CSE 548: Analysis of Algorithms

Lecture 12
(Analyzing Parallel Algorithms)

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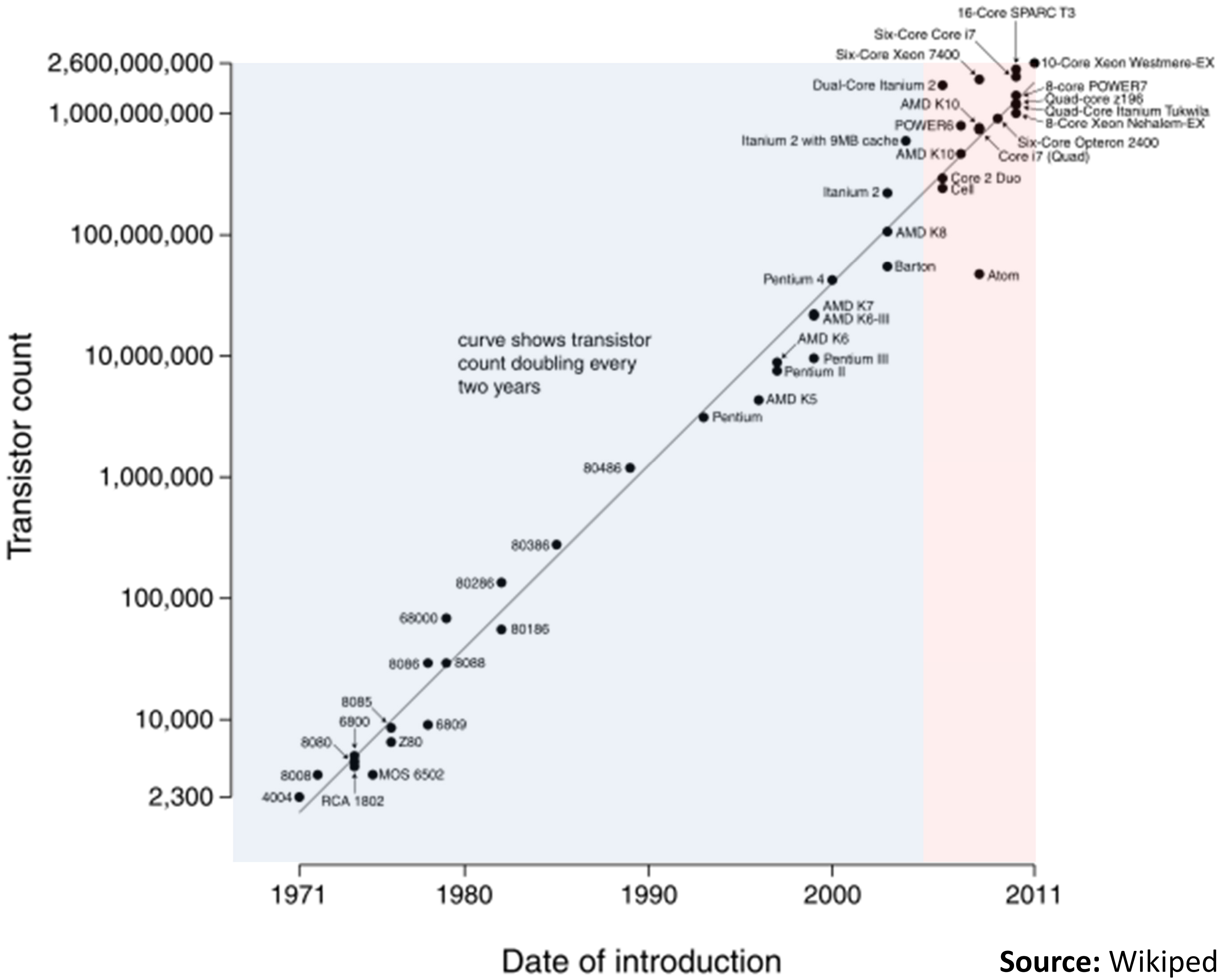
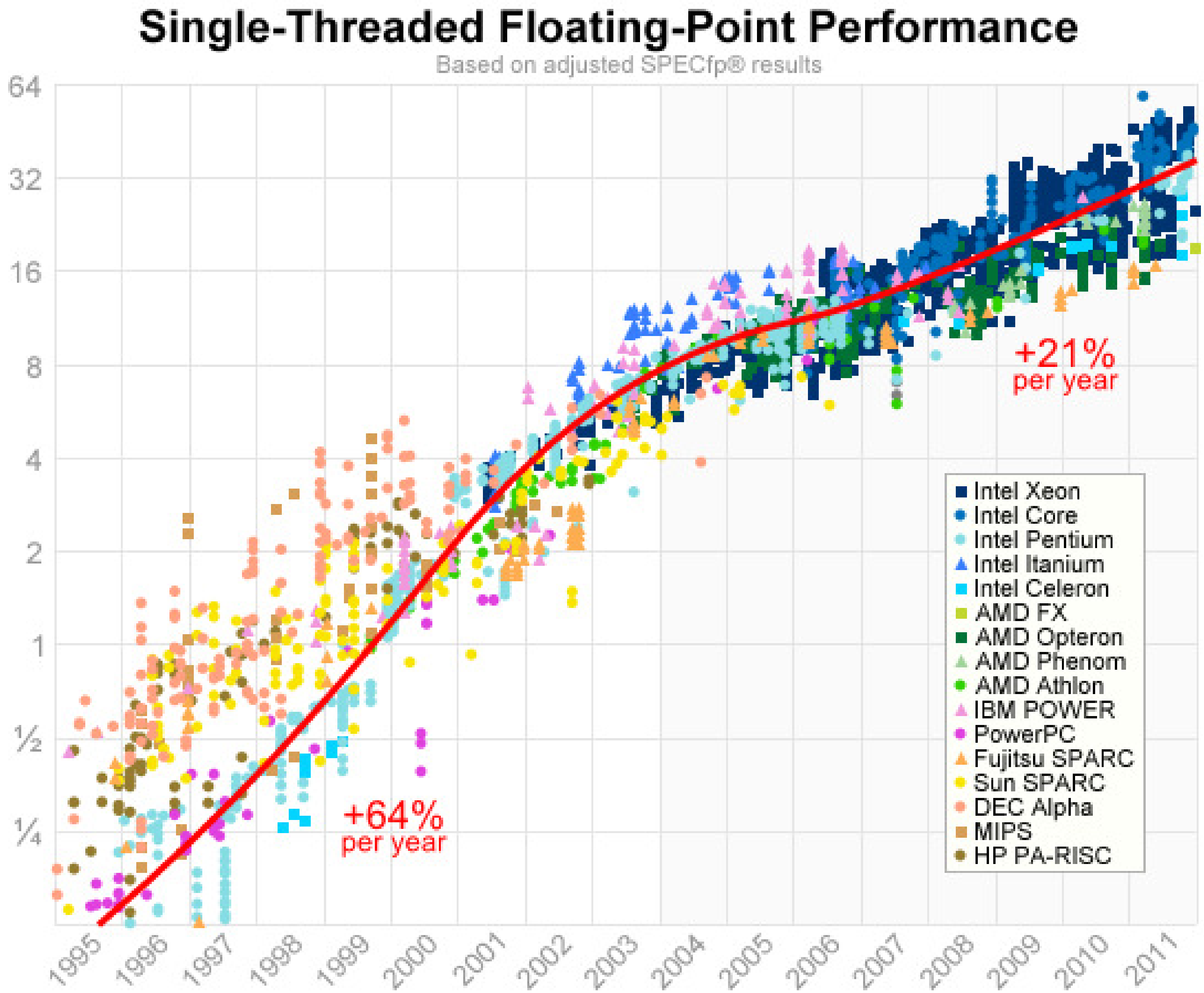
Why Parallelism?

Moore's Law



Source: Wikipedia

Unicore Performance



Source: Jeff Preshing, 2012, http://preshing.com/20120208/a-look-back-at-single-threaded-cpu-performance/

Unicore Performance Has Hit a Wall!

Some Reasons

- Lack of additional ILP
  (Instruction Level Hidden Parallelism)
- High power density
- Manufacturing issues
- Physical limits
- Memory speed

Unicore Performance: No Additional ILP

"Everything that can be invented has been invented."

