CSE 548: Analysis of Algorithms

Lecture 2
(Divide-and-Conquer Algorithms: Integer Multiplication)

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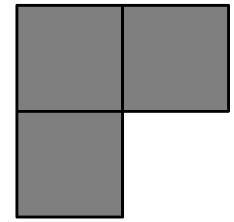
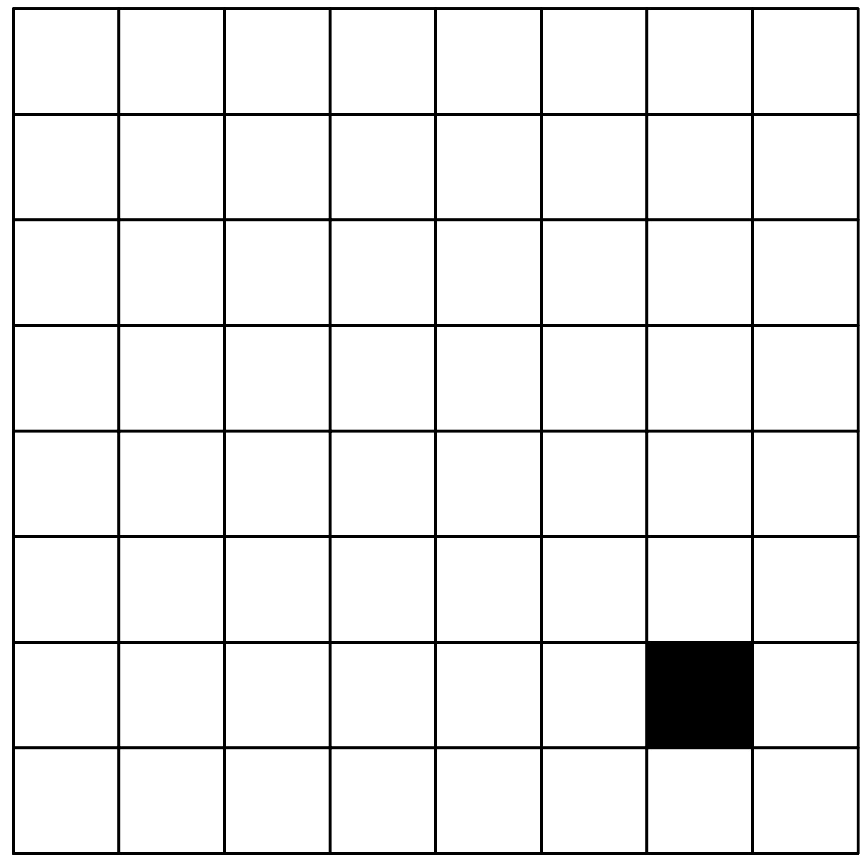
Tromino Cover

A *right tromino* is an L-shaped tile formed by three adjacent squares.



Puzzle: You are given a $2^n \times 2^n$ board with one missing square.

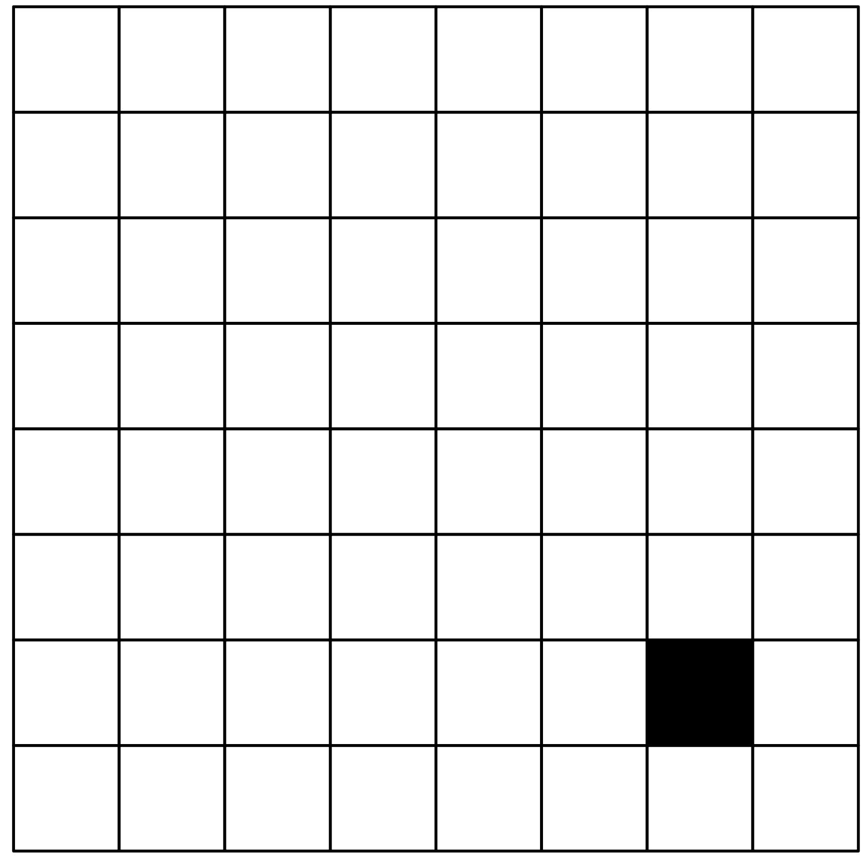
- you must cover all squares except the missing one exactly using right trominoes
- the trominoes must not overlap



