CSE 638: Advanced Algorithms

Supplemental Material
(Akra-Bazzi Recurrences)

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Akra-Bazzi Recurrences

Consider the following recurrence:

$$T(x) = \begin{cases} \Theta(1), & \text{if } 1 \le x \le x_0, \\ \displaystyle\sum_{i=1}^{k} a_i T(b_i x) + g(x), & \text{if } x > x_0; \end{cases}$$

where,

1. $k \ge 1$ is an integer constant
2. $a_i > 0$ is a constant for $1 \le i \le k$
3. $b_i \in (0,1)$ is a constant for $1 \le i \le k$
4. $x \ge 1$ is a real number
5. $x_0 \ge \max\left\{\frac{1}{b_i}, \frac{1}{1-b_i}\right\}$ is a constant for $1 \le i \le k$
6. $g(x)$ is a nonnegative function that satisfies a *polynomial-growth condition* (to be specified soon)

Polynomial-Growth Condition

We say that $g(x)$ satisfies the *polynomial-growth condition* if there exist positive constants c_1 and c_2 such that for all $x \geq 1$, for all $1 \leq i \leq k$, and for all $u \in [b_i x, x]$,

$$c_1 g(x) \leq g(u) \leq c_2 g(x),$$

where x, k, b_i and $g(x)$ are as defined in the previous slide.

The Akra-Bazzi Solution

Consider the recurrence given in the previous two slides under the conditions specified there:

$$T(x) = \begin{cases} \Theta(1), & if\ 1 \leq x \leq x_0, \\ \displaystyle\sum_{i=1}^{k} a_i T(b_i x) + g(x), & if\ x > x_0. \end{cases}$$

Let p be the unique real number for which $\sum_{i=1}^{k} a_i b_i^p = 1$. Then

$$T(x) = \Theta\left(x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du\right)\right)$$

Examples of Akra-Bazzi Recurrences

Example 1: $T(x) = 2T\left(\frac{x}{4}\right) + 3T\left(\frac{x}{6}\right) + \Theta(x \log x)$

Then $p = 1$ and $T(x) = \Theta\left(x\left(1 + \int_1^x \frac{u \log u}{u^2} du\right)\right) = \Theta(x \log^2 x)$

Example 2: $T(x) = 2T\left(\frac{x}{2}\right) + \frac{8}{9}T\left(\frac{3x}{4}\right) + \Theta\left(\frac{x^2}{\log x}\right)$

Then $p = 2$ and $T(x) = \Theta\left(x^2\left(1 + \int_1^x \frac{u^2/\log u}{u^3} du\right)\right) = \Theta\left(\frac{x^2}{\log \log x}\right)$

Example 3: $T(x) = T\left(\frac{x}{2}\right) + \Theta(\log x)$

Then $p = 0$ and $T(x) = \Theta\left(1 + \int_1^x \frac{\log u}{u} du\right) = \Theta(\log^2 x)$

Example 4: $T(x) = \frac{1}{2}T\left(\frac{x}{2}\right) + \Theta\left(\frac{1}{x}\right)$

Then $p = -1$ and $T(x) = \Theta\left(\frac{1}{x}\left(1 + \int_1^x \frac{1}{u} du\right)\right) = \Theta\left(\frac{\log x}{x}\right)$

A Helping Lemma

Lemma: If $g(x)$ is a nonnegative function that satisfies the polynomial-growth condition, then there exist positive constants c_3 and c_4 such that for $1 \le i \le k$ and all $x \ge 1$,

$$c_3 g(x) \le x^p \int_{b_i x}^{x} \frac{g(u)}{u^{p+1}} du \le c_4 g(x).$$

Proof:

$$b_i x \le u \le x$$

$$\Rightarrow \frac{1}{\max\{(b_i x)^{p+1}, x^{p+1}\}} \le \frac{1}{u^{p+1}} \le \frac{1}{\min\{(b_i x)^{p+1}, x^{p+1}\}}$$

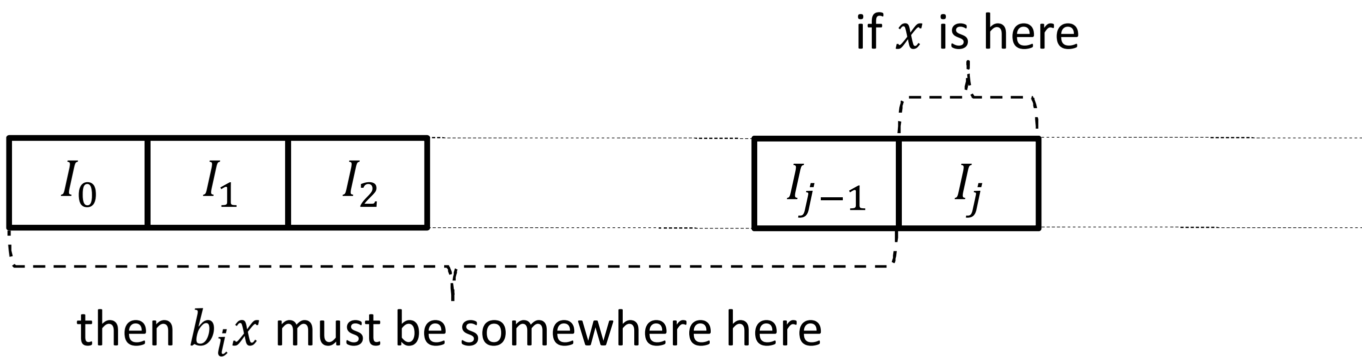
$$\Rightarrow \frac{x^p c_1 g(x)}{\max\{(b_i x)^{p+1}, x^{p+1}\}} \int_{b_i x}^{x} du \le x^p \int_{b_i x}^{x} \frac{g(u)}{u^{p+1}} du \le \frac{x^p c_2 g(x)}{\min\{(b_i x)^{p+1}, x^{p+1}\}} \int_{b_i x}^{x} du$$