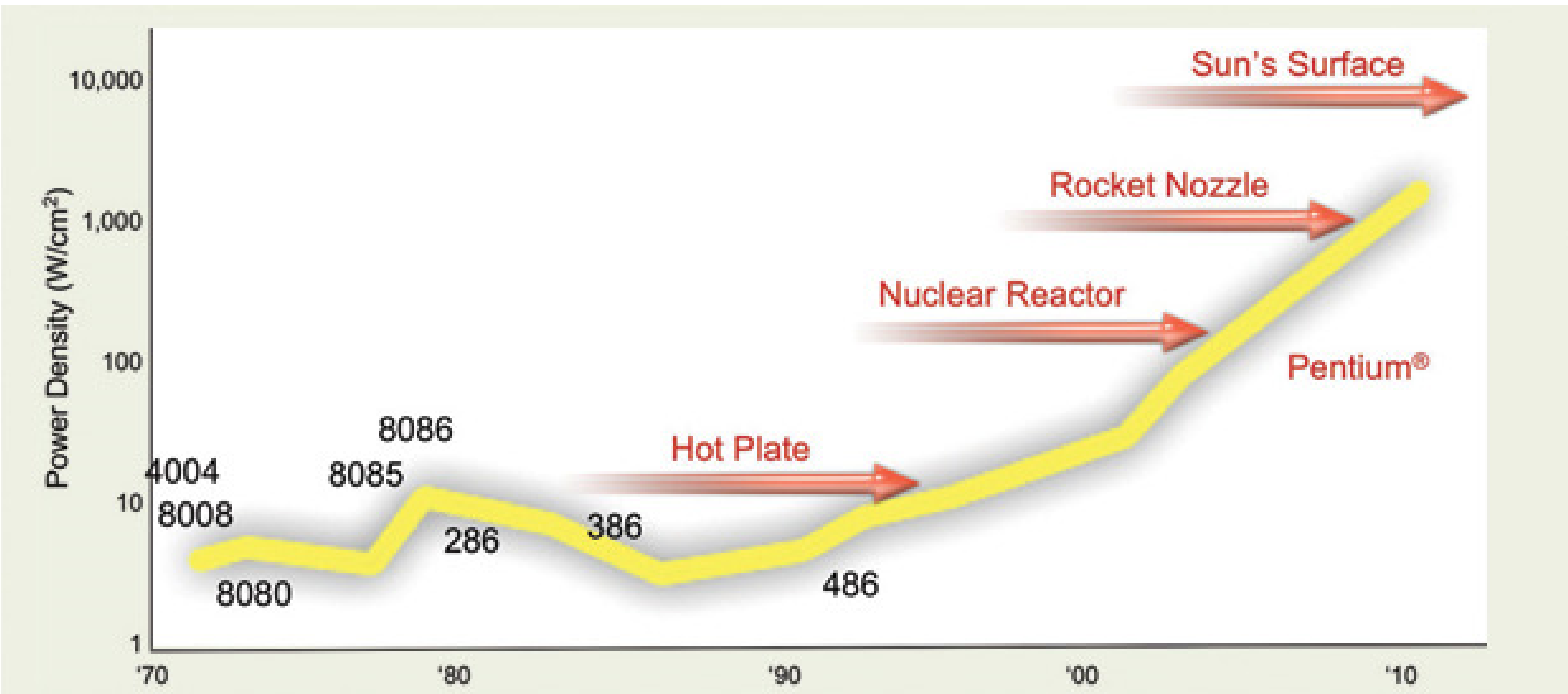
— Charles H. Duell
Commissioner, U.S. patent office, 1899

Exhausted all ideas to exploit hidden parallelism?

- Multiple simultaneous instructions
- Instruction Pipelining
- Out-of-order instructions
- Speculative execution
- Branch prediction
- Register renaming, etc.

Unicore Performance: High Power Density

- Dynamic power, $P_d \propto V^2 f C$
    - *V = supply voltage*
    - *f = clock frequency*
    - *C = capacitance*
- But $V \propto f$
- Thus $P_d \propto f^3$



Source: Patrick Gelsinger, Intel Developer Forum, Spring 2004 (Simon Floyd)

Unicore Performance: Manufacturing Issues

- Frequency, $f \propto 1 / s$
  - *s = feature size (transistor dimension)*
- Transistors / unit area $\propto 1 / s^2$
- Typically, die size $\propto 1 / s$
- So, what happens if feature size goes down by a factor of x?
  - Raw computing power goes up by a factor of x^4 !
  - Typically most programs run faster by a factor of x^3 without any change!

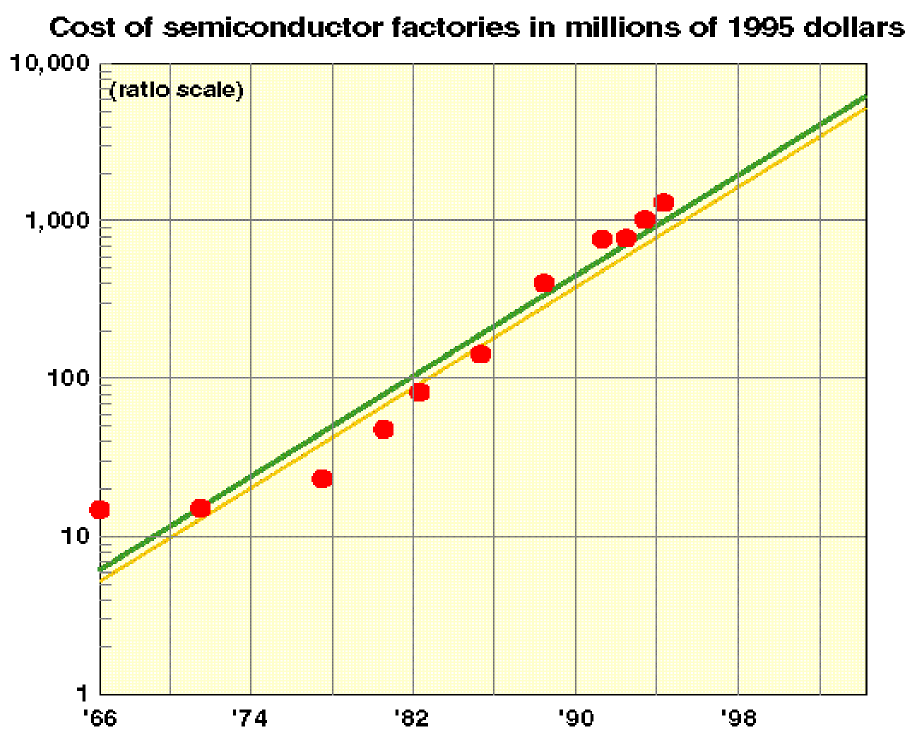
Source: Kathy Yelick and Jim Demmel, UC Berkeley

Unicore Performance: Manufacturing Issues

– Manufacturing cost goes up as feature size decreases

  – Cost of a semiconductor fabrication plant doubles every 4 years (Rock's Law)

– CMOS feature size is limited to 5 nm (at least 10 atoms)



Source: Kathy Yelick and Jim Demmel, UC Berkeley

Unicore Performance: Physical Limits

Execute the following loop on a serial machine in 1 second:

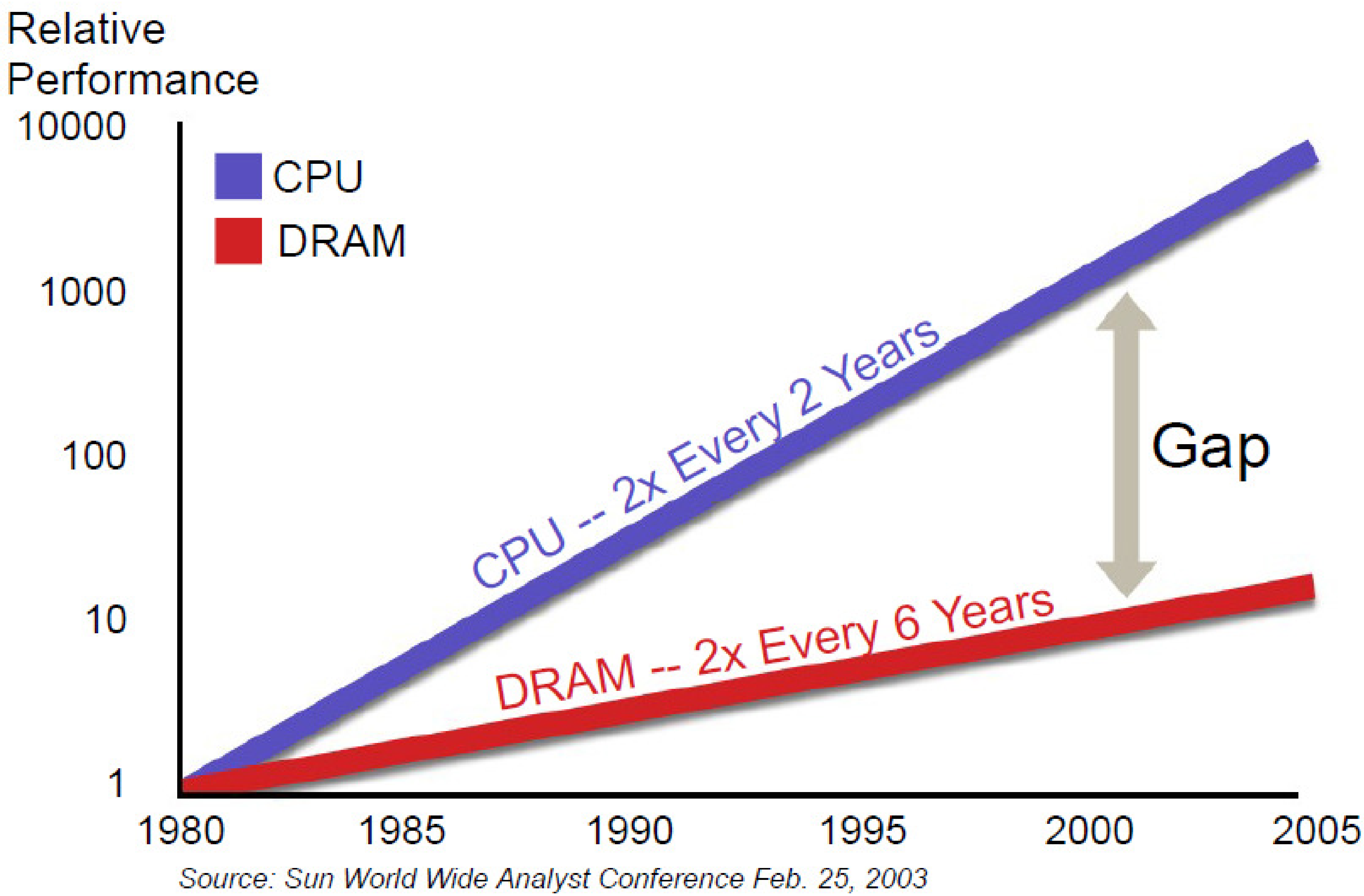
$$\textit{for}\ (\ i = 0;\ i < 10^{12};\ {+}{+}i\ )$$
$$z[\ i\ ] = x[\ i\ ] + y[\ i\ ];$$