$2^3 \times 2^3$ board

Tromino Cover

Steps

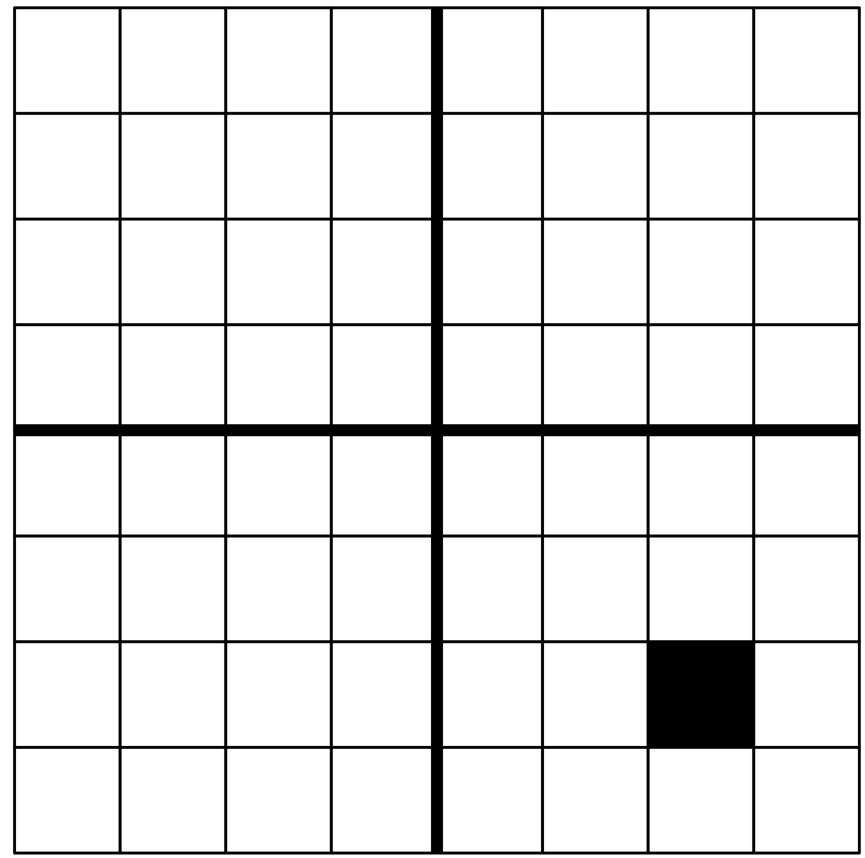


$2^3 \times 2^3$ board

Tromino Cover

Steps

– Divide the $2^n \times 2^n$ board into 4 disjoint $2^{n-1} \times 2^{n-1}$ subboards.

