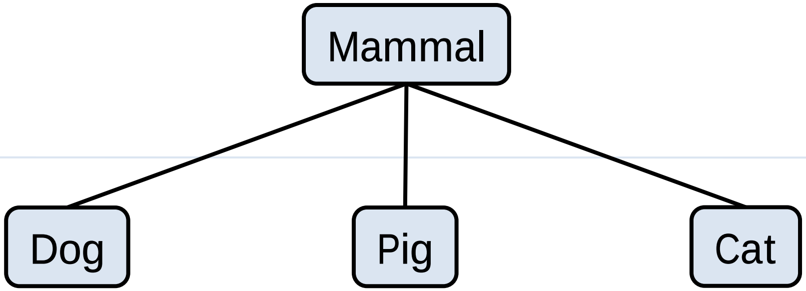
TREES



Presentation for use with the textbook Data Structures and Algorithms in Java, 6th edition, by M. T. Goodrich, R. Tamassia, and M. H. Goldwasser, Wiley, 2014

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WHAT IS A TREE

In computer science, a tree is an abstract model of hierarchical structure (a type of nonlinear data structure)

- Trees consists of nodes with a parent-child relation
- Trees also provide a natural or ganization for data,
  - Organization charts
  - File systems
  - Programming environment s

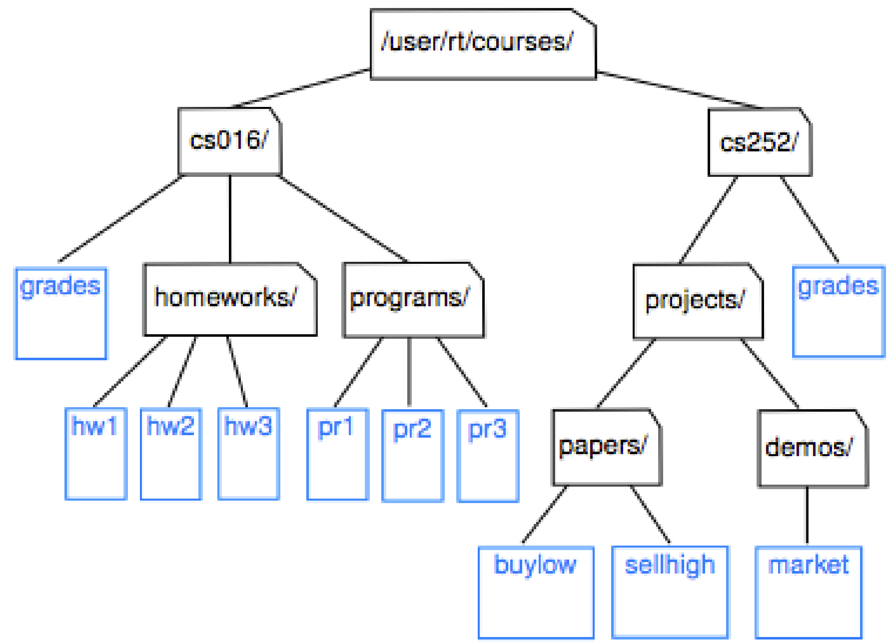

Figure 8.3: Tree representing a portion of a file system.

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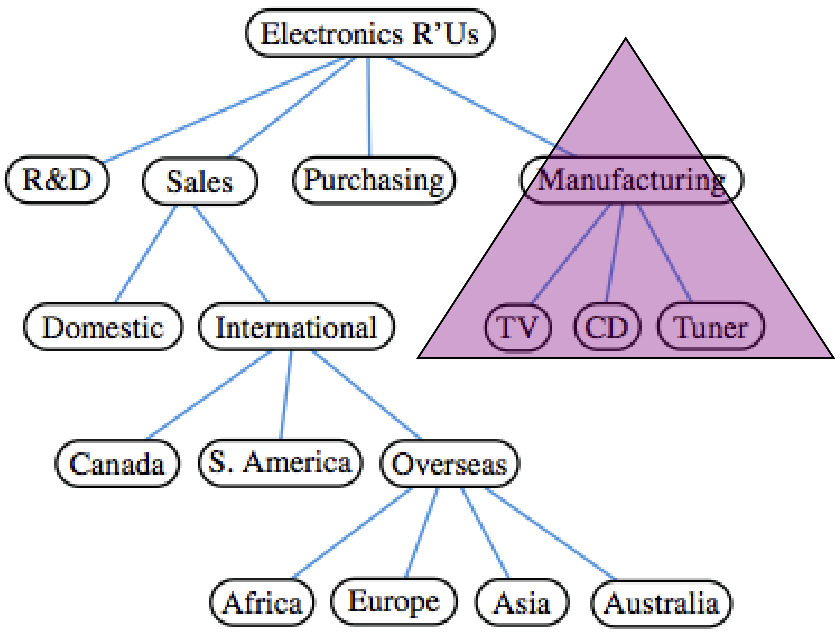
TREE TERMINOLOGY

- Root: node without parent (A)
- Internal node: node with at least one child (A, B, C, F)
- External node (a.k.a. leaf): node without children (E, I, J, K, G, H, D)
- Ancestors of a node: parent, grandparent, grand-grandparent, etc.
- Descendant of a node: child, grandchild, grand-grandchild, etc.
- Subtree: tree consisting of a node and its descendants



organization of a fictitious corporation.

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TREE TERMINOLOGY CONT

- Siblings: two nodes that are children of the same parent
- Depth of a node: number of ancestors
- Height of a tree: maximum depth of any node (3)
- Edge: a pair of nodes (*u*,*v*) such that *u* is the parent of *v*, or vice versa.
- *Path:* a sequence of nodes such that any two consecutive nodes in the sequence form an edge

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FORMAL TREE DEFINITION

- Formally, we define a ***tree*** *T* as a set of ***nodes*** storing elements such that the nodes have a ***parent-child*** relationship that satisfies the following properties:
  - If *T* is nonempty, it has a special node, called the ***root*** of *T* , that has no parent.
  - Each node *v* of *T* different from the root has a unique ***parent*** node *w*; every node with parent *w* is a ***child*** of *w*.

- Note: a tree can be empty (no nodes)

  => Tree can be defined recursively such that a tree *T* is either empty or consists of a node *r*, called the root of *T*, and a (possibly empty) set of subtrees whose roots are the children of *r*.

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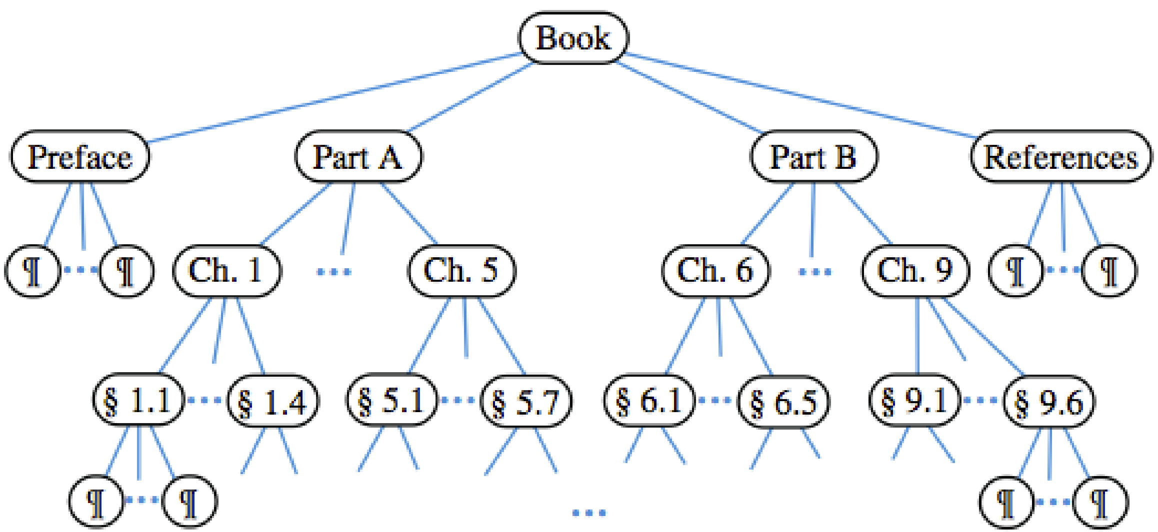
ORDERED TREES

- A tree is *ordered* if there is a meaningful linear order among the children of each node;
  - An order is usually visualized by arranging siblings left to right, according to their order.



An ordered tree associated with a book.