- We will have to access 3×10^{12} data items in one second
- Speed of light is, $c \approx 3\times10^{8}$ m/s
- So each data item must be within $c\ /\ 3\times10^{12} \approx 0.1$ mm from the CPU on the average
- All data must be put inside a 0.2 mm × 0.2 mm square
- Each data item (≥ 8 bytes) can occupy only 1 Å^2 space!

  (size of a small atom!)

Source: Kathy Yelick and Jim Demmel, UC Berkeley

Unicore Performance: Memory Wall



Source: Rick Hetherington, Chief Technology Officer, Microelectronics, Sun Microsystems

Unicore Performance Has Hit a Wall!

Some Reasons

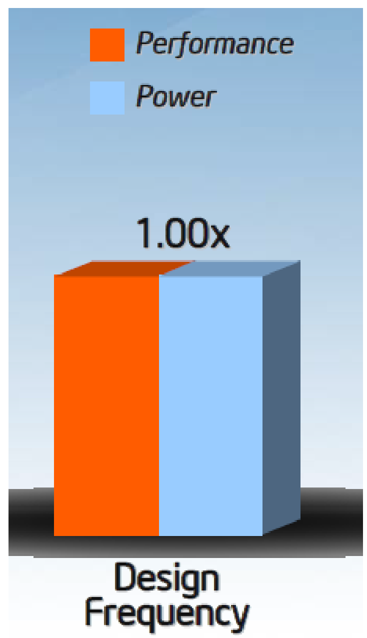
- Lack of additional ILP
  (Instruction Level Hidden Parallelism)
- High power density
- Manufacturing issues
- Physical limits
- Memory speed

"Oh Sinnerman, where you gonna run to?"

— Sinnerman (recorded by Nina Simone)

Where You Gonna Run To?

- Changing f by 20% changes performance by 13%
- So what happens if we overclock by 20%?



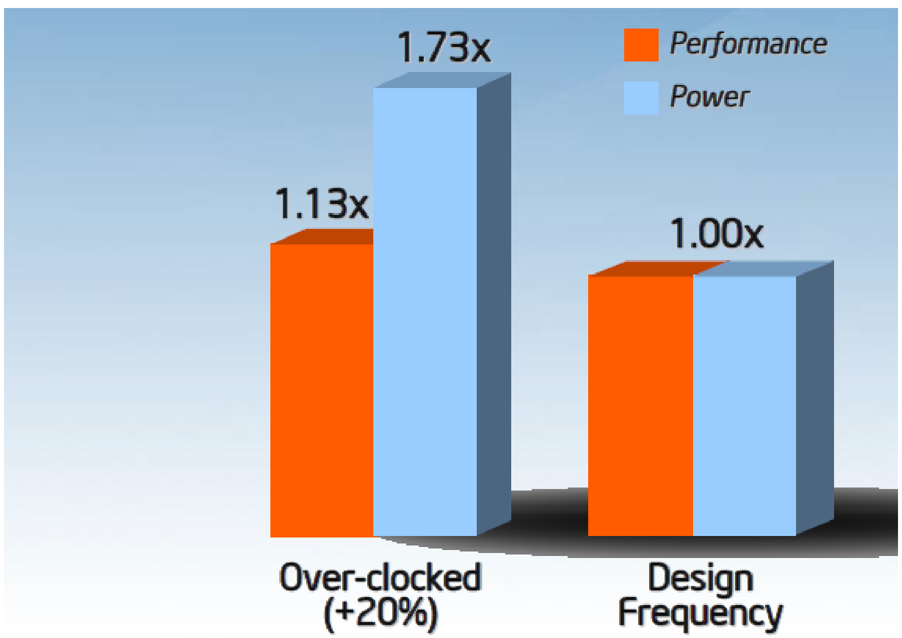
Source: Andrew A. Chien, Vice President of Research, Intel Corporation

Where You Gonna Run To?

– Changing f by 20% changes performance by 13%

– So what happens if we overclock by 20%?

– And underclock by 20%?



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Where You Gonna Run To?

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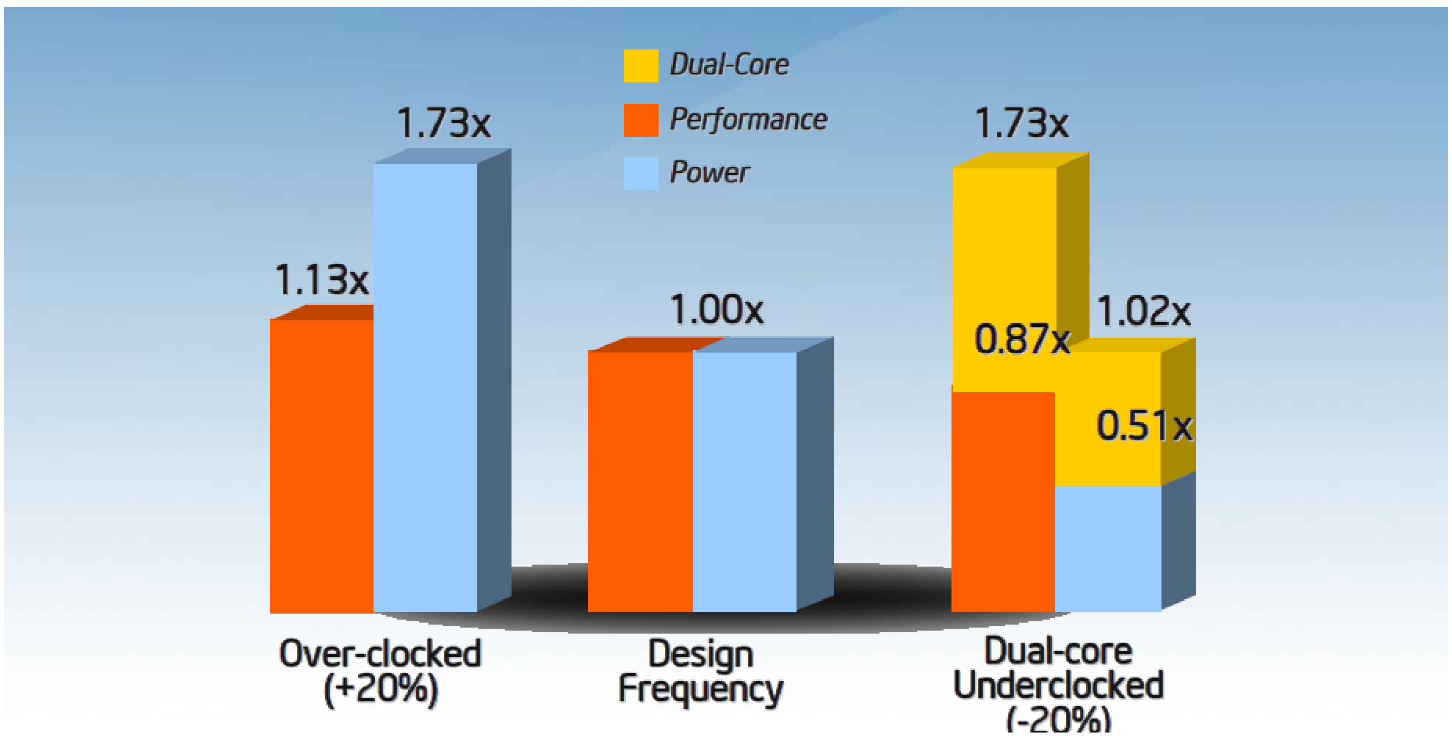

