CSE 548 Fall 2012: Homework #2

Due October 15th in class.

First, two corrections to the lecture:

• During lecture, I mixed up the definition of “universal” hash functions and “2-universal” hash functions. I gave the definition for “universal”, but called it “2-universal”. I will only use the notion of “universal” hash functions in future homeworks/mid-terms/exams. My apologies for the error.

• In the cook-book version of the Chernoff bounds, I stated that if

\[ X = \sum_{i=1}^{n} X_i \]

where \( X_i \) are independent 0/1 random variables, and \( \mu = E[X] \), then

\[ X < \mu + O(\sqrt{\mu \log n}) \text{ w.h.p.} \]

However, the proof reveals an extra assumption I forgot to mention in class: this bound only applies when \( \mu = \Omega(\log n) \). For example, if \( X_i = 1 \) with probability 1/2 for all \( i \), then \( \mu = n/2 \), and the above statement holds. However, if \( \mu \) is very small, e.g. if \( \mu = 1 \), then the above statement may not hold. A more general bound is that, with high probability,

\[ X < \mu + \begin{cases} 
O\left(\sqrt{\mu \log n}\right) & \text{if } \mu = \Omega(\log n) \\
O\left(\frac{\log n}{\log \log n - \log \mu}\right) & \text{if } \mu = \Omega(1) \text{ and } \mu = o(\log n) 
\end{cases} \]

Problem 1

What does the (corrected) Chernoff bound given above become when \( \mu = 1 \)?

Problem 2

Use your result from problem 1 to prove, using Chernoff bounds, that after throwing \( n \) balls randomly into \( n \) bins, the fullest bin will contain \( O\left(\frac{\log n}{\log \log n}\right) \) balls w.h.p.

Problem 3

Use the Chernoff bounds to prove that, after flipping a fair coin \( n \) times, the number of heads will be less than 0.6n w.h.p.
Problem 4

Show that after throwing \(O(n \log n)\) balls into \(n\) bins, every bin has at least one ball w.h.p. (Hint: don’t use Chernoff bounds, just prove it directly).

Problem 5

How many times must we flip a coin to get \(\Omega(n)\) heads with high probability?

Problem 6

If I roll a 6-sided die, I expect to roll six times before I get a one. How many times must I roll to get a one with high probability?

What if it is a \(k\)-sided die; how many times must I roll to get a one with high probability?