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TREE ADT

- We use positions as an abstraction for a node of a tree
- A position object for a tree supports the method:
  - getElement(): Returns the element stored at this position.

- ***Accessor methods*** for navigating through positions of a tree *T*
  - root(): Returns the position of the root of the tree (or null if empty).
  - parent(*p*): Returns the position of the parent of position *p* (or null if *p* is the root).
  - children(*p*): Returns an iterable collection containing the children of position *p* (if any).
  - numChildren(*p*): Returns the number of children of position *p*.

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TREE ADT CONT.

- ***Query methods,*** **which are often used with** conditionals statements:
  - isInternal(*p*): Returns true if position *p* has at least one child.
  - isExternal(*p*): Returns true if position *p* does not have any children.
  - isRoot(*p*): Returns true if position *p* is the root of the tree.
- General methods, unrelated to the specific structure of the tree:
  - size(): Returns the number of positions (and hence elements) that are contained in the tree.
  - isEmpty(): Returns true if the tree does not contain any positions (and thus no elements).
  - iterator(): Returns an iterator for all elements in the tree (so that the tree itself is Iterable).
  - positions(): Returns an iterable collection of all positions of the tree.
- Additional update methods may be defined by data structures implementing the Tree ADT. (Discussed later)

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A TREE INTERFACE IN JAVA

Methods for a Tree interface:

```
/** An interface for a tree where nodes can have an arbitrary number of children. */
public interface Tree<E> extends Iterable<E> {
  Position<E> root();
  Position<E> parent(Position<E> p) throws IllegalArgumentException;
  Iterable<Position<E>> children(Position<E> p)
                                  throws IllegalArgumentException;
  int numChildren(Position<E> p) throws IllegalArgumentException;
  boolean isInternal(Position<E> p) throws IllegalArgumentException;
  boolean isExternal(Position<E> p) throws IllegalArgumentException;
  boolean isRoot(Position<E> p) throws IllegalArgumentException;
  int size();
  boolean isEmpty();
  Iterator<E> iterator();
  Iterable<Position<E>> positions();
}
```

Accessor methods

Query methods

General method s

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AN ABSTRACTTREE BASE CLASS IN JAVA

- If a concrete implementation provides three fundamental methods—*root(), parent(p),* and *children(p)*— all other behaviors of the Tree interface can be derived within the AbstractTree base class.

```
/** An abstract base class providing some functionality of the Tree interface. */
public abstract class AbstractTree<E> implements Tree<E> {
  public boolean isInternal(Position<E> p) { return numChildren(p) > 0; }
  public boolean isExternal(Position<E> p) { return numChildren(p) == 0; }
  public boolean isRoot(Position<E> p) { return p == root( ); }
  public boolean isEmpty( ) { return size( ) == 0; }
}
```

An initial implementation of the AbstractTree base class. (We add additional functionality to this class as the chapter continues.)

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COMPUTING DEPTH

- Let *p* be a position within tree *T*. The ***depth*** of *p* is the number of ancestors of *p*, other than *p* itself.
- The depth of *p* can also be recursively defined as follows:
  - If *p* is the root, then the depth of *p* is 0.
  - Otherwise, the depth of *p* is one plus the depth of the parent of *p*.

```
/** Returns the number of levels separating Position p from the root. */
public int depth(Position<E> p) {
  if (isRoot(p))
    return 0;
  else
    return 1 + depth(parent(p));
}
```

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COMPUTING HEIGHT

- We next define the ***height*** of a tree to be equal to the maximum of the depths of its positions (or zero, if the tree is empty).

- If using the definition as is, the height computation become inefficient:

```
/** Returns the height of the tree. */
private int heightBad() {
  int h = 0;
  for (Position<E> p : positions())
    if (isExternal(p))
      h = Math.max(h, depth(p));
  return h;
}
```

Analysis:
Positions(p): can be implemented to run in O(n);
Because heightBad calls algorithm depth(p) on each leaf of T, its running time is

$$O(n + \sum_{p \in L}(d_p + 1)),$$

where L is the set of leaf positions of T.
In the worst case, the sum $\sum_{p \in L}(d_p + 1)$ is proportional to n^2.
Thus, algorithm heightBad runs in $O(n^2)$ worst-case time.

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COMPUTING HEIGHT CONT

- Recursive definition to compute height.
- Define the ***height*** of a position *p* in a tree *T* as follows:
  - If *p* is a leaf, then the height of *p* is 0.
  - Otherwise, the height of *p* is one more than the maximum of the heights of *p*'s children.

- The height of the root of a nonempty tree *T*, according to the recursive definition, equals the maximum depth among all leaves of tree *T* .

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COMPUTING HEIGHT CONT

```
/** Returns the height of the subtree rooted at Position p. */
public int height(Position<E> p) {
  int h = 0;                                   // base case if p is external
  for (Position<E> c : children(p))
    h = Math.max(h, 1 + height(c));
  return h;
}
```

$O(n)$ worst-case time

- The overall height of a nonempty tree can be computed by sending the root of the tree as a parameter.

- Assuming that children(p) executes in $O(c_p + 1)$ time, where c_p denotes the number of children of p. Algorithm height(p) spends $O(c_p + 1)$ time at each position p to compute the maximum, and its overall running time is

$$O(\sum_p (c_p + 1)) = O(n + \sum_p c_p).$$

Let T be a tree with n positions, and let c_p denote the number of children of a position p of T. Then, summing over the positions of T, $\sum_p c_p = n - 1$.

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BINARY TREES

- A ***binary tree*** is an ordered tree with the following properties:
  - Every node has at most two children.
  - Each child node is labeled as being either a ***left child*** or a ***right child***.
  - A left child precedes a right child in the order of children of a node.
- The subtree rooted at a left or right child of an internal node *v* is called a ***left subtree*** or ***right subtree***, respectively, of *v*.
- A binary tree is ***proper (full)*** if each node has either zero or two children.
  - Every internal node has exactly two children.
- A binary tree that is not proper is ***improper***
- Alternative recursive definition: a binary tree is either
  - a tree consisting of a single node, or
  - a tree whose root has an ordered pair of children, each of which is a binary tree

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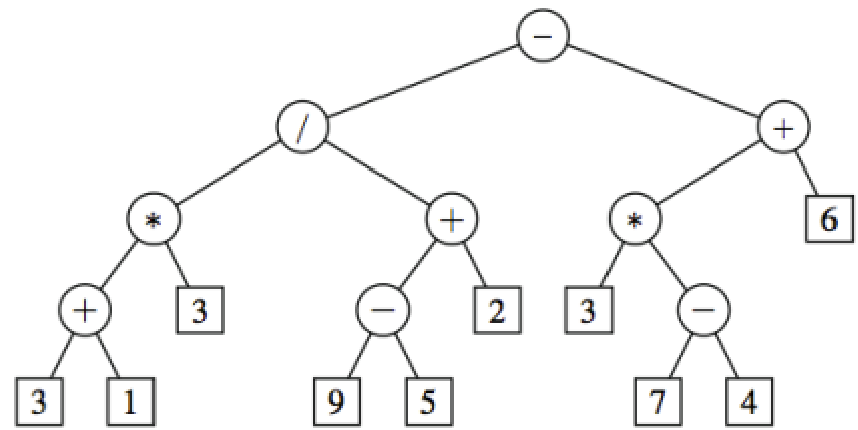
BINARY TREES CONT. : ARITHMETIC EXPRESSION TREE

- Leaves are associated with variables or constants
- Internal nodes are associated with one of the operators +, −, ∗, and /
- Each node in such a tree has a value associated with it.
  - If a node is leaf, then its value is that of its variable or constant.
  - If a node is internal, then its value is defined by applying its operation to the values of its children.
- A typical arithmetic expression tree is a proper binary tree,
- If allowed unary operators, like negation (−), then tree is improper binary



tree represents the expression
$((((3+1)*3)/((9-5)+2))-((3*(7-4))+6))$
The value associated with the internal node labeled “/” is 2.