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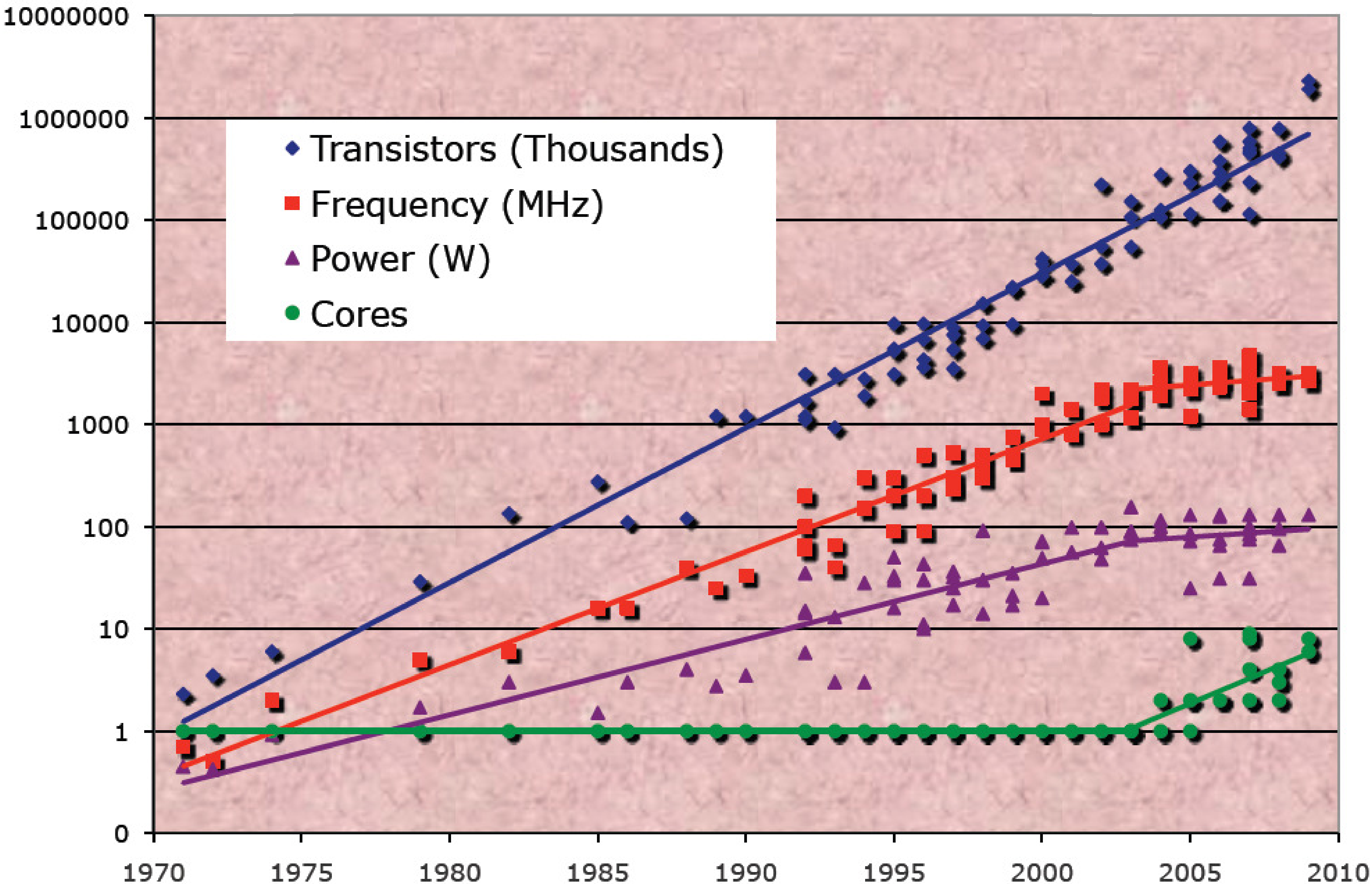
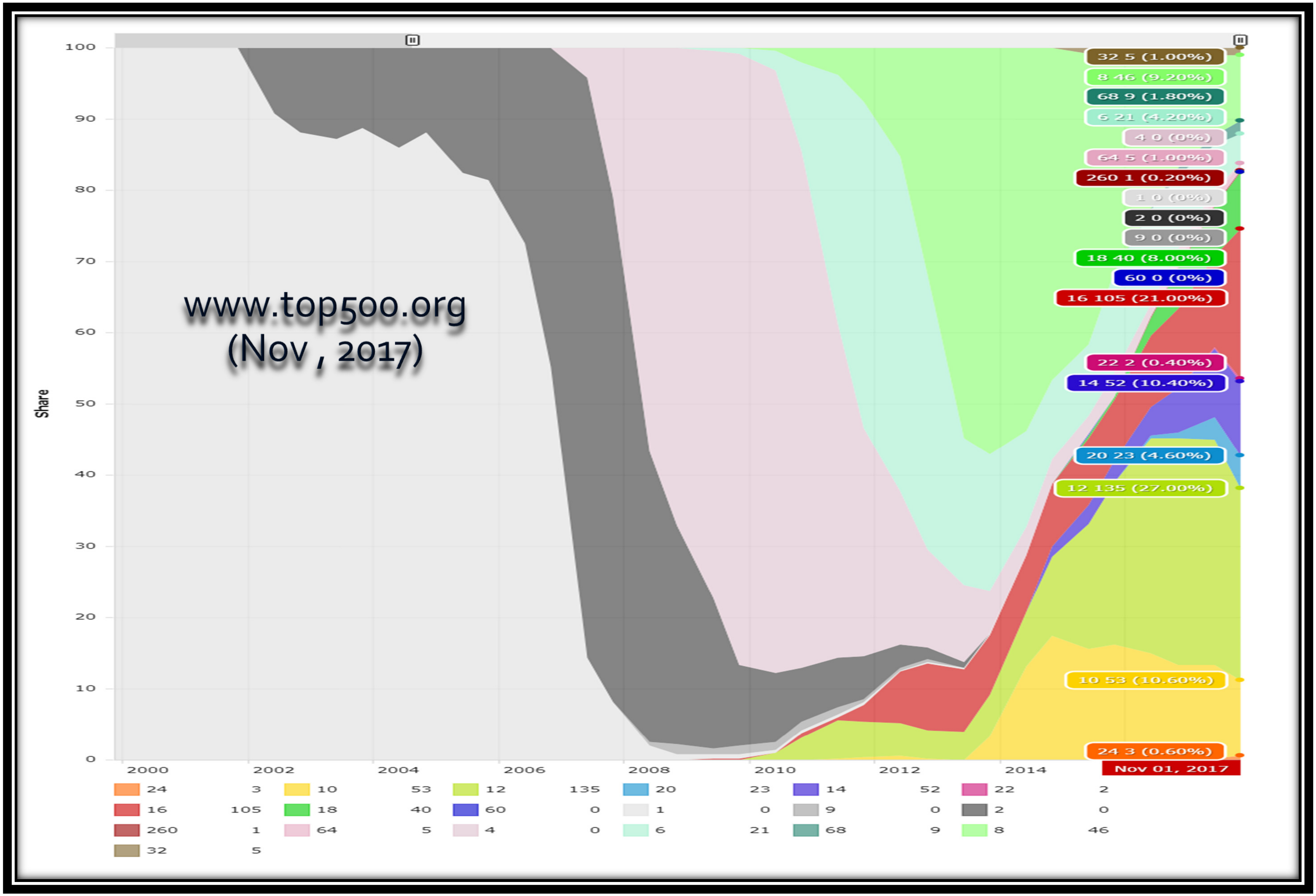
Moore's Law Reinterpreted

