Problem 1

Recall that for a function \( f : A \to B \) and \( X \subseteq A \) and \( Y \subseteq B \),
\[
f(X) = \{ y \in B | y = f(x), x \in X \}
\]
and
\[
f^{-1}(Y) = \{ x \in A | f(x) \in Y \}
\]
For each pair of sets listed below, indicated whether the first is always a subset of (\( \subseteq \)), strict subset of (\( \subset \)), superset of (\( \supseteq \)), strict superset of (\( \supset \)), equal to (\( = \)), or not equal to (\( \neq \)) the second subset, for each type of function. Make the strongest claim that you can.

<table>
<thead>
<tr>
<th>Set 1</th>
<th>Set 2</th>
<th>Any function</th>
<th>Injection</th>
<th>Surjection</th>
<th>Bijection</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(f^{-1}(Y)) )</td>
<td>( Y )</td>
<td>( X \subseteq Y )</td>
<td>( \subseteq )</td>
<td>( \subset )</td>
<td>( \supseteq )</td>
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<tr>
<td>( f^{-1}(f(X)) )</td>
<td>( X )</td>
<td>( X \subseteq f^{-1}(f(X)) )</td>
<td>( \subseteq )</td>
<td>( \subset )</td>
<td>( \supseteq )</td>
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<tr>
<td>( f(f^{-1}(f(X))) )</td>
<td>( f(X) )</td>
<td>( f(X) \subseteq f(f^{-1}(f(X))) )</td>
<td>( \subseteq )</td>
<td>( \subset )</td>
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</tr>
<tr>
<td>( f^{-1}(f(f^{-1}(Y))) )</td>
<td>( f^{-1}(Y) )</td>
<td>( f^{-1}(Y) \subseteq f^{-1}(f(f^{-1}(Y))) )</td>
<td>( \subseteq )</td>
<td>( \subset )</td>
<td>( \supseteq )</td>
</tr>
<tr>
<td>( f(X_1 \cup X_2) )</td>
<td>( f(X_1) \cup f(X_2) )</td>
<td>( f(X_1 \cup X_2) \subseteq (f(X_1) \cup f(X_2)) )</td>
<td>( \subseteq )</td>
<td>( \subset )</td>
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</table>

Problem 2

An Eulerian cycle is a path through a graph that crosses each edge exactly once. Suppose \( G \) is a connected graph. Prove that \( G \) has an Eulerian cycle if and only if every vertex of \( G \) has even degree.

Problem 3

Let \( k \) be a fixed positive integer. Put the following functions in increasing asymptotic order. Indicate which functions, if any, are asymptotically equivalent (i.e. \( \Theta() \)).

\[
\begin{align*}
f_1(n) & = (\log n)^{\log n} \\
f_2(n) & = 5^n \\
f_3(n) & = \sum_{i=0}^{k} n^i \\
f_4(n) & = n^{k+1} \\
f_5(n) & = (\log n)^{\log \log n} \\
f_6(n) & = 2^n \\
f_7(n) & = \sum_{i=0}^{n} i^k \\
f_8(n) & = \binom{n}{k}
\end{align*}
\]
Problem 4

As you have noticed, a proof of a theorem consists of a finite sequence of letters and symbols. Let $S$ be the set of all symbols that can occur in a proof, e.g. $S$ would contain all the English letters and punctuation, Greek letters, and various mathematical symbols. Argue that if some theorem has a proof, you can always find the proof eventually.

Problem 5

If $A[0, \ldots, n-1]$ is a permutation of $0, \ldots, n-1$, then we can view $A$ as a function $A : \{0, \ldots, n-1\} \rightarrow \{0, \ldots, n-1\}$. Write an algorithm that finds $B[0, \ldots, n-1]$ such that $A \circ B$ is the identity function.

Problem 6

Use the partition algorithm to write a function `find-kth-smallest(A, n, k)` that, given an array, $A$, of $n$ integers, returns the $k$th smallest integer in the array. In other words, your algorithm should return the integer that would be in position $k$ after the array is sorted. A trivial solution to this problem is

```plaintext
procedure find-kth-smallest(A, n, k)
    qsort(A, n)
    return A[k]
```

This algorithm would have running time $O(n \log n)$. Your algorithm should be faster.