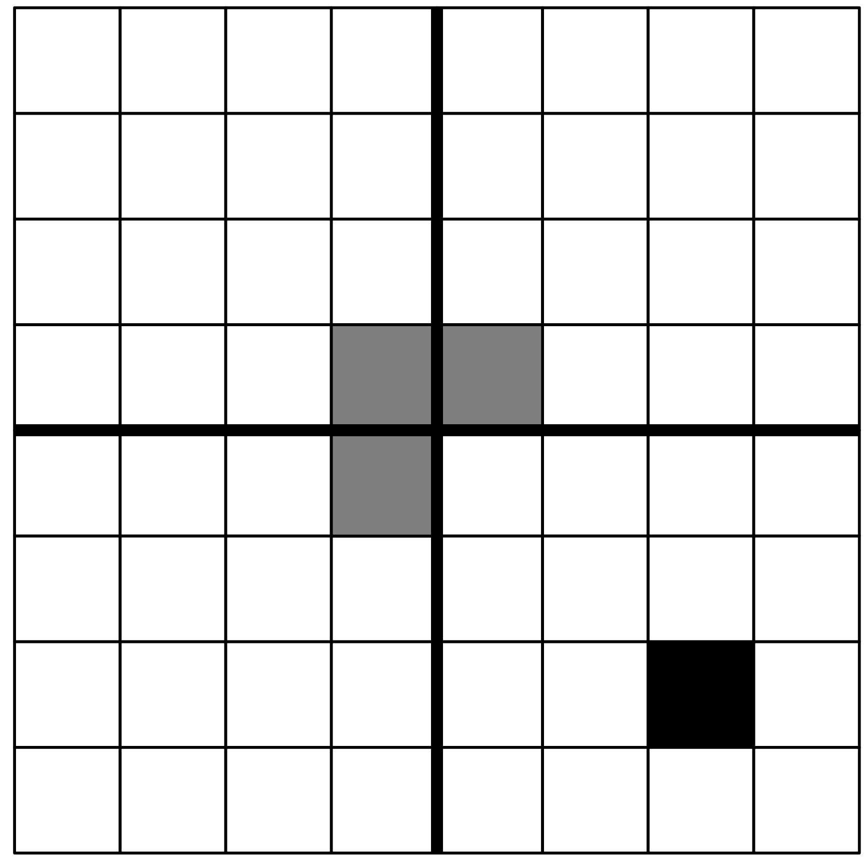


$2^3 \times 2^3$ board

Tromino Cover

Steps

- Divide the $2^n \times 2^n$ board into 4 disjoint $2^{n-1} \times 2^{n-1}$ subboards.
- Place a tromino at the center so that it fully covers one square from each of the three (3) subboards with no missing square, and misses the fourth subboard completely.



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Tromino Cover

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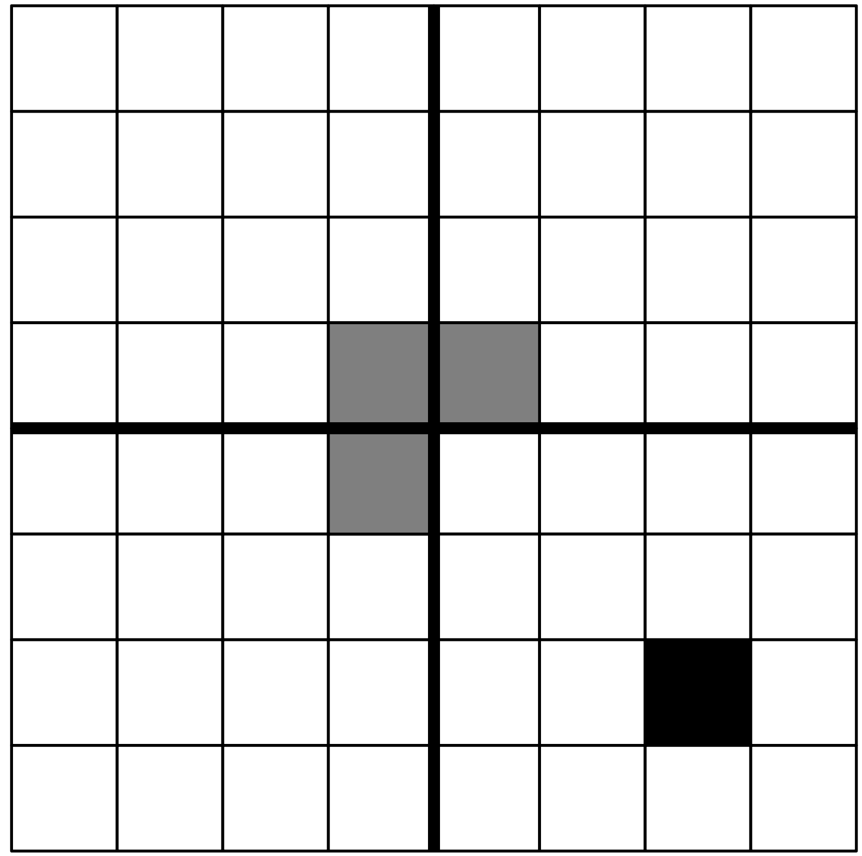
This reduces the original problem into 4 smaller instances of the same problem!