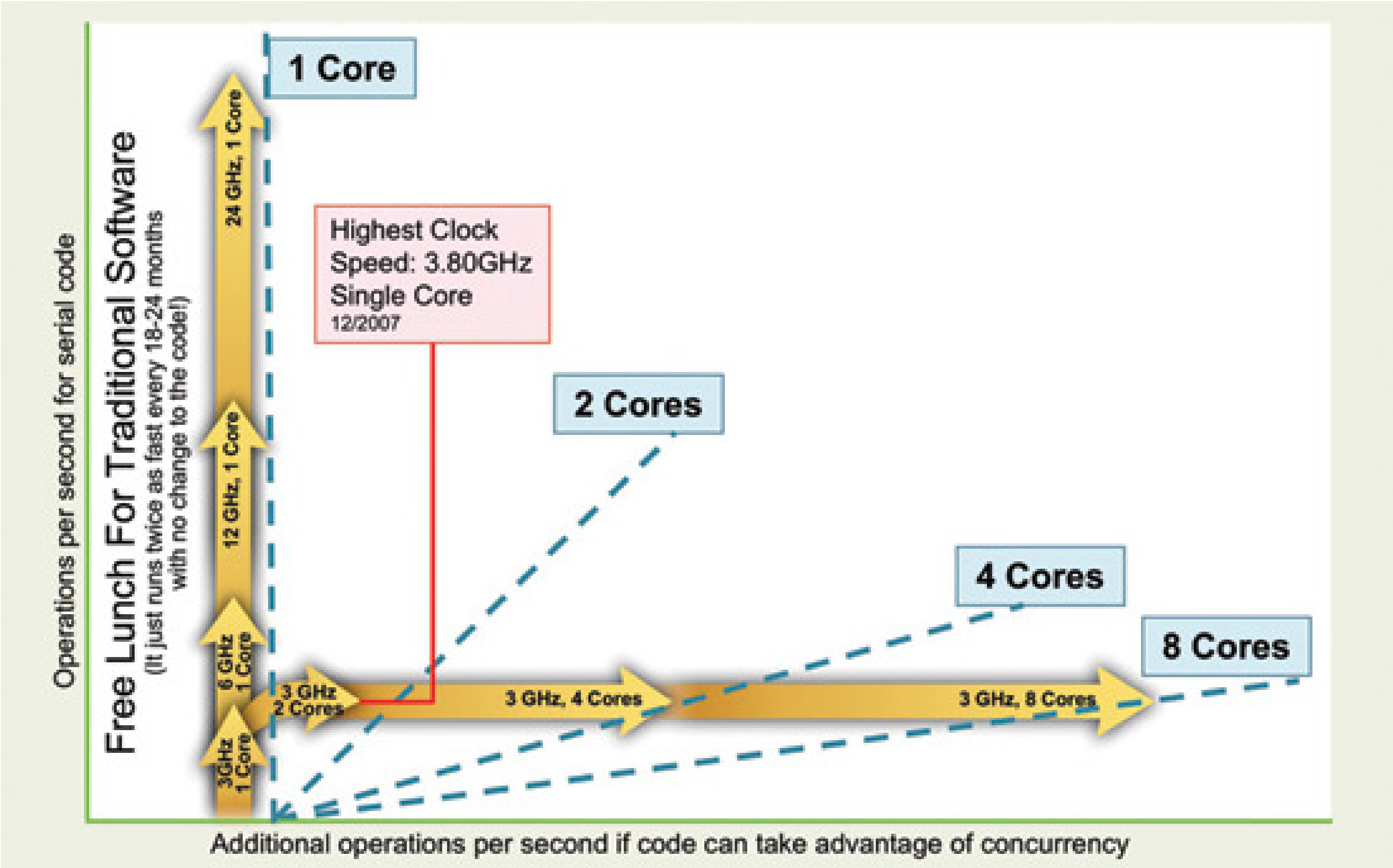


Source: Report of the 2011 Workshop on Exascale Programming Challenges

Top 500 Supercomputing Sites (Cores / Socket)



No Free Lunch for Traditional Software



Source: Simon Floyd, Workstation Performance: Tomorrow's Possibilities (Viewpoint Column)

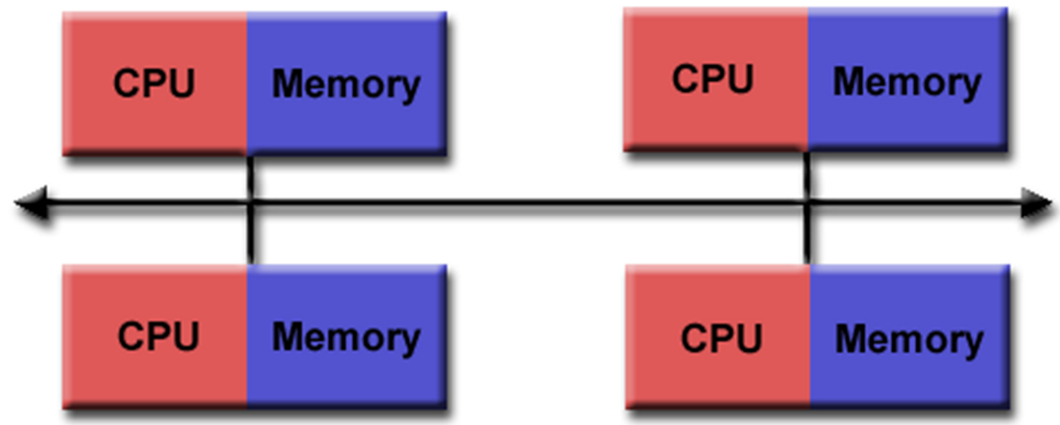
Insatiable Demand for Performance

Source: Patrick Gelsinger, Intel Developer Forum, 2008

Some Useful Classifications of Parallel Computers

Parallel Computer Memory Architecture (Distributed Memory)

- Each processor has its own local memory — no global address space

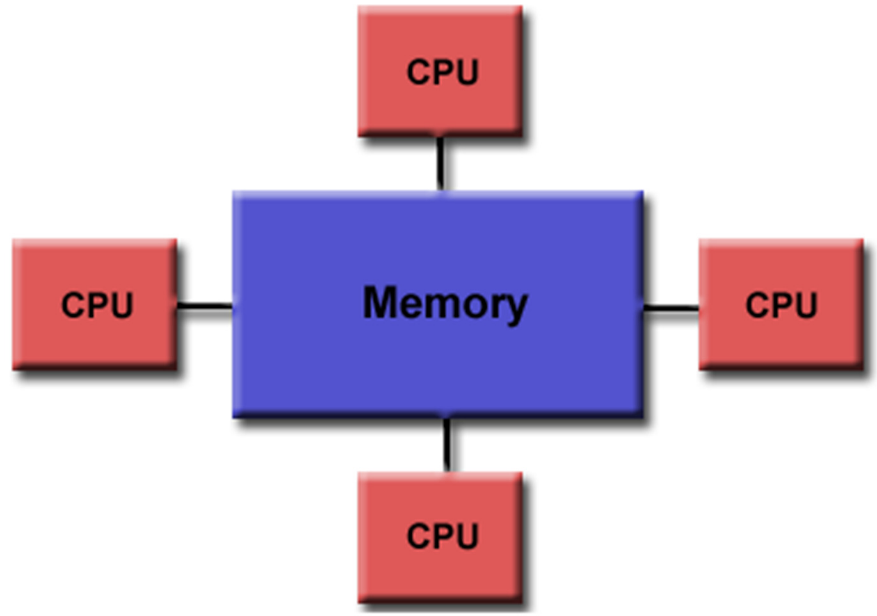


Source: Blaise Barney, LLNL

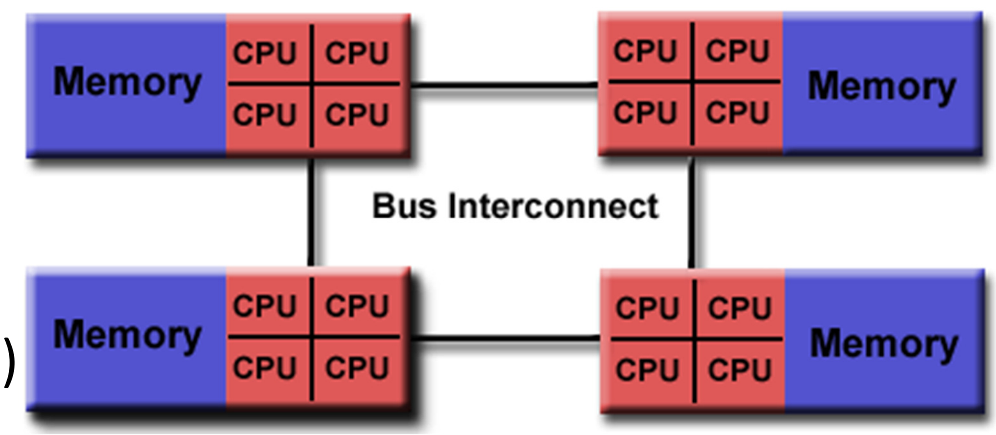
- Changes in local memory by one processor have no effect on memory of other processors
- Communication network to connect inter-processor memory
- Programming
  - Message Passing Interface (MPI)
  - Many once available: PVM, Chameleon, MPL, NX, etc.

Parallel Computer Memory Architecture (Shared Memory)

- All processors access all memory as global address space
- Changes in memory by one processor are visible to all others
- Two types
  - Uniform Memory Access (UMA)
  - Non-Uniform Memory Access (NUMA)
- Programming
  - Open Multi-Processing (OpenMP)
  - Cilk/Cilk++ and Intel Cilk Plus
  - Intel Thread Building Block (TBB), etc.