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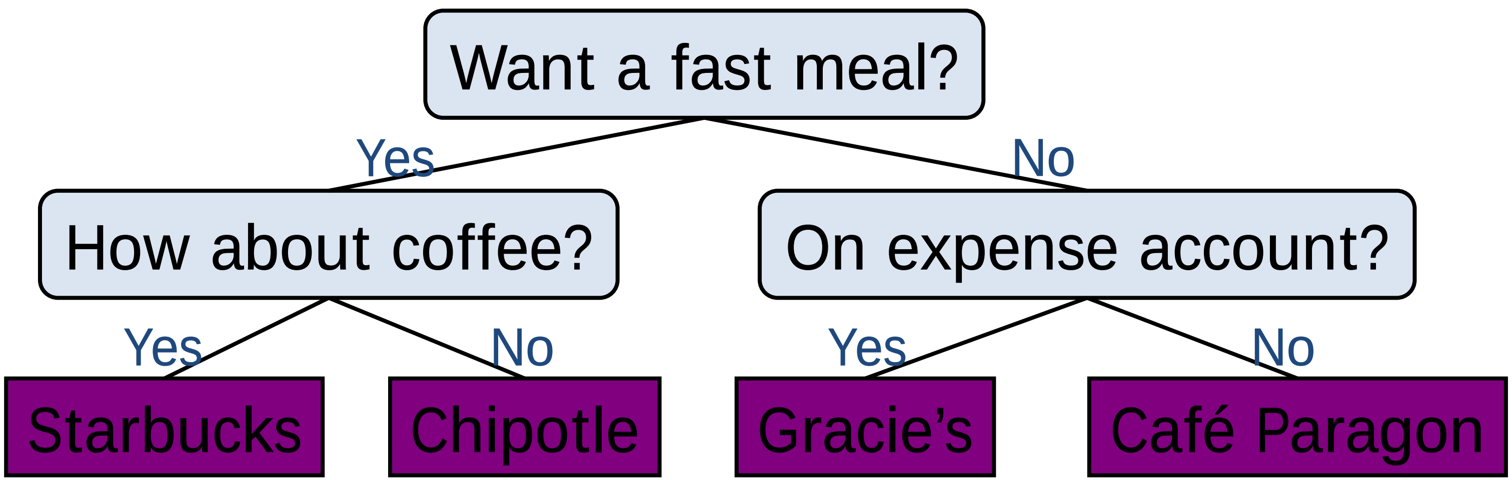
BINARY TREES CONT. : DECISION TREE

- Binary tree associated with a decision process
  - internal nodes: questions with yes/no answer
  - external nodes: decisions
- Example: dining decision



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BINARY TREE ABSTRACT DATA TYPE

- Binary tree is a specialization of a tree that supports three additional accessor methods:
  - left(*p*): Returns the position of the left child of *p* (or null if *p* has no left child).
  - right(*p*): Returns the position of the right child of *p* (or null if *p* has no right child).
  - sibling(*p*): Returns the position of the sibling of *p* (or null if *p* has no sibling).

- We again defer the definition and implementation of specialized update methods for binary trees.

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BINARY TREE ADT: BINARYTREE INTERFACE

```
/** An interface for a binary tree, in which each node has at most two children. */
public interface BinaryTree<E> extends Tree<E> {
  /** Returns the Position of p's left child (or null if no child exists). */
  Position<E> left(Position<E> p) throws IllegalArgumentException;
  /** Returns the Position of p's right child (or null if no child exists). */
  Position<E> right(Position<E> p) throws IllegalArgumentException;
  /** Returns the Position of p's sibling (or null if no sibling exists). */
  Position<E> sibling(Position<E> p) throws IllegalArgumentException;
}
```

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BINARY TREE ADT: ABSTRACTBINARYTREE BASE CLASS

```
/** An abstract base class providing some functionality of the BinaryTree interface.*/
public abstract class AbstractBinaryTree<E> extends AbstractTree<E>
                                            implements BinaryTree<E> {
  /** Returns the Position of p's sibling (or null if no sibling exists). */
  public Position<E> sibling(Position<E> p) {
    Position<E> parent = parent(p);
    if (parent == null) return null;              // p must be the root
    if (p == left(parent))                        // p is a left child
      return right(parent);                       // (right child might be null)
    else                                          // p is a right child
      return left(parent);                        // (left child might be null)
  }
```

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BINARY TREE ADT: ABSTRACTBINARYTREE BASE CLASS

```
  /** Returns the number of children of Position p. */
  public int numChildren(Position<E> p) {
    int count=0;
    if (left(p) != null)
      count++;
    if (right(p) != null)
      count++;
    return count;
  }
  /** Returns an iterable collection of the Positions representing p's children. */
  public Iterable<Position<E>> children(Position<E> p) {
    List<Position<E>> snapshot = new ArrayList<>(2);    // max capacity of 2
    if (left(p) != null)
      snapshot.add(left(p));
    if (right(p) != null)
      snapshot.add(right(p));
    return snapshot;
  }
}
```

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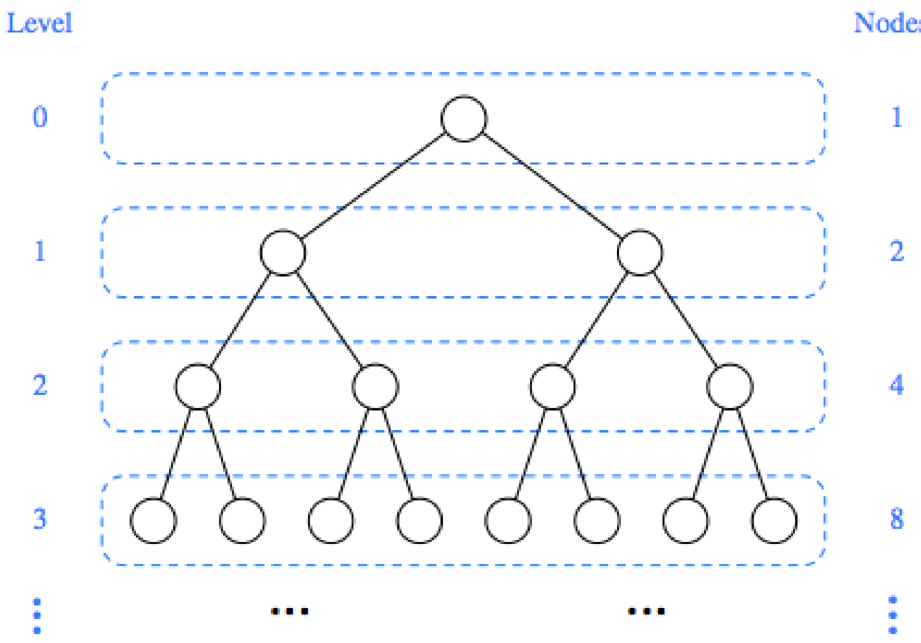
PROPERTIES OF PROPER BINARY TREES

Proposition: Let T be a nonempty binary tree, and let n, n_E , n_I , and h denote the number of nodes, number of external nodes, number of internal nodes, and height of T , respectively. Then T has the following properties:

1. $h+1 \leq n \leq 2^{h+1}-1$
2. $1 \leq n_E \leq 2^h$
3. $h \leq n_I \leq 2^h-1$
4. $\log(n+1)-1 \leq h \leq n-1$

Also, if T is proper, then T has the following properties:

1. $2h+1 \leq n \leq 2^{h+1}-1$
2. $h+1 \leq n_E \leq 2^h$
3. $h \leq n_I \leq 2^h-1$
4. $\log(n+1)-1 \leq h \leq (n-1)/2$



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- Relating Internal Nodes to External Nodes in a Proper Binary Tree
- Proposition: In a nonempty proper binary tree T , with n_E external nodes and n_I internal nodes, we have $n_E = n_I + 1$.

