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The Bounds of Min-Max Pair Heap Construction

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Abstract—In this paper, lower and upper bounds for min-max pair heap construction has been presented. It has been shown that the construction of a min-max pair heap with n elements requires at least $2.07n$ element comparisons. A new algorithm for creating min-max pair heap has been devised that lowers the upper bound to $2.43n$.

Keywords—Algorithm, Heap, Min-max pair heap, Min-max heap, Double-ended priority queue.

1. INTRODUCTION

The MinMax Heap structure, introduced by Strothotte [1], is a structure for the implementation of double-ended priority queue. It is based on heap structure under the notion of min-max ordering: values stored at nodes on even (odd) levels are smaller than or equal to (respectively, greater than) values stored at their descendants. This structure can be constructed in linear time. *FindMin, FindMax* operations are performed in constant time and *Insert*(x), *DeleteMin*, and *DeleteMax* in logarithmic time using this structure. A sublinear merging algorithm for this structure is given with the relaxation of strict ordering [2].

The min-max pair heap, introduced by Olariu *et al.* [3], has the advantage that his double-ended priority queue supports merging in sublinear time. Recently, Rahman *et al.* [4] have improved the algorithms for min-max pair heap. These improved algorithms for min-max pair heap outperform the original algorithms of Strothotte in all aspects excepting creation, for which the latter requires $2.15n$ comparisons [1], whereas the improved min-max pair heap creation requires $2.566n$ comparisons [4]. This paper will investigate further to obtain the bounds of min-max pair heap construction.

2. MIN-MAX PAIR HEAPS

Definition. *A min-max pair heap is a binary tree H featuring the heap-shape property, such that every node in $H[i]$ has two fields, called the min field and the max field, and such that H*

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has min-max ordering: for every i ($1 \le i \le n$), the value stored in the min field of $H[i]$ is the smallest of all values in the subtree of H rooted at $H[i]$; similarly the value stored in the max field of $H[i]$ is the largest key stored in the subtree of H rooted at $H[i]$.

However, we can consider those two heaps separately by taking min(max) elements. We name the min heap as A and max heap as B. Then, we can show their relationship by a Hasse diagram. For example, a min-max pair heap is shown in Figure 1. Its corresponding Hasse diagram is shown in Figure 2.

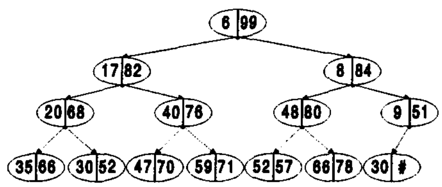


Figure 1. A sample min-max pair heap of height 3.

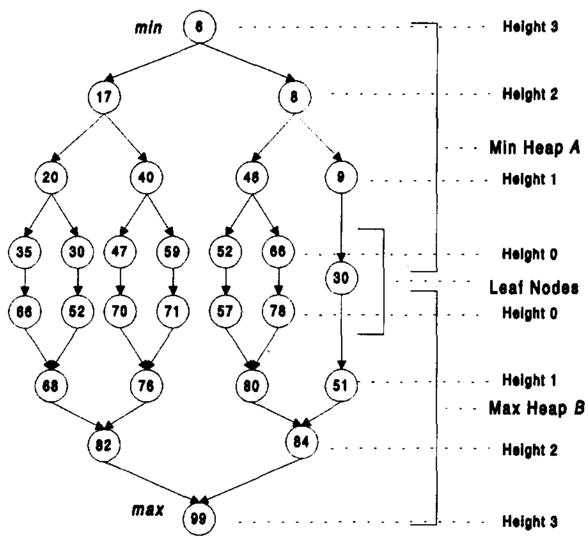


Figure 2. The Hasse diagram for min-max pair heap of Figure 1.

3. WORST CASE COMPLEXITIES OF MIN, MIN-MAX, AND MIN-MAX PAIR HEAPS

The known complexity for min heaps, min-max heaps, and min-max pair heaps is shown in Table 1 [1,4]. The results of [4] can be easily obtained by analyzing the Hasse diagram for min-max pair heap. In Table 1, the function $g(x)$ is defined as follows: $g(x) = 0$ for $x \le 1$ and $g(n) = g(\lceil \lg(n) \rceil) + 1$.

Table 1. Worst-case complexities for min-heaps, min-max heaps, and min-max pair heaps.