UMA

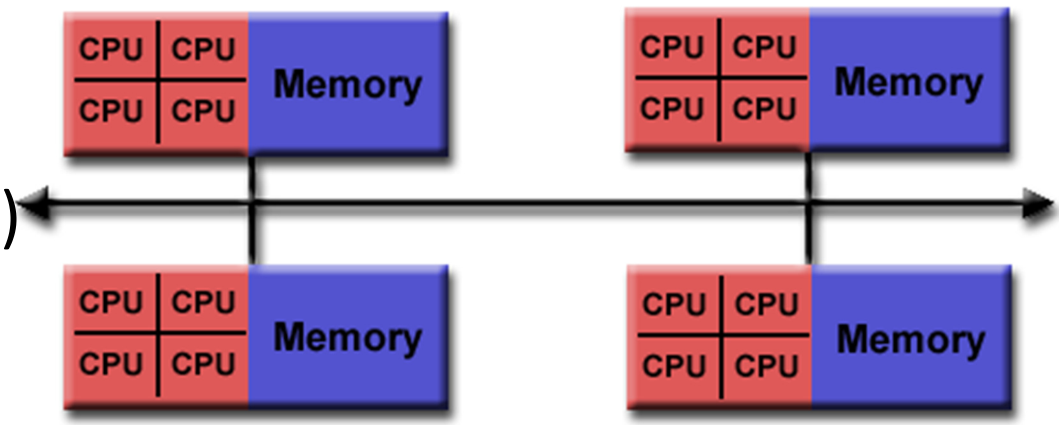


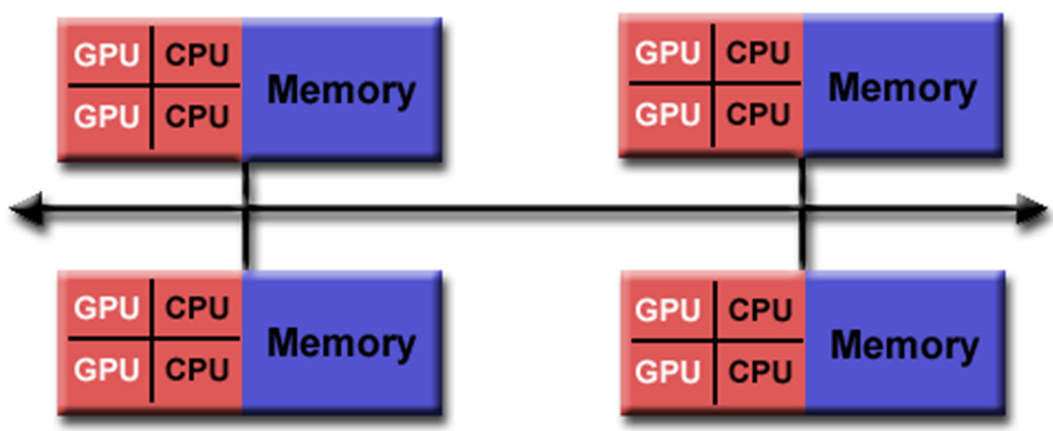
NUMA

Source: Blaise Barney, LLNL

Parallel Computer Memory Architecture (Hybrid Distributed-Shared Memory)

- The share-memory component can be a cache-coherent SMP or a Graphics Processing Unit (GPU)
- The distributed-memory component is the networking of multiple SMP/GPU machines
- Most common architecture for the largest and fastest computers in the world today
- Programming
  - OpenMP / Cilk + CUDA / OpenCL + MPI, etc.





Source: Blaise Barney, LLNL

Types of Parallelism

Nested Parallelism

$$ {}^{n}C_{r} = {}^{n-1}C_{r-1} + {}^{n-1}C_{r} $$

```
int comb ( int n, int r )
{
   if ( r > n ) return 0;
   if ( r == 0 || r == n ) return 1;

   int x, y;

   x = comb( n - 1, r - 1 );
   y = comb( n - 1, r );

   return ( x + y );
}
```

Serial Code

Control cannot pass this point until all spawned children have returned.

Grant permission to execute the called (spawned) function in parallel with the caller.

```
int comb ( int n, int r )
{
   if ( r > n ) return 0;
   if ( r == 0 || r == n ) return 1;

   int x, y;

   x = spawn comb( n - 1, r - 1 );
   y = comb( n - 1, r );

   sync;

   return ( x + y );
}
```

Parallel Code

Loop Parallelism

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \xrightarrow[\text{transpose}]{\text{in-place}} \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

```
for ( int i = 1; i < n; ++i )
    for ( int j = 0; j < i; ++j )
        {
          double t = A[ i ][
          A[ i ][ j ] = A[ j
          A[ j ][ i ] = t;
        }
```

Serial Code

Allows all iterations of the loop to be executed in parallel.

Can be converted to spawns and syncs using recursive divide-and-conquer.

```
parallel for ( int i = 1; i < n; ++i )
    for ( int j = 0; j < i; ++j )
        {
          double t = A[ i ][ j ];
          A[ i ][ j ] = A[ j ][ i ];
          A[ j ][ i ] = t;
        }
```

Parallel Code

Recursive D&C Implementation of Parallel Loops

```
parallel for ( int i = s; i < t; ++i )
        BODY( i );
```

divide-and-conquer implementation

```
void recur( int lo, int hi )
{
   if ( hi - lo > GRAINSIZE )
     {
        int mid = lo + ( hi - lo ) / 2;
        spawn recur( lo, mid );
        recur( mid, hi );
        sync;
     }
   else
     {
        for ( int i = lo; i < hi; ++i )
           BODY( i );
     }
}

recur( s, t );
```

Analyzing Parallel Algorithms

Speedup

Let T_p = running time using p identical processing elements

Speedup, $S_p = \frac{T_1}{T_p}$

Theoretically, $S_p \leq p$

Perfect or *linear* or *ideal* speedup if $S_p = p$

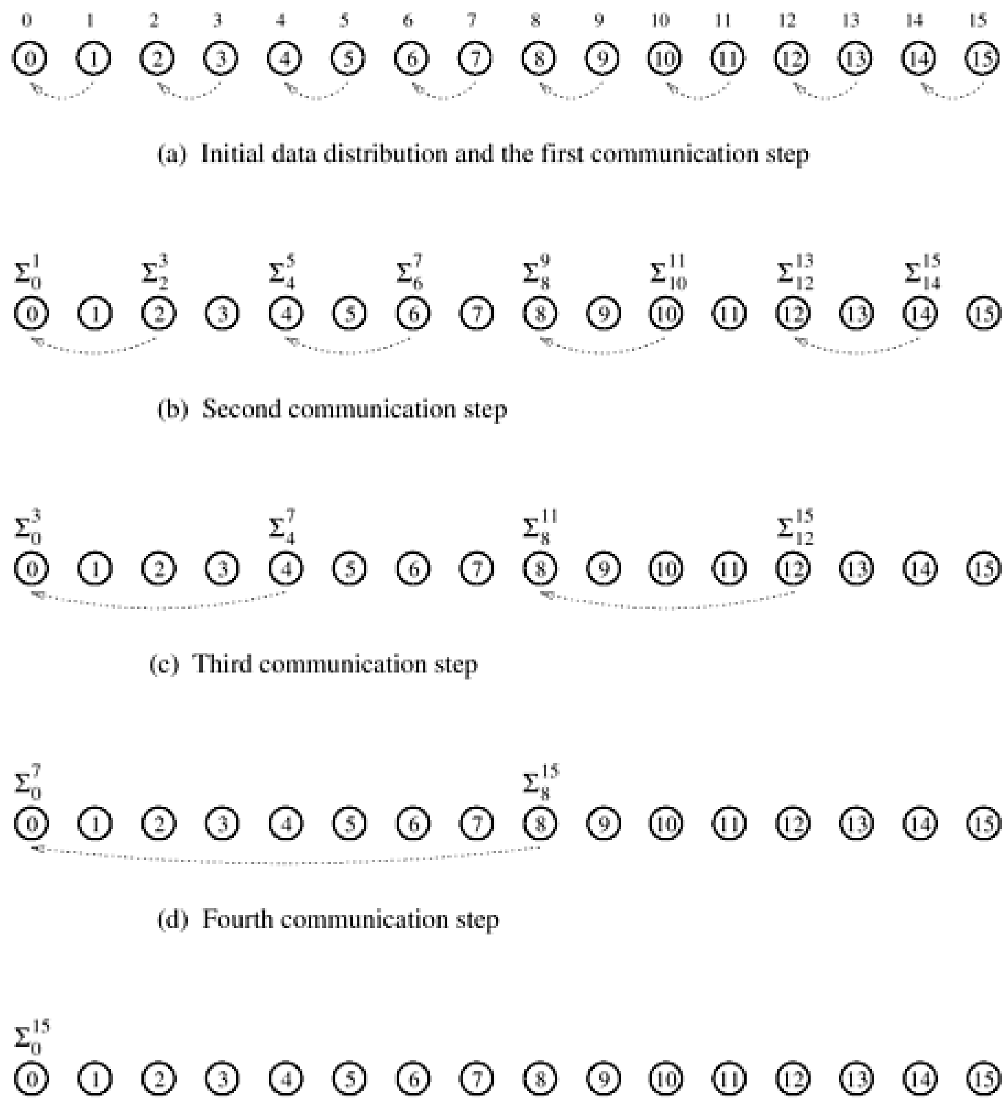
Speedup

Consider adding n numbers using n identical processing elements.

Serial runtime, $T = \Theta(n)$

Parallel runtime, $T_n = \Theta(\log n)$

Speedup, $S_n = \frac{T_1}{T_n} = \Theta\left(\frac{n}{\log n}\right)$



(a) Initial data distribution and the first communication step

(b) Second communication step

(c) Third communication step

(d) Fourth communication step

(e) Accumulation of the sum at processing element 0 after the final communication

Source: Grama et al., "Introduction to Parallel Computing", 2nd Edition

Superlinear Speedup

Theoretically, $S_p \leq p$

But in practice *superlinear speedup* is sometimes observed, that is, $S_p > p$ (why?)