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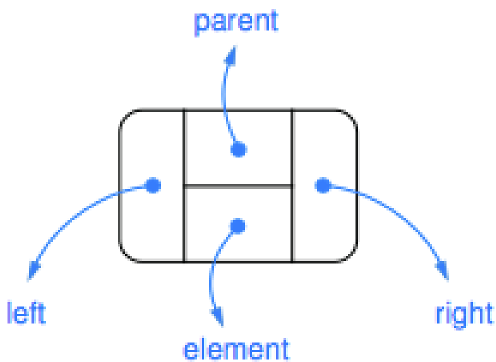
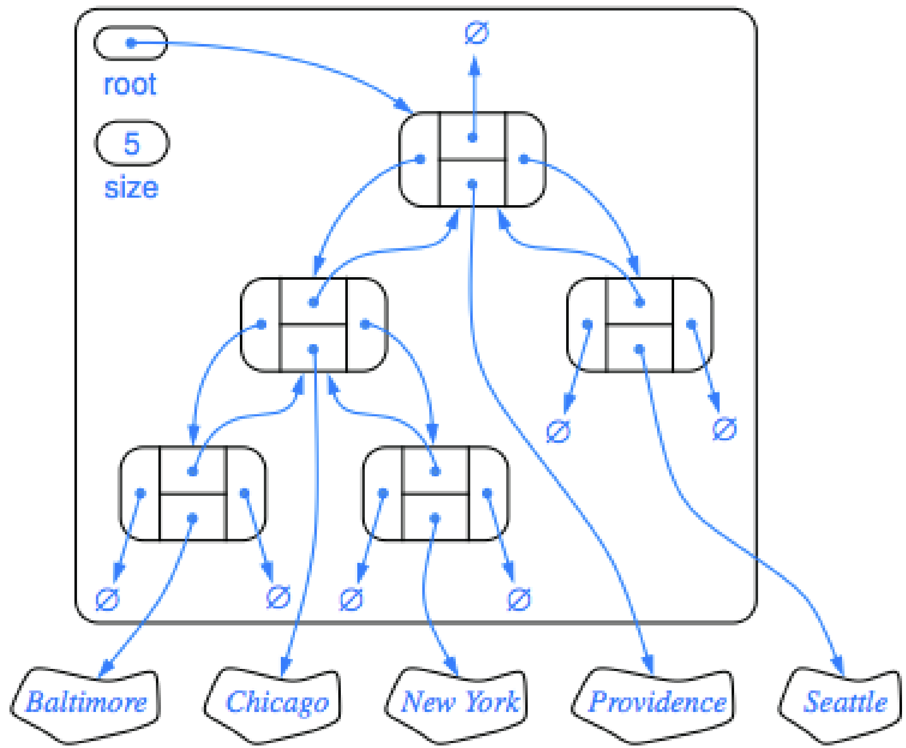
LINKED STRUCTURE FOR BINARY TREES

linked structure, with a node that maintains references to the element stored at a position *p* and to the nodes associated with the children and parent of *p*.



(a)



(b)

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OPERATIONS FOR UPDATING A LINKED BINARY TREE

- Means for changing the structure of content of a tree.
- Suggested update methods for a linked binary tree:
  - addRoot(*e*): Creates a root for an empty tree, storing *e* as the element, and returns the position of that root; an error occurs if the tree is not empty.
  - addLeft(*p*, *e*): Creates a left child of position *p*, storing element *e*, and returns the position of the new node; an error occurs if *p* already has a left child.
  - addRight(*p*, *e*): Creates a right child of position *p*, storing element *e*, and returns the position of the new node; an error occurs if *p* already has a right child.
  - set(*p*, *e*): Replaces the element stored at position *p* with element *e*, and returns the previously stored element.
  - attach(*p*, *T*1, *T*2): Attaches the internal structure of trees *T*1 and *T*2 as the respective left and right subtrees of leaf position *p* and resets *T*1 and *T*2 to empty trees; an error condition occurs if *p* is not a leaf.
  - remove(*p*): Removes the node at position *p*,replacing it with its child (if any), and returns the element that had been stored at *p*; an error occurs if *p* has two children.

Each can be implemented in $O(1)$ worst-case time with our linked representation

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JAVA IMPLEMENTATION OF A LINKED BINARY TREE STRUCTURE 1

```
/** Concrete implementation of a binary tree using a node-based, linked structure. */
public class LinkedBinaryTree<E> extends AbstractBinaryTree<E> {

  //---------------- nested Node class ----------------
  protected static class Node<E> implements Position<E> {
    private E element;                // an element stored at this node
    private Node<E> parent;           // a reference to the parent node (if any)
    private Node<E> left;             // a reference to the left child (if any)
    private Node<E> right;            // a reference to the right child (if any)
    /** Constructs a node with the given element and neighbors. */
    public Node(E e, Node<E> above, Node<E> leftChild, Node<E> rightChild) {
      element = e;
      parent = above;
      left = leftChild;
      right = rightChild;
    }
    // accessor methods
    public E getElement() { return element; }
    public Node<E> getParent() { return parent; }
    public Node<E> getLeft() { return left; }
    public Node<E> getRight() { return right; }
    // update methods
    public void setElement(E e) { element = e; }
    public void setParent(Node<E> parentNode) { parent = parentNode; }
    public void setLeft(Node<E> leftChild) { left = leftChild; }
    public void setRight(Node<E> rightChild) { right = rightChild; }
  } //----------- end of nested Node class -----------
```

nested Node class, which implements the Position interface.

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JAVA IMPLEMENTATION OF A LINKED BINARY TREE STRUCTURE 2

createNode: returns a new node instance. This ***factory method pattern*** allowing us to later subclass our tree in order to use a specialized node type. (Discussed latter)

```
/** Factory function to create a new node storing element e. */
protected Node<E> createNode(E e, Node<E> parent,
                                            Node<E> left, Node<E> right) {
  return new Node<E>(e, parent, left, right);
}

// LinkedBinaryTree instance variables
protected Node<E> root = null;        // root of the tree
private int size = 0;                 // number of nodes in the tree

// constructor
public LinkedBinaryTree() { }         // constructs an empty binary tree
```

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JAVA IMPLEMENTATION OF A LINKED BINARY TREE STRUCTURE 3

```
  // nonpublic utility
  /** Validates the position and returns it as a node. */
  protected Node<E> validate(Position<E> p) throws IllegalArgumentException {
    if (!(p instanceof Node))
      throw new IllegalArgumentException("Not valid position type");
    Node<E> node = (Node<E>) p;                 // safe cast
    if (node.getParent( ) == node)              // our convention for defunct node
      throw new IllegalArgumentException("p is no longer in the tree");
    return node;
  }

  // accessor methods (not already implemented in AbstractBinaryTree)
  /** Returns the number of nodes in the tree. */
  public int size( ) {
    return size;
  }

  /** Returns the root Position of the tree (or null if tree is empty). */
  public Position<E> root( ) {
    return root;
  }

```

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JAVA IMPLEMENTATION OF A LINKED BINARY TREE STRUCTURE 4

```
/** Returns the Position of p's parent (or null if p is root). */
public Position<E> parent(Position<E> p) throws IllegalArgumentException {
  Node<E> node = validate(p);
  return node.getParent( );
}

/** Returns the Position of p's left child (or null if no child exists). */
public Position<E> left(Position<E> p) throws IllegalArgumentException {
  Node<E> node = validate(p);
  return node.getLeft( );
}

/** Returns the Position of p's right child (or null if no child exists). */
public Position<E> right(Position<E> p) throws IllegalArgumentException {
  Node<E> node = validate(p);
  return node.getRight( );
}
```

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JAVA IMPLEMENTATION OF A LINKED BINARY TREE STRUCTURE 5

```
// update methods supported by this class
/** Places element e at the root of an empty tree and returns its new Position. */
public Position<E> addRoot(E e) throws IllegalStateException {
  if (!isEmpty()) throw new IllegalStateException("Tree is not empty");
  root = createNode(e, null, null, null);
  size = 1;
  return root;
}

/** Creates a new left child of Position p storing element e; returns its Position. */
public Position<E> addLeft(Position<E> p, E e)
                        throws IllegalArgumentException {
  Node<E> parent = validate(p);
  if (parent.getLeft() != null)
    throw new IllegalArgumentException("p already has a left child");
  Node<E> child = createNode(e, parent, null, null);
  parent.setLeft(child);
  size++;
  return child;
}
```

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JAVA IMPLEMENTATION OF A LINKED BINARY TREE STRUCTURE 6

```
/** Creates a new right child of Position p storing element e; returns its Position. */
public Position<E> addRight(Position<E> p, E e)
                          throws IllegalArgumentException {
  Node<E> parent = validate(p);
  if (parent.getRight( ) != null)
    throw new IllegalArgumentException("p already has a right child");
  Node<E> child = createNode(e, parent, null, null);
  parent.setRight(child);
  size++;
  return child;
}

/** Replaces the element at Position p with e and returns the replaced element. */
public E set(Position<E> p, E e) throws IllegalArgumentException {
  Node<E> node = validate(p);
  E temp = node.getElement( );
  node.setElement(e);
  return temp;
}
```

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JAVA IMPLEMENTATION OF A LINKED BINARY TREE STRUCTURE 7

```
/** Attaches trees t1 and t2 as left and right subtrees of external p. */
public void attach(Position<E> p, LinkedBinaryTree<E> t1,
                   LinkedBinaryTree<E> t2) throws IllegalArgumentException {
  Node<E> node = validate(p);
  if (isInternal(p)) throw new IllegalArgumentException("p must be a leaf");
  size += t1.size() + t2.size();
  if (!t1.isEmpty()) {                  // attach t1 as left subtree of node
    t1.root.setParent(node);
    node.setLeft(t1.root);
    t1.root = null;
    t1.size = 0;
  }
  if (!t2.isEmpty()) {                  // attach t2 as right subtree of node
    t2.root.setParent(node);
    node.setRight(t2.root);
    t2.root = null;
    t2.size = 0;
  }
}
```