$$\Rightarrow \frac{(1 - b_i) c_1}{\max\{1, b_i^{p+1}\}} g(x) \le x^p \int_{b_i x}^{x} \frac{g(u)}{u^{p+1}} du \le \frac{(1 - b_i) c_2}{\min\{1, b_i^{p+1}\}} g(x)$$

$$\Rightarrow c_3 g(x) \le x^p \int_{b_i x}^{x} \frac{g(u)}{u^{p+1}} du \le c_4 g(x)$$

Partitioning the Domain of x

Let $I_0 = [1, x_0]$ and $I_j = [x_0 + j - 1, x_0 + j]$ for $j \geq 1$.

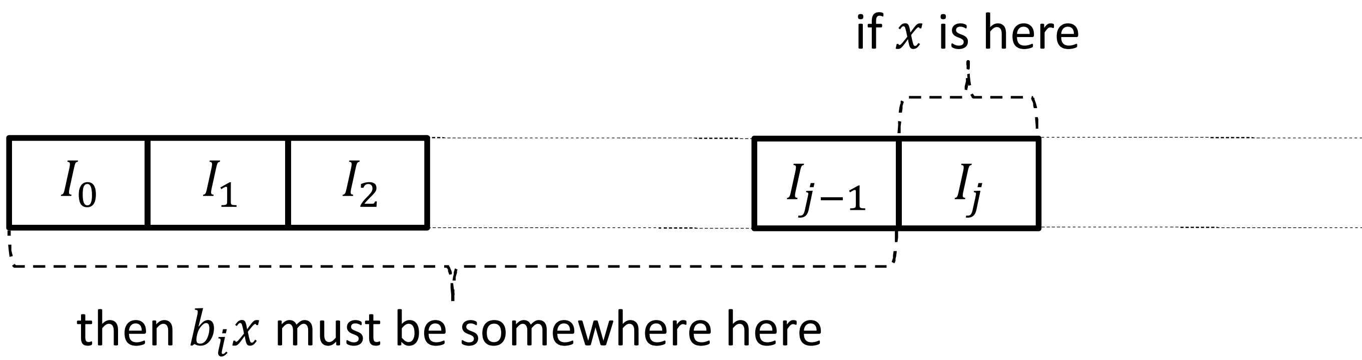


That allows us to use induction in the proof of:

$$T(x) = \begin{cases} \Theta(1), & if\ 1 \leq x \leq x_0, \\ \sum_{i=1}^{k} a_i T(b_i x) + g(x), & if\ x > x_0. \end{cases}$$

Partitioning the Domain of x

Let $I_0 = [1, x_0]$ and $I_j = [x_0 + j - 1, x_0 + j]$ for $j \geq 1$.



Proof:

$$x_0 + j - 1 < x \leq x_0 + j$$

$$\Rightarrow b_i(x_0 + j - 1) < b_i x \leq b_i(x_0 + j)$$

$$\Rightarrow b_i x_0 < b_i x \leq b_i x_0 + j$$

$$\Rightarrow 1 < b_i x \leq x_0 + j - (1 - b_i)x_0$$

$$\Rightarrow 1 < b_i x \leq x_0 + j - 1$$

Derivation of the Akra-Bazzi Solution

Lower Bound: There exists a constant $c_5 > 0$ such that for all $x > x_0$,

$$T(x) \geq c_5 x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du\right).$$

Proof: By induction on the interval I_j containing x.

Base case ($j = 0$) follows since $T(x) = \Theta(1)$ when $x \in I_0 = [1, x_0]$.

Induction: $T(x) = \sum_{i=1}^{k} a_i T(b_i x) + g(x) \geq \sum_{i=1}^{k} a_i c_5 (b_i x)^p \left(1 + \int_1^{b_i x} \frac{g(u)}{u^{p+1}} du\right) + g(x)$

$$= c_5 x^p \sum_{i=1}^{k} a_i b_i^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du - \int_{b_i x}^x \frac{g(u)}{u^{p+1}} du\right) + g(x)$$

$$\geq c_5 x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du - \frac{c_4}{x^p} g(x)\right) \sum_{i=1}^{k} a_i b_i^p + g(x)$$

$$= c_5 x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du\right) + (1 - c_4 c_5) g(x) \geq c_5 x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du\right)$$

(assuming $c_4 c_5 \leq 1$)

Derivation of the Akra-Bazzi Solution

Upper Bound: There exists a constant $c_6 > 0$ such that for all $x > x_0$,

$$T(x) \leq c_6 x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du\right).$$

Proof: Similar to the lower bound proof.