Reasons for superlinear speedup

- Cache effects
- Exploratory decomposition

Parallelism & Span Law

We defined, T_p = runtime on p identical processing elements

Then span, T_∞ = runtime on an infinite number of identical processing elements

Parallelism, $P = \frac{T_1}{T_\infty}$

Parallelism is an upper bound on speedup, i.e., $S_p \leq P$

Span Law

$$T_p \geq T_\infty$$

Work Law

The cost of solving (or work performed for solving) a problem:

On a Serial Computer: is given by T_1

On a Parallel Computer: is given by pT_p

Work Law

$$T_p \geq \frac{T_1}{p}$$

Bounding Parallel Running Time (T_p)

A *runtime/online scheduler* maps tasks to processing elements dynamically at runtime.

A *greedy scheduler* never leaves a processing element idle if it can map a task to it.

Theorem [Graham'68, Brent'74]: For any greedy scheduler,

$$\boxed{T_p \le \frac{T_1}{p} + T_\infty}$$

Corollary: For any greedy scheduler,

$$\boxed{T_p \le 2T_p^*,}$$

where T_p^* is the running time due to optimal scheduling on *p* processing elements.

Work Optimality

Let T_s = runtime of the optimal or the fastest known serial algorithm

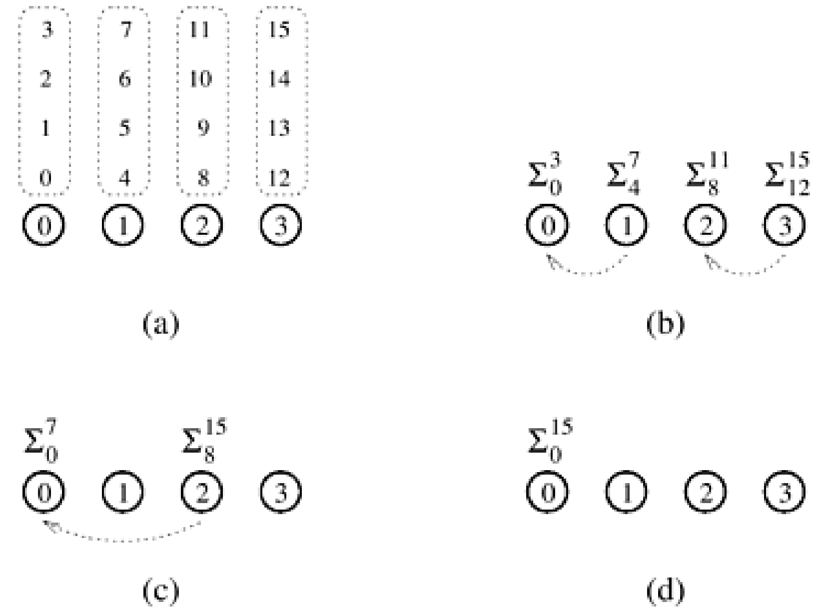
A parallel algorithm is *cost-optimal* or *work-optimal* provided

$$pT_p = \Theta(T_s)$$

Our algorithm for adding n numbers using n identical processing elements is clearly not work optimal.

Adding *n* Numbers Work-Optimality

We reduce the number of processing elements which in turn increases the granularity of the subproblem assigned to each processing element.



Source: Grama et al., "Introduction to Parallel Computing", 2nd Edition

Suppose we use p processing elements.

First each processing element locally adds its $\frac{n}{p}$ numbers in time $\Theta\left(\frac{n}{p}\right)$.

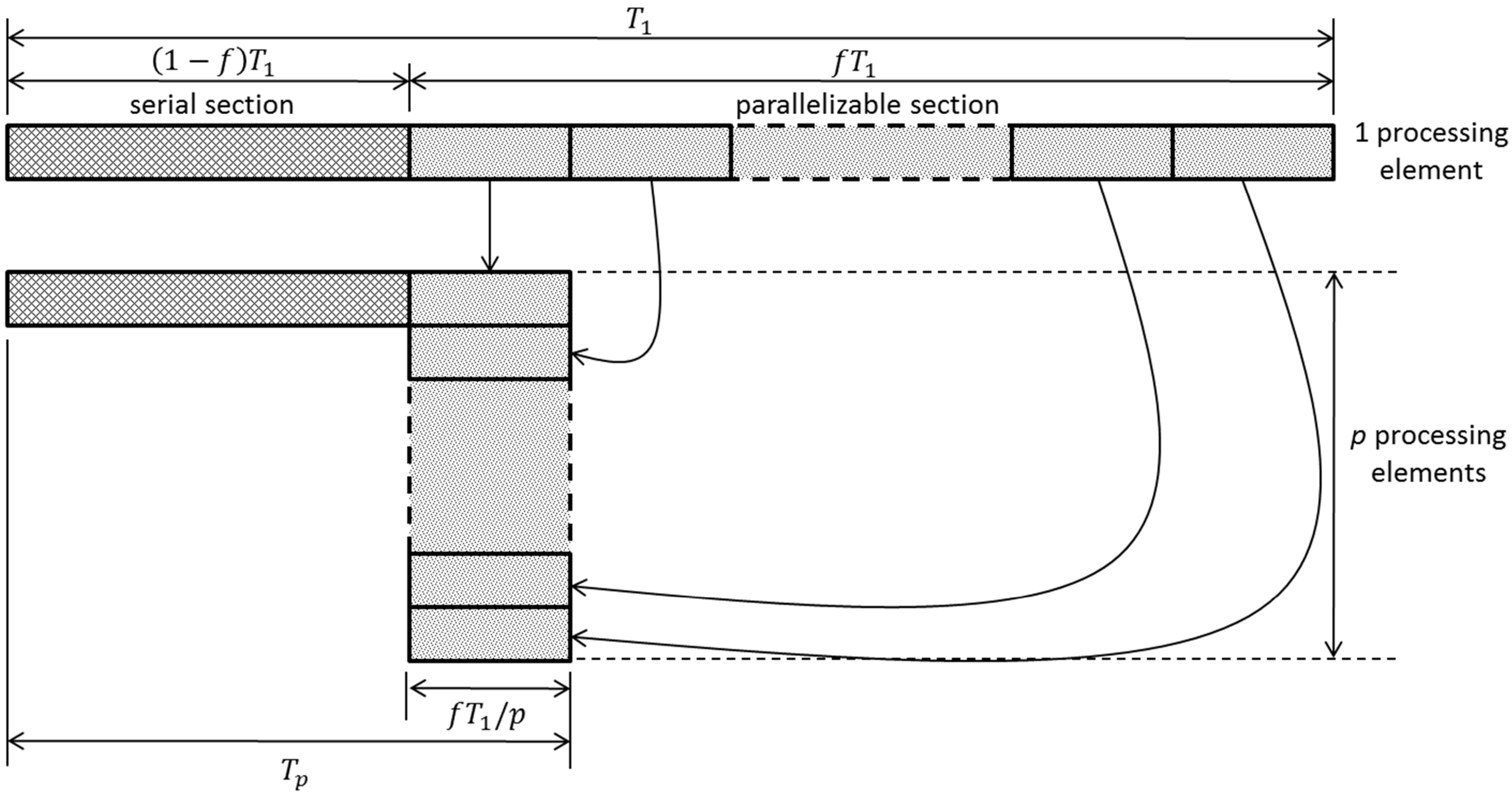
Then p processing elements adds these p partial sums in time $\Theta(\log p)$.

Thus $T_p = \Theta\left(\frac{n}{p} + \log p\right)$, and $T_s = \Theta(n)$.

So the algorithm is work-optimal provided $n = \Omega(p \log p)$.

Scaling Law

Scaling of Parallel Algorithms (Amdahl's Law)



Suppose only a fraction *f* of a computation can be parallelized.