JAVA IMPLEMENTATION OF A LINKED BINARY TREE STRUCTURE 8

```
  /** Removes the node at Position p and replaces it with its child, if any. */
  public E remove(Position<E> p) throws IllegalArgumentException {
    Node<E> node = validate(p);
    if (numChildren(p) == 2)
      throw new IllegalArgumentException("p has two children");
    Node<E> child = (node.getLeft( ) != null ? node.getLeft( ) : node.getRight( ) );
    if (child != null)
      child.setParent(node.getParent( ));    // child's grandparent becomes its parent
    if (node == root)
      root = child;                          // child becomes root
    else {
      Node<E> parent = node.getParent( );
      if (node == parent.getLeft( ))
        parent.setLeft(child);
      else
        parent.setRight(child);
    }
    size--;
    E temp = node.getElement( );
    node.setElement(null);                   // help garbage collection
    node.setLeft(null);
    node.setRight(null);
    node.setParent(node);                    // our convention for defunct node
    return temp;
  }
} //---------- end of LinkedBinaryTree class ----------
```

remove(p): intentionally sets the parent field of a deleted node to refer to itself, in accordance with our conventional representation of a defunct node (as detected within the validate method).

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PERFORMANCE OF THE LINKED BINARY TREE IMPLEMENTATION

Method	Running Time
size, isEmpty	$O(1)$
root, parent, left, right, sibling, children, numChildren	$O(1)$
isInternal, isExternal, isRoot	$O(1)$
addRoot, addLeft, addRight, set, attach, remove	$O(1)$
depth(p)	$O(d_p+1)$
height	$O(n)$

Running times for the methods of an n-node binary tree implemented with a linked structure. The space usage is $O(n)$. d_p is depth of node.

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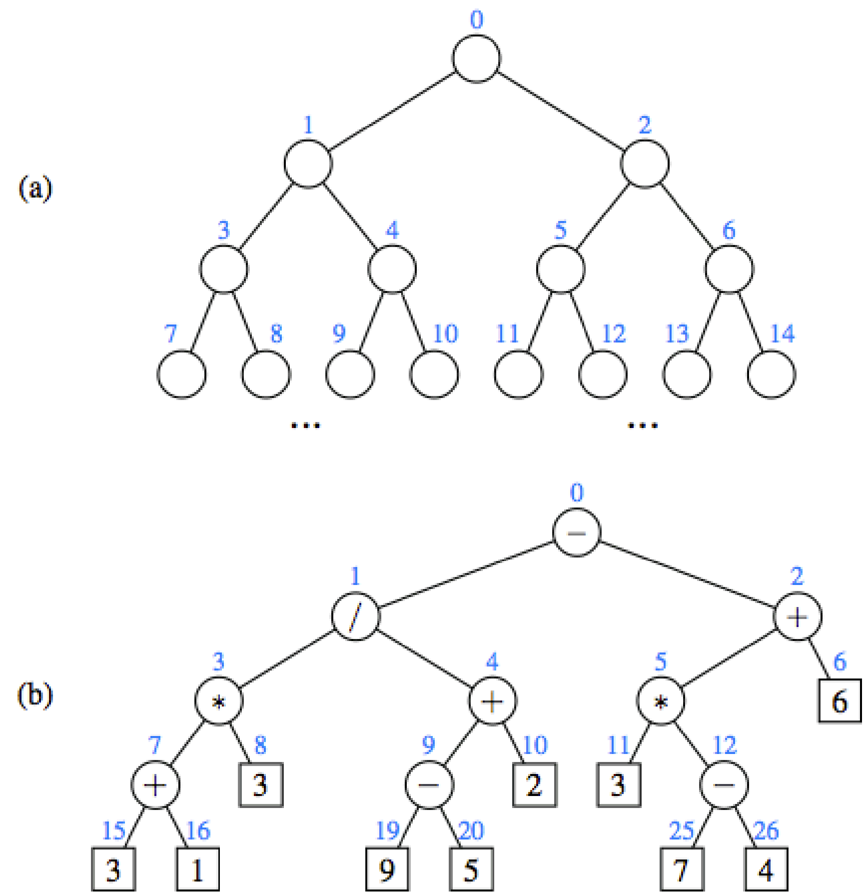
ARRAY-BASED REPRESENTATION OF A BINARY TREE

- Utilize the way of numbering the positions of T.
- For every position *p* of *T* , let *f(p)* be the of integer defined as follows.
  - If *p* is the root of *T*, then $f(p)=0$.
  - If *p* is the left child of position *q*, then $f(p) = 2f(q)+1$.
  - If *p* is the right child of position *q*, then $f(p) = 2f(q)+2$.
- *f* is known as ***level numbering*** of the positions in a binary tree *T* , for it numbers the positions on each level of *T* in increasing order from left to right.



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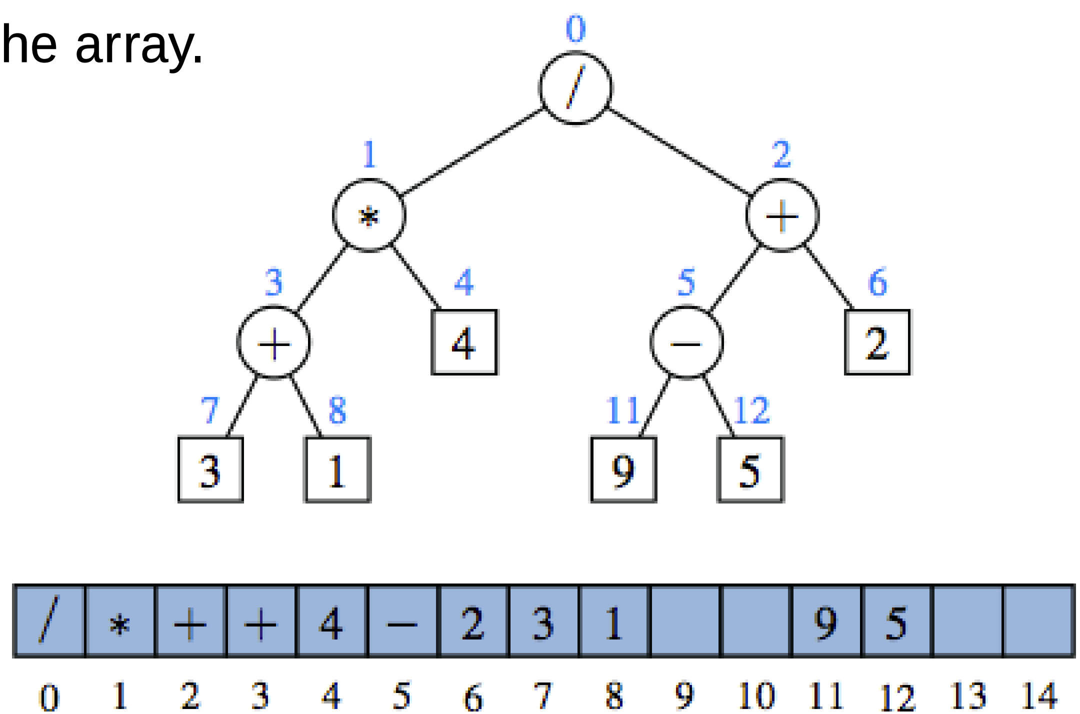
ARRAY-BASED REPRESENTATION OF A BINARY TREE 2

an array-based structure *A*, with the element at position *p* of *T* stored at index *f*(*p*) of the array.



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ARRAY-BASED REPRESENTATION OF A BINARY TREE 3

- Advantage:
  - position *p* can be represented by the single integer $f(p)$,
  - position-based methods such as root, parent, left, and right can be implemented using simple arithmetic operations on the number $f(p)$.
    - The left child of *p* has index $2f(p)+1$,
    - the right child of *p* has index $2f(p)+2$,
    - the parent of *p* has index $\lfloor (f(p) - 1)/2 \rfloor$.
- Disadvantage:
  - space usage of an array-based representation depends greatly on the shape of the tree;
    - worst case space usage: $N = 2^n - 1$, where n is the number of nodes in T
  - many update operations for trees cannot be efficiently supported.
    - EX> removing a node and promoting its child takes $O(n)$ time: the node and all it's descendants.