	Min-Heaps	Min-Max Heaps	Min-Max Pair Heaps
Create	$1.625n$	$2.15\ldots n$	$2.566\ldots n$
Insert	$\lg(\lg(n+1))$	$\lg(\lg(n+1))$	$\lg(\lg(n+2))$
DeleteMin	$\lg(n) + g(n)$	$1.5\lg(n) + \lg(\lg(n))$	$\lg(n+2) + \lg(\lg(n+2))$
DeleteMax	$0.5n + \lg(\lg(n))$	$1.5\lg(n) + \lg(\lg(n))$	$\lg(n+2) + \lg(\lg(n+2))$

4. LOWER BOUND ANALYSIS

First, we will calculate the information theoretic lower bound for the construction of min-max pair heaps.

THEOREM 1. *The number of comparisons necessary to construct a complete min-max pair heap of height h is at least $2.07286n$, where $n(=2^{h+2}-2)$ is the number of elements in the heap.*

PROOF. Let $N(h)$ be the number of distinct min-max pair heaps that can be constructed from $2^{h+2}-2$ elements. This min-max pair heap can be viewed as the minimum and maximum element connecting to two min-max pair heaps of height $h-1$ each. Hence,

$$N(h) = \binom{2^{h+2}-4}{2^{h+1}-2}[N(h-1)]^2 = \frac{(2^{h+2}-4)!}{(2^{h+1}-2)!\,(2^{h+1}-2)!}[N(h-1)]^2,$$

$$\Rightarrow \frac{N(h)}{(2^{h+2}-2)!} = \frac{1}{(2^{h+2}-2)\,(2^{h+2}-3)}\left\{\frac{N(h-1)}{(2^{h+1}-2)!}\right\}^2,$$

$$\Rightarrow \frac{(2^{h+2}-2)!}{N(h)} = \left(2^{h+2}-2\right)\left(2^{h+2}-3\right)\left\{\frac{(2^{h+1}-2)!}{N(h-1)}\right\}^2.$$

By the information-theoretic lower bound, we know that the minimum number of comparisons, on the average, needed to build a min-max pair heap of n elements is at least

$$\lg\left(\frac{(2^{h+2}-2)!}{N(h)}\right) = \lg\left(2^{h+2}-2\right) + \lg\left(2^{h+2}-3\right) + 2\lg\left(\frac{(2^{h+1}-2)!}{N(h-1)}\right).$$

Assuming

$$C(h) = \lg\left(\frac{(2^{h+2}-2)!}{N(h)}\right),$$

we get

$$C(h) = \lg\left(2^{h+2}-2\right) + \lg\left(2^{h+2}-3\right) + 2C(h-1) = \sum_{i=0}^{h} 2^i\left[\lg\left(2^{h+2-i}-2\right) + \lg\left(2^{h+2-i}-3\right)\right]$$

$$= \sum_{i=0}^{h-l} 2^i\left[\lg\left(2^{h+2-i}-2\right) + \lg\left(2^{h+2-i}-3\right)\right]$$

$$+ \sum_{i=h-l+1}^{h} 2^i\left[\lg\left(2^{h+2-i}-2\right) + \lg\left(2^{h+2-i}-3\right)\right], \qquad [h \geq l \geq 0],$$

$$\leq \sum_{i=0}^{h-l} 2^{i+1}(h+2-i) + \sum_{j=2}^{l+1}\frac{2^{h+2}}{2^j}\lg\left\{\left(2^j-2\right)\left(2^j-3\right)\right\}$$

$$= 2^{h+2}\left[\frac{l+3}{2^l} - \frac{h-4}{2^{h+1}} + \sum_{j=2}^{l+1}\frac{1}{2^j}\lg\left\{\left(2^j-2\right)\left(2^j-3\right)\right\}\right].$$

Hence,

$$\lim_{h\to\infty} C(h) \leq \lim_{h\to\infty}\frac{2^{h+2}}{2^{h+2}-2}\left[\frac{l+3}{2^l} - \frac{h-4}{2^{h+1}} + \sum_{j=2}^{l+1}\frac{1}{2^j}\lg\left\{\left(2^j-2\right)\left(2^j-3\right)\right\}\right]n$$

$$= \left[\frac{l+3}{2^l} + \sum_{j=2}^{l+1}\frac{1}{2^j}\lg\left\{\left(2^j-2\right)\left(2^j-3\right)\right\}\right]n.$$