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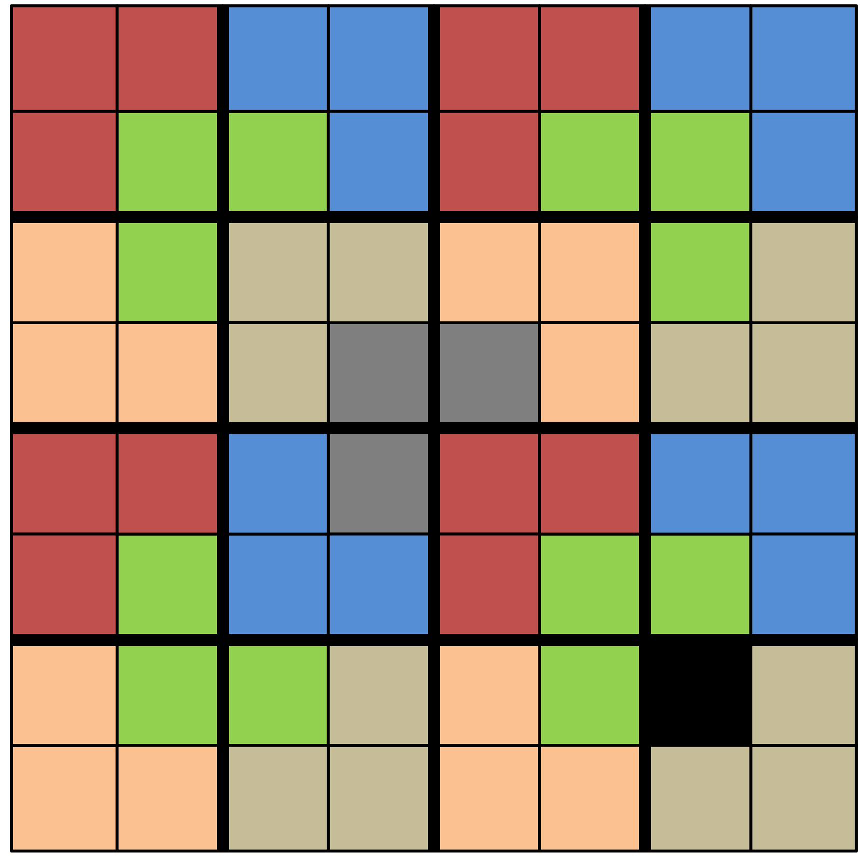
  This reduces the original problem into 4 smaller instances of the same problem!
- Solve each smaller subproblem recursively using the same technique.



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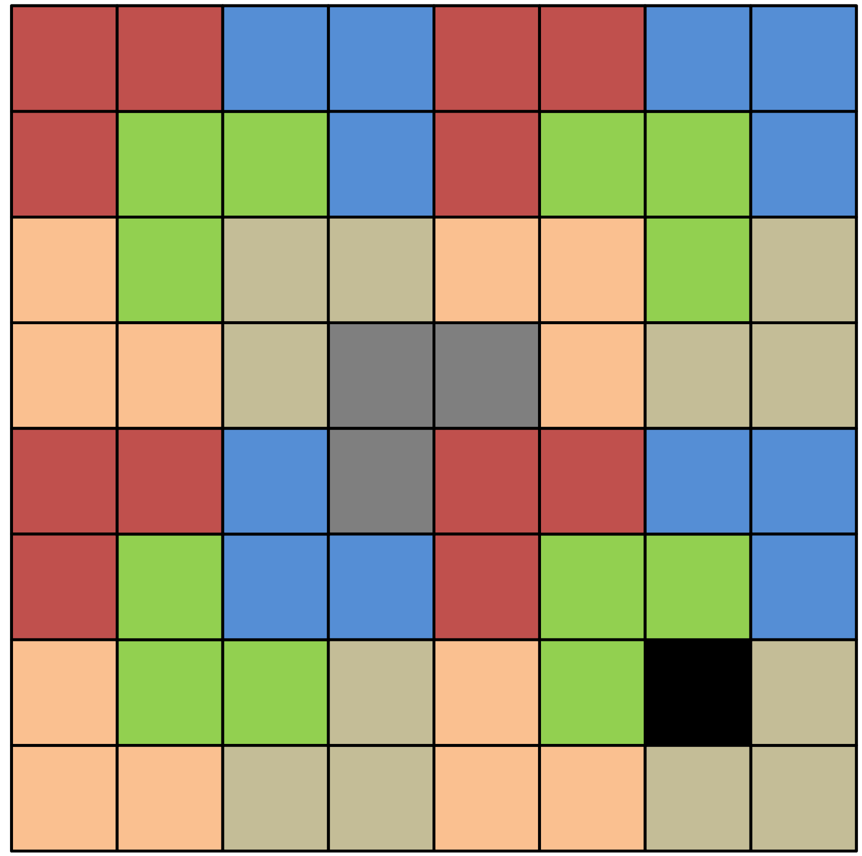

$2^3 \times 2^3$ board

– Solve each smaller subproblem recursively using the same technique.