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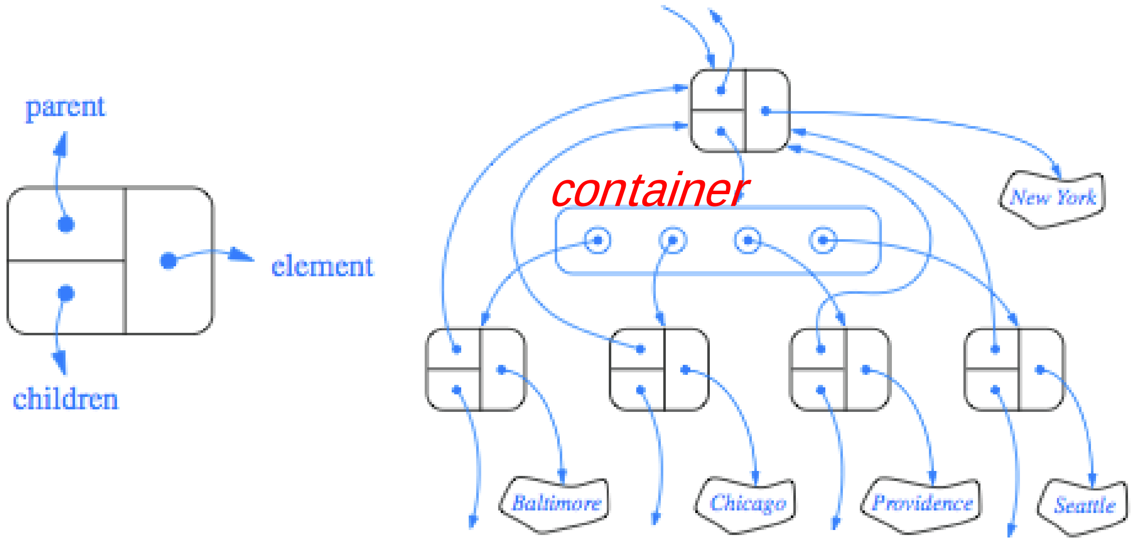
LINKED STRUCTURE FOR GENERAL TREES



(a) (b)

- General trees have no a priori limit on the number of children t hat a node may have
- each node store a single *container* of references to its children

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LINKED STRUCTURE FOR GENERAL TREES 2

Method	Running Time
size, isEmpty	$O(1)$
root, parent, isRoot, isInternal, isExternal	$O(1)$
numChildren(p)	$O(1)$
children(p)	$O(c_p + 1)$
depth(p)	$O(d_p + 1)$
height	$O(n)$

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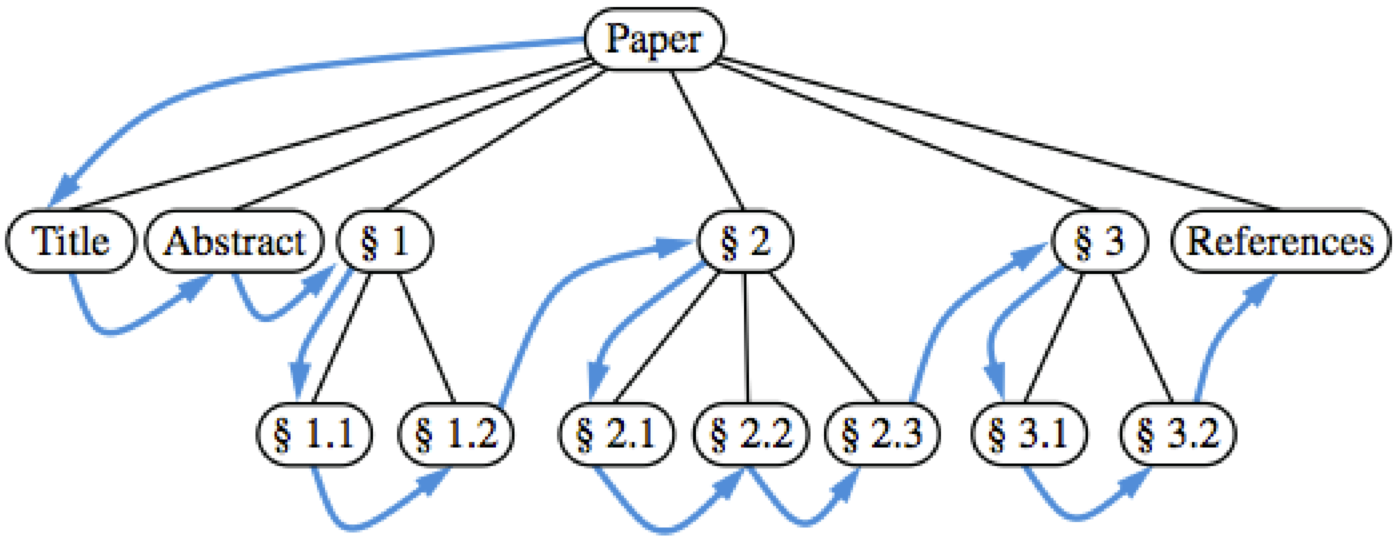
DEPTH-FIRST TREE TRAVERSAL 1: PREORDER TRAVERSAL

- A traversal visits the nodes of a tree in a systematic manner
- In a preorder traversal, a node is visited before its descendants
- Application: print a structured document

Algorithm *preOrder*(*v*)
  visit(*v*)
  for each child *w* of *v*
    preorder (*w*)



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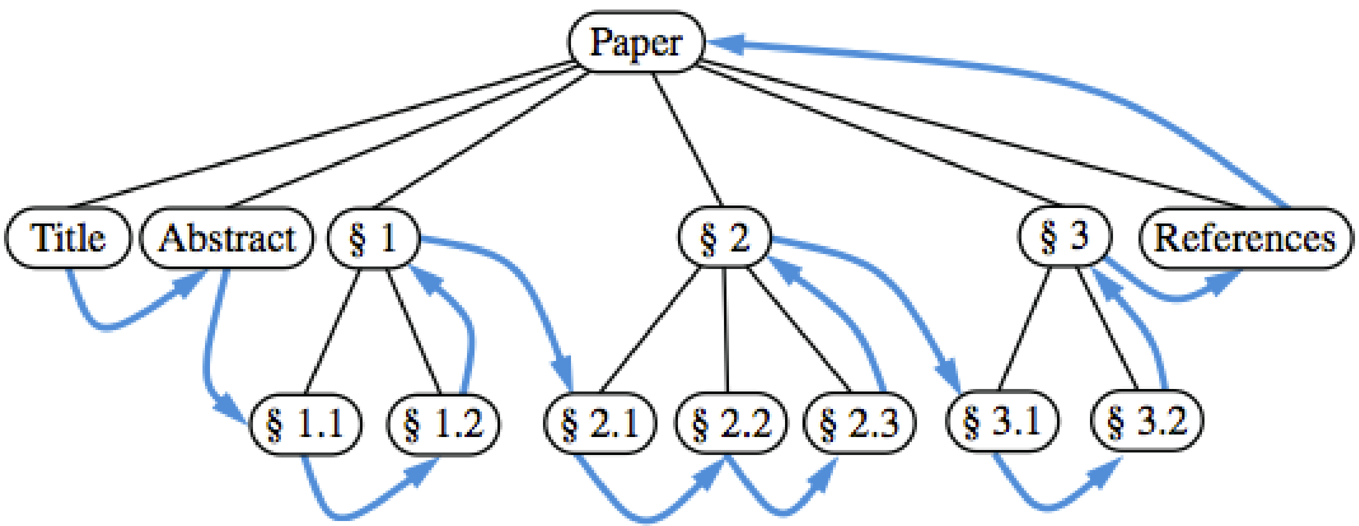
DEPTH-FIRST TREE TRAVERSAL 2: POSTORDER TRAVERSAL

- In a postorder traversal, a node is visited after its descendants
- Application: compute space used by files in a directory and its subdirectories

Algorithm ***postOrder***(*v*)
    for each child *w* of *v*
        postOrder (*w*)
    visit(*v*)