Taking $l=10$, $\lim_{h\to\infty} C(h) \leq 2.07286n$. ∎

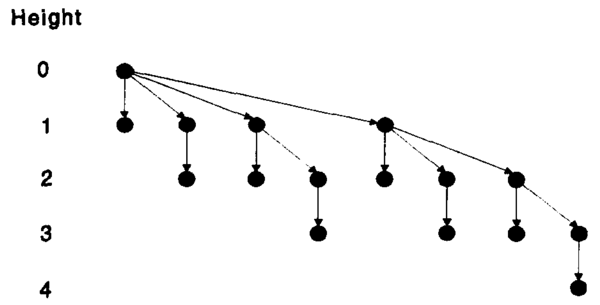


Figure 3. Binomial tree of height 4.

5. UPPER BOUND FOR MIN-MAX PAIR HEAP

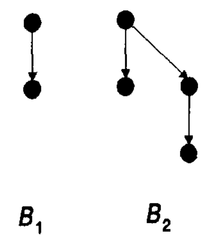
THEOREM 2. *A min-max pair heap of* $n(= 2^{h+2})$ *nodes can be constructed in at most* $2.4311n$ *comparisons in the worst case.*

PROOF. We describe an algorithm for the creation of min-max pair heap first and then analyze it to show that in the worst case it requires no more than $2.4311n$ comparisons.

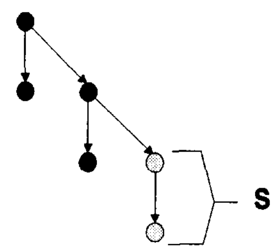
We start by constructing a binomial tree structure B_k of height k from n nodes. By R_k we denote the root of the binomial tree B_k. (See Figure 3.)

(1) Then, the smallest element is identified. It is the root of the whole structure.

(2) If the child of the root is labeled from $0 \ldots k-1$ from left to right, then the child i is the root of the subtree B_i.



(a) B_1 and B_2.



(b) The case when $R_1 < R_2$.

Figure 4.

(3) Figure 4a shows B_1 and B_2. Now, we compare R_1 with R_2 to get a $B_1 + B_2$ structure shown in Figure 5. If $R_1 \geq R_2$, then we make R_1 as a child of R_2 and easily get the $B_1 + B_2$ structure. If $R_1 < R_2$, then we obtain a structure shown in Figure 4b. Moving the subtree S to the root, we obtain the $B_1 + B_2$ structure.

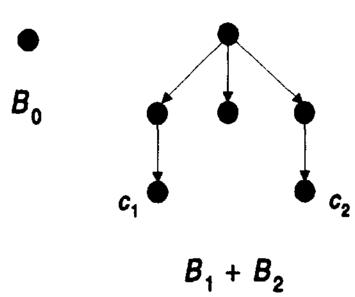


Figure 5.

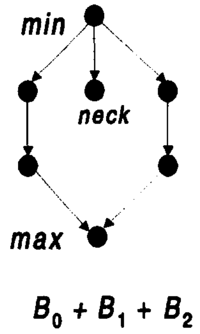


Figure 6.

(4) We next compare C_1 and C_2 in Figure 5 to obtain the chain of maximum sons. Then, we insert R_0 into that chain using binary insertion. Thus, we have now a min-max pair structure as is shown in Figure 6, with the exception of one node hanging from the root. We call this node the *neck* of the structure. Steps 3 and 4 require four comparisons.

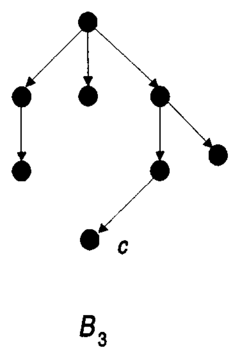


Figure 7. Redrawn B_3.

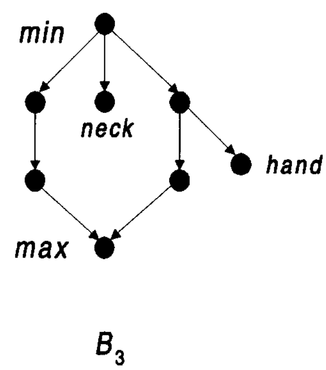


Figure 8. Min-max pair heap with *neck* and *hand*.