– This algorithm design technique is called *recursive divide & conquer.*

A Latin Phrase

"Divide et impera"

(meaning: "divide and rule" or "divide and conquer")

— Philip II, king of Macedon (382-336 BC),
describing his policy toward the Greek city-states
(some say the Roman emperor Julius Caesar,
100-44 BC, is the source of this phrase)

The strategy is to break large power alliances into smaller ones that are easier to manage (or subdue).

This is a combination of political, military and economic strategy of gaining and maintaining power.

Unsurprisingly, this is also a very powerful problem solving strategy in computer science.

Divide-and-Conquer

1. **Divide:** divide the original problem into smaller subproblems that are easier are to solve
2. **Conquer:** solve the smaller subproblems (perhaps recursively)
3. **Merge:** combine the solutions to the smaller subproblems to obtain a solution for the original problem

Integer Multiplication

Multiplying Two n-bit Numbers

$$x = \underbrace{\overbrace{\boxed{\;x_L\;}}^{\frac{n}{2}\,bits}\overbrace{\boxed{\;x_R\;}}^{\frac{n}{2}\,bits}}_{n\ bits} = 2^{n/2}x_L + x_R$$

$$y = \boxed{\;y_L\;}\boxed{\;y_R\;} = 2^{n/2}y_L + y_R$$

$$xy = \left(2^{n/2}x_L + x_R\right)\left(2^{n/2}y_L + y_R\right) = 2^n x_L y_L + 2^{n/2}(x_L y_R + x_R y_L) + x_R y_R$$

So # $\frac{n}{2}$-bit products: 4

bit shifts (by n or $\frac{n}{2}$ bits): 2

additions (at most $2n$ bits long) : 3

We can compute the $\frac{n}{2}$-bit products recursively.

Let $T(n)$ be the overall running time for n-bit inputs. Then

$$T(n) = \begin{cases} \Theta(1) & if\ n = 1, \\ 4T\left(\frac{n}{2}\right) + \Theta(n) & otherwise. \end{cases} = \Theta\left(n^2\right) \quad \text{(how? derive)}$$

Multiplying Two n-bit Numbers Faster (Karatsuba's Algorithm)

$$\begin{array}{rcc l}
 & \overbrace{\hphantom{x_L x_L x_L}}^{\frac{n}{2}\,bits} & \overbrace{\hphantom{x_R x_R x_R}}^{\frac{n}{2}\,bits} & \\
x = & \boxed{\quad x_L \quad} & \boxed{\quad x_R \quad} & = 2^{n/2}x_L + x_R \\
y = & \boxed{\quad y_L \quad} & \boxed{\quad y_R \quad} & = 2^{n/2}y_L + y_R \\
 & \underbrace{\hphantom{y_L y_L y_L y_L y_L y_L y_L}}_{n\ bits} & &
\end{array}$$

$$xy = \left(2^{n/2}x_L + x_R\right)\left(2^{n/2}y_L + y_R\right)$$

$$= 2^n x_L y_L + 2^{n/2}(x_L y_R + x_R y_L) + x_R y_R$$

$$= 2^n x_L y_L + 2^{n/2}\left((x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R\right) + x_R y_R$$

So # $\frac{n}{2}$- or $\left(\frac{n}{2} + 1\right)$ -bit products: 3

Then the overall running time for n-bit inputs:

$$T(n) = \begin{cases} \Theta(1) & if\ n = 1, \\ 3T\left(\frac{n}{2}\right) + \Theta(n) & otherwise. \end{cases}$$

$$= \Theta\left(n^{\log_2 3}\right) = O\left(n^{1.59}\right)$$ (how? derive)

Algorithms for Multiplying Two *n*-bit Numbers

Inventor	Year	Complexity
Classical	—	$\Theta(n^2)$
Anatolii Karatsuba	1960	$\Theta(n^{\log_2 3})$
Andrei Toom & Stephen Cook (generalization of Karatsuba's algorithm)	1963 – 66	$\Theta\left(n2^{\sqrt{2\log_2 n}}\log n\right)$
Arnold Schönhage & Volker Strassen (Fast Fourier Transform)	1971	$\Theta(n \log n \log\log n)$
Martin Fürer (Fast Fourier Transform)	2005	$n \log n \, 2^{O(\log^* n)}$

Lower bound: $\Omega(n)$ (why?)