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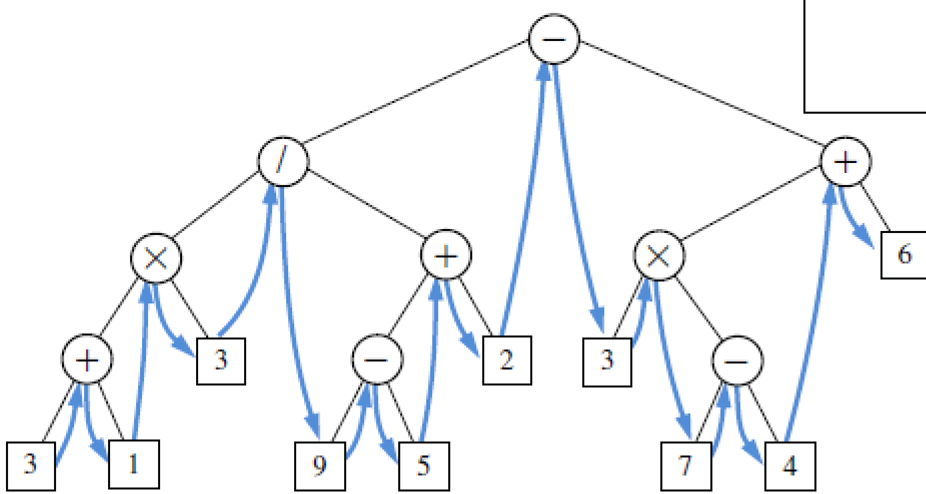
DEPTH-FIRST TREE TRAVERSAL 3: IN-ORDER (SYMMETRIC) SEARCH

- In an in-order traversal a node is visited after its left subtree and before its right subtree
- Application: draw a binary tree
  - x(v) = in-order rank of v
  - y(v) = depth of v

Algorithm *inOrder*(v)
    if *left* (v) $\neq$ **null**
        inOrder (*left* (v))
    visit(v)
    if *right*(v) $\neq$ **null**
        inOrder (*right* (v))



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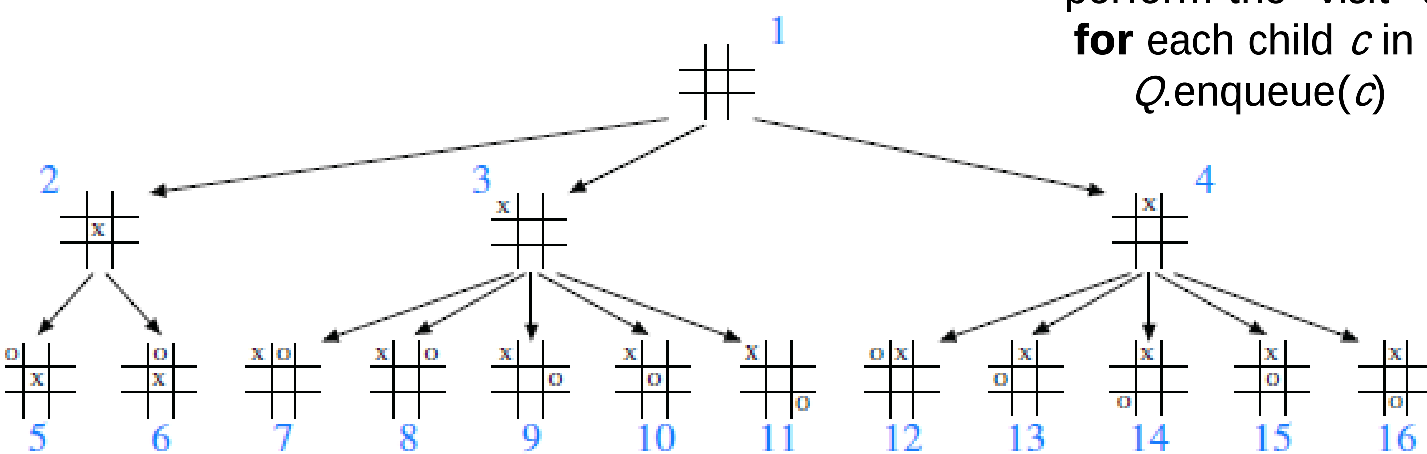
BREADTH-FIRST TREE TRAVERSAL

- *Breadth-first traversal*: traverses a tree so that we visit all the positions at depth *d* before we visit the positions at depth *d*+1.

Algorithm breadthfirst( ):
  Initialize queue *Q* to contain root( )
  while *Q* not empty **do**
    p = *Q*.dequeue( )
    perform the “visit” action for position *p*
    for each child *c* in children(*p*) **do**
      Q.enqueue(*c*)



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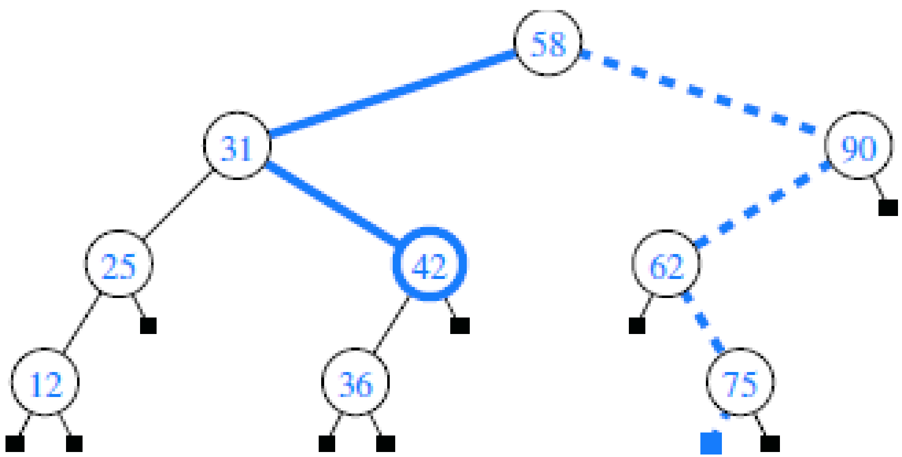
BINARY SEARCH TREES

- Let *S* be a set whose unique elements have an order relation. A **binary search tree** for *S* is a proper binary tree *T* such that, for each internal position *p* of *T*:
  - Position *p* stores an element of *S*, denoted as *e*(*p*).
  - Elements stored in the left subtree of *p* (if any) are less than *e*(*p*).
  - Elements stored in the right subtree of *p* (if any) are greater than *e*(*p*).



running time of searching in a binary search tree *T* is proportional to the height of *T*.

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IMPLEMENTING TREE TRAVERSALS IN JAVA

- We can implement the iterator( ) method by adapting an iteration produced by the positions( ) method.

```
//---------------- nested ElementIterator class ----------------
/* This class adapts the iteration produced by positions() to return elements. */
private class ElementIterator implements Iterator<E> {
  Iterator<Position<E>> posIterator = positions().iterator();
  public boolean hasNext() { return posIterator.hasNext(); }
  public E next() { return posIterator.next().getElement(); } // return element!
  public void remove() { posIterator.remove(); }
}

/** Returns an iterator of the elements stored in the tree. */
public Iterator<E> iterator() { return new ElementIterator(); }
```

- We first need to choose tree traversal algorithms to implement the positions( ) method.
- Ex> `public Iterable<Position<E>> positions() { return preorder(); }`

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IMPLEMENTING PREORDER TRAVERSALS IN JAVA

- private **preorderSubtree** method allows us to parameterize the recursive process with a specific position of the tree that serves as the root of a subtree to traverse

```
/** Adds positions of the subtree rooted at Position p to the given snapshot. */
private void preorderSubtree(Position<E> p, List<Position<E>> snapshot) {
  snapshot.add(p);          // for preorder, we add position p before exploring subtrees
  for (Position<E> c : children(p))
    preorderSubtree(c, snapshot);
}
```

- public **preorder** method: has the responsibility of creating an empty list for the snapshot buffer, and invoking the recursive method at the root of the tree

```
/** Returns an iterable collection of positions of the tree, reported in preorder. */
public Iterable<Position<E>> preorder() {
  List<Position<E>> snapshot = new ArrayList<>();
  if (!isEmpty())
    preorderSubtree(root(), snapshot);      // fill the snapshot recursively
  return snapshot;
}
```

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IMPLEMENTING POSTORDER TRAVERSALS IN JAVA

```
/** Adds positions of the subtree rooted at Position p to the given snapshot. */
private void postorderSubtree(Position<E> p, List<Position<E>> snapshot) {
  for (Position<E> c : children(p))
    postorderSubtree(c, snapshot);
  snapshot.add(p);        // for postorder, we add position p after exploring subtrees
}
/** Returns an iterable collection of positions of the tree, reported in postorder. */
public Iterable<Position<E>> postorder() {
  List<Position<E>> snapshot = new ArrayList<>();
  if (!isEmpty())
    postorderSubtree(root(), snapshot);       // fill the snapshot recursively
  return snapshot;
}
```

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IMPLEMENTING IN-ORDER TRAVERSALS IN JAVA