(5) For the subtree B_3, we have a structure that is redrawn in Figure 7. Then, we insert C into the leftmost three-element chain from root to produce a structure as shown in Figure 8. Here, we also have a *neck* and another node branching from the min-max pair heap. We call this node the *hand*. This step requires two comparisons.

(6) Now, we have two structures, one that is shown in Figure 6 and the other shown in Figure 8. In general, we need to construct a min-max pair heap with neck of height $h + 1$ from a min-max pair heap with neck of height h and a min-max pair heap with both neck and hand.

We compare the two *min* elements to find the element that will become the *min* element of the structure of height $h + 1$. Then, we find out the chain of minimum sons from the same subheap whose *min* element has been chosen. Thus, $h + 1$ comparisons are made. We insert *hand* into that chain excluding minimum since already we know that it is smaller than *hand*. This requires $\lceil \lg(2h + 2) \rceil$ comparisons.

Next, we compare the *max* elements, and the maximum becomes the *max* element of the structure of height $h + 1$. Then, we find out the chain of maximum sons in the chosen subheap. This requires $h + 1$ comparisons. Then, we will insert the *neck* of the chosen subheap into the chain excluding minimum. This requires $\lceil \lg(2h + 2) \rceil$ comparisons. The neck that has not been inserted is surely greater than the min element, and it becomes the neck of current structure. Thus, we obtain another min-max pair heap of height $h + 1$ having *neck*.

Thus, this step is performed recursively until we reach the root of the binomial tree and requires $2(h + 1) + 2\lceil \lg(2h + 2) \rceil$ comparisons at height h.

(7) Thus, we continue merging and reach to merge at the root of the binomial tree. Since the minimum is already there, no *min* element adjustment is required. We just perform *max* element adjustment. Then, the unused *neck* and *hand* become the *neck* and *hand* of the whole structure. Thus, at the root we obtain a min-max pair heap having *neck* and *hand*. This requires a total of $h + \lceil \lg(2h) \rceil$ comparisons.

(8) At last, we need to insert the *neck* and *hand* of the structure to the min-max pair heap. This step is performed only at the root and requires $2(h + 1) + 2\lceil \lg(2h + 2) \rceil$ comparisons. Thus, this step will not affect the order of the construction.

Now, to construct a min-max pair heap of height h, a binomial tree of height $h+2$ is constructed first. This step requires $2^{h+2} - 1$ comparisons. Let $C(h)$ be the additional cost to make a min-max pair heap from the B_{h+2}. We can easily construct a recurrence relation for $C(h)$ from the above description as

$$C(h) = 4 + \sum_{i=1}^{h-1} C(i) + \sum_{i=1}^{h-2} 2\{i + 1 + \lceil \lg(2i+2) \rceil\} + h + \lceil \lg(2h) \rceil.$$

Subtracting $C(h)$ from $C(h+1)$,

$$C(h+1) - C(h) = C(h) + 2h + 1 + 2\lceil \lg(2h) \rceil + \lceil \lg(2h+2) \rceil - \lceil \lg(2h) \rceil,$$
$$\Rightarrow C(h+1) = 2C(h) + 2h + 1 + \lceil \lg(2h) \rceil + \lceil \lg(2h+2) \rceil.$$

With boundary condition $C(1) = 2$, we have

$$C(h) = 2^h + \sum_{i=1}^{h-1} \left[2^{i-1}\{2(h-i) + 1 + \lceil \lg(2h-2i) \rceil + \lceil \lg(2h-2i+2) \rceil\} \right],$$
$$\Rightarrow C(h) = \frac{7}{2}2^h - 2h - 3 + \sum_{i=1}^{h-1} \left[2^{i-1}\{\lceil \lg(2h-2i) \rceil + \lceil \lg(2h-2i+2) \rceil\} \right],$$
$$\therefore \lim_{h \to \infty} C(h) = 2.4311n.$$

Thus, the theorem is proved. ∎

6. CONCLUDING REMARKS

We have analyzed the lower and upper bounds for min-max pair heap creation. We have given a constructive upper bound by presenting a new algorithm for its construction. This appears to be the first attempt to give bounds for the creation of min-max pair heaps. A gap between the information theoretic lower bound and the attained upper bound indicates the possibility of further improvement of the results.

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