- The inorder traversal algorithm, because it explicitly relies on the notion of a left and right child of a node, only applies to binary trees. (define it in AbstractBinaryTree class.)

```
/** Adds positions of the subtree rooted at Position p to the given snapshot. */
private void inorderSubtree(Position<E> p, List<Position<E>> snapshot) {
  if (left(p) != null)
    inorderSubtree(left(p), snapshot);
  snapshot.add(p);
  if (right(p) != null)
    inorderSubtree(right(p), snapshot);
}
/** Returns an iterable collection of positions of the tree, reported in inorder. */
public Iterable<Position<E>> inorder() {
  List<Position<E>> snapshot = new ArrayList<>();
  if (!isEmpty())
    inorderSubtree(root(), snapshot);      // fill the snapshot recursively
  return snapshot;
}
/** Overrides positions to make inorder the default order for binary trees. */
public Iterable<Position<E>> positions() {
  return inorder();
}
```

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IMPLEMENTING BREADTHFIRST TRAVERSALS IN JAVA

```java
/** Returns an iterable collection of positions of the tree in breadth-first order. */
public Iterable<Position<E>> breadthfirst() {
  List<Position<E>> snapshot = new ArrayList<>();
  if (!isEmpty()) {
    Queue<Position<E>> fringe = new LinkedQueue<>();
    fringe.enqueue(root());                    // start with the root
    while (!fringe.isEmpty()) {
      Position<E> p = fringe.dequeue();        // remove from front of the queue
      snapshot.add(p);                         // report this position
      for (Position<E> c : children(p))
        fringe.enqueue(c);                     // add children to back of queue
    }
  }
  return snapshot;
}
```

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APPLICATIONS OF TREE TRAVERSALS: TABLE OF CONTENTS

- Preorder traversal of the tree can be used to produce a table of contents for the document

unindented version

```
for (Position<E> p : T.preorder())
  System.out.println(p.getElement());
```

indented version

```
/** Prints preorder representation of subtree of T rooted at p having depth d. */
public static <E> void printPreorderIndent(Tree<E> T, Position<E> p, int d) {
  System.out.println(spaces(2*d) + p.getElement());       // indent based on d
  for (Position<E> c : T.children(p))
    printPreorderIndent(T, c, d+1);                        // child depth is d+1
}
```

```
Paper
Title
Abstract
§1
§1.1
§1.2
§2
§2.1
...
```

unindented version

```
Paper
  Title
  Abstract
  §1
    §1.1
    §1.2
  §2
    §2.1
    ...
```

indented version

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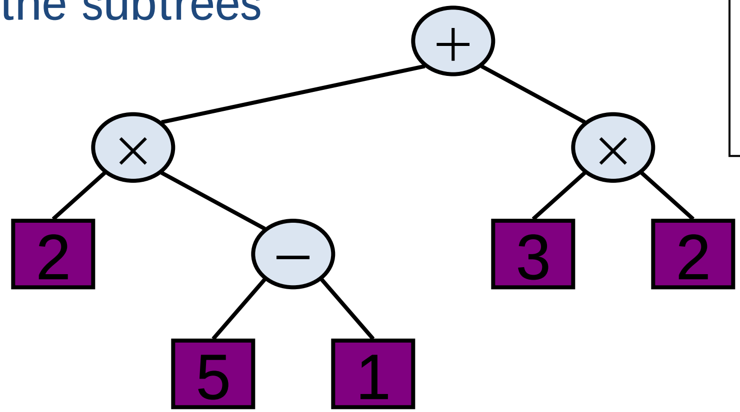
APPLICATIONS OF TREE TRAVERSALS: EVALUATE ARITHMETIC EXPRESSIONS

- Specialization of a post-order traversal
  - recursive method returning the value of a subtree
  - when visiting an internal node, combine the values of the subtrees

Algorithm *evalExpr*(*v*)
  if *isExternal* (*v*)
    return *v.element* ()
  else
    x ← *evalExpr*(*left*(*v*))
    y ← *evalExpr*(*right*(*v*))
    ◊ ← operator stored at ***v***
    return *x* ◊ *y*



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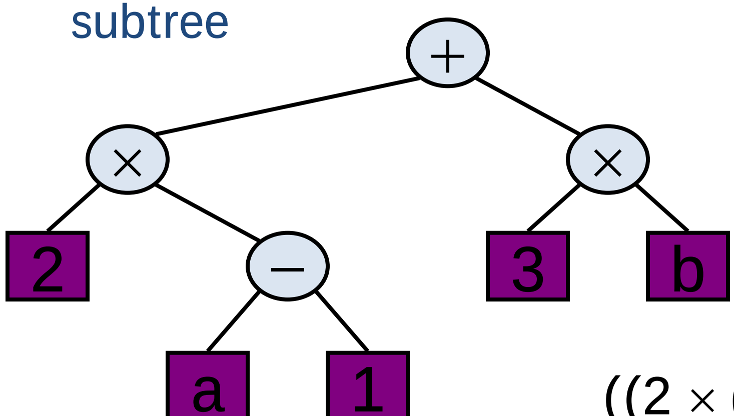
APPLICATIONS OF TREE TRAVERSALS: PRINT ARITHMETIC EXPRESSIONS

- Specialization of an in-order traversal
  - print operand or operator when visiting node
  - print "(" before traversing left subtree
  - print ")" after traversing right subtree



Algorithm *printExpression*(*v*)
  if *left* (*v*) ≠ **null**
    print("(")
    inOrder (*left*(*v*))
  print(*v.element* ())
  if *right*(*v*) ≠ **null**
    inOrder (*right*(*v*))
    print (")")

$((2 \times (a - 1)) + (3 \times b))$

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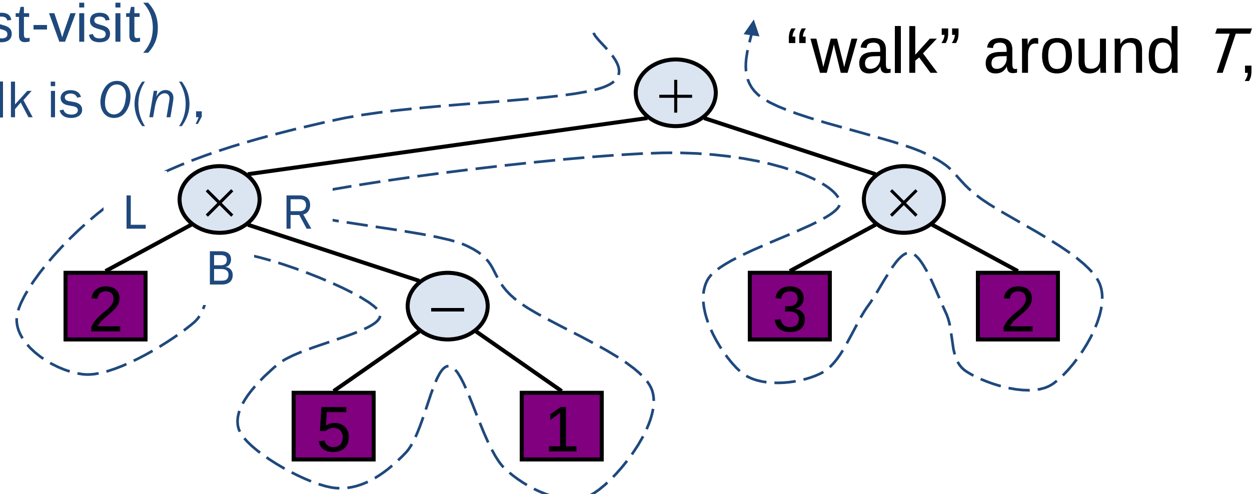
EULER TOUR TRAVERSAL

- *Euler tour traversal* are generic traversal of a tree
- Includes a special cases the preorder, postorder and inorder travers als
- Walk around the tree and visit each node three times:
  - on the left (pre-visit)
  - from below (in-visit)
  - on the right (post-visit)
- Complexity of the walk is $O(n)$,



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EULER TOUR TRAVERSAL CONT.

Algorithm eulerTour(T, p):
  perform the "pre visit" action for position p
  for each child c in T.children(p) **do**
    eulerTour(T, c)   { recursively tour the subtree rooted at c }
  perform the "post visit" action for position p

Algorithm eulerTourBinary(T, p):
  perform the "pre visit" action for position p
  if p has a left child lc **then**
    eulerTourBinary(T, lc)   { recursively tour the left subtree of p }
  perform the "in visit" action for position p
  if p has a right child rc **then**
    eulerTourBinary(T, rc)   { recursively tour the right subtree of p }
  perform the "post visit" action for position p