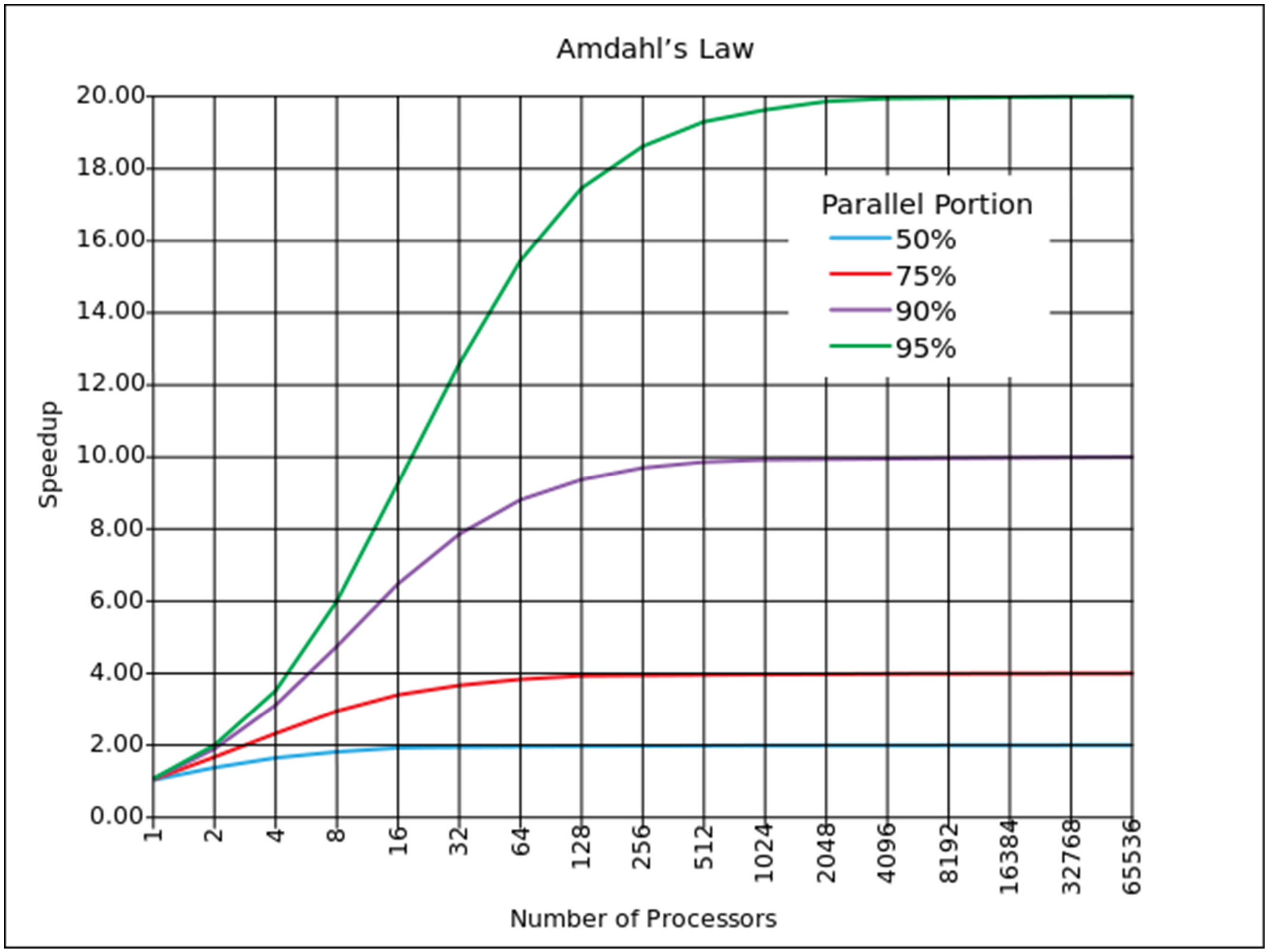
Then parallel running time, $T_p \geq (1-f)T_1 + f\,\frac{T_1}{p}$

Speedup, $S_p = \frac{T_1}{T_p} \leq \frac{p}{f+(1-f)p} = \frac{1}{(1-f)+\frac{f}{p}} \leq \frac{1}{1-f}$

Scaling of Parallel Algorithms (Amdahl's Law)

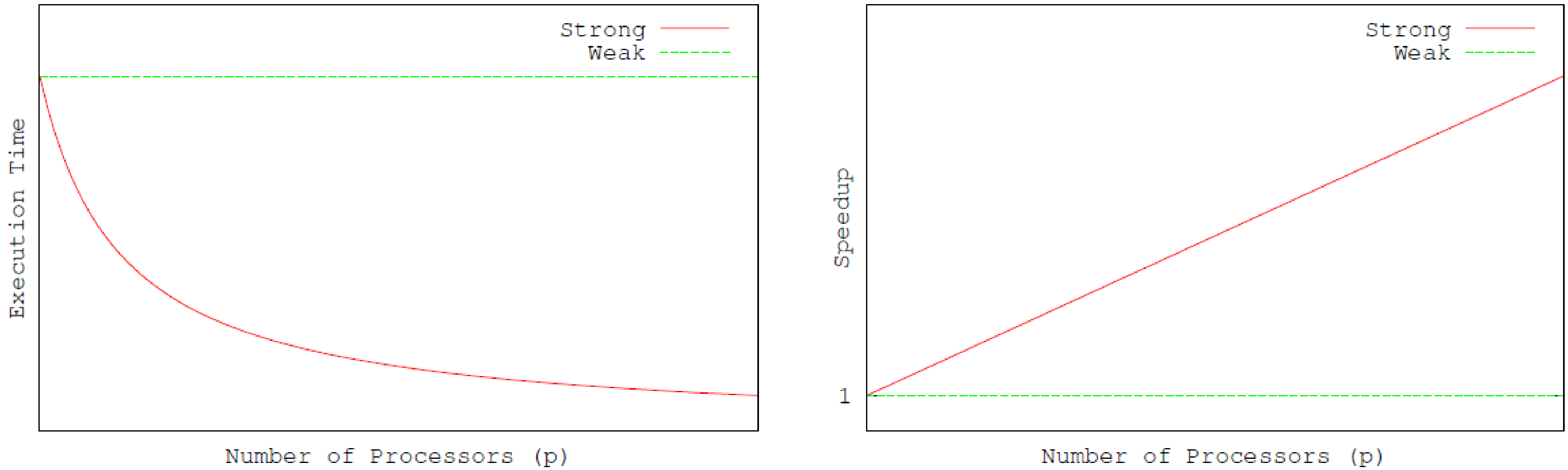
Suppose only a fraction f of a computation can be parallelized.

Speedup, $S_p = \frac{T_1}{T_p} \le \frac{1}{(1-f)+\frac{f}{p}} \le \frac{1}{1-f}$



Source: Wikipedia

Strong Scaling vs. Weak Scaling



Source: Martha Kim, Columbia University

Strong Scaling

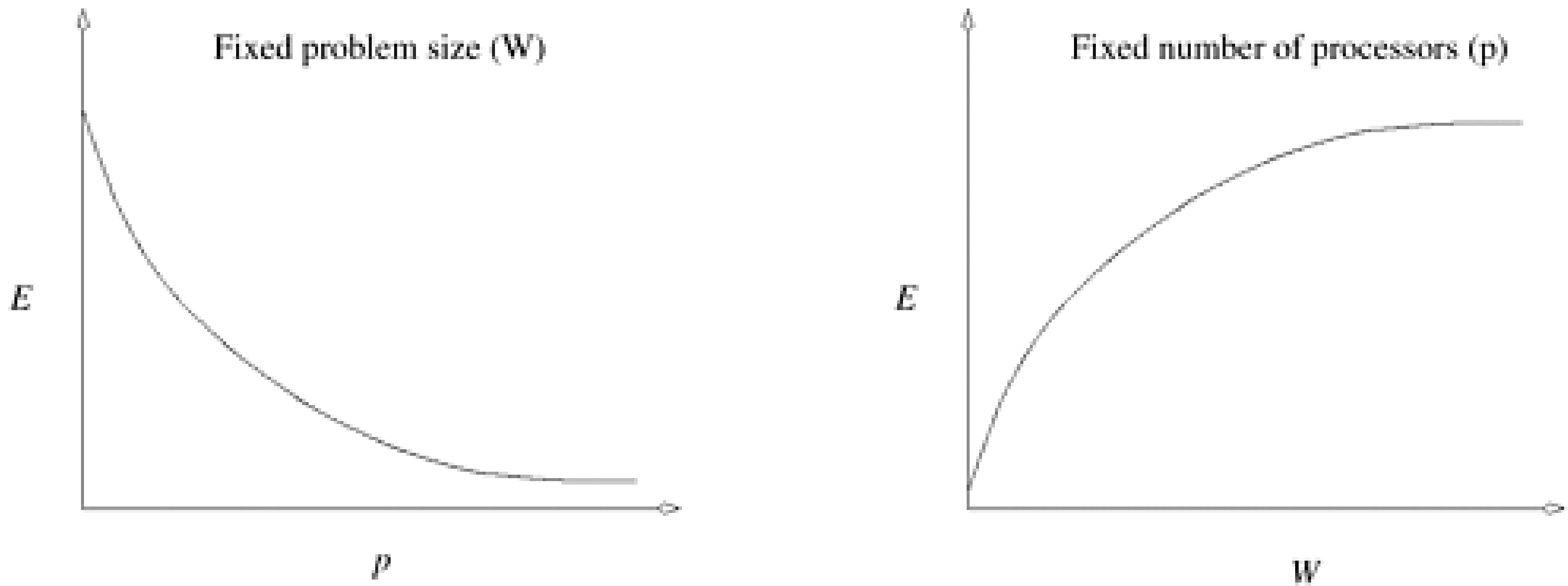
How T_p (or S_p) varies with p when the problem size is fixed.

Weak Scaling

How T_p (or S_p) varies with p when the problem size per processing element is fixed.

Scalable Parallel Algorithms

Efficiency, $E_p = \frac{S_p}{p} = \frac{T_1}{pT_p}$



Source: Grama et al., "Introduction to Parallel Computing", 2nd Edition

A parallel algorithm is called *scalable* if its efficiency can be maintained at a fixed value by simultaneously increasing the number of processing elements and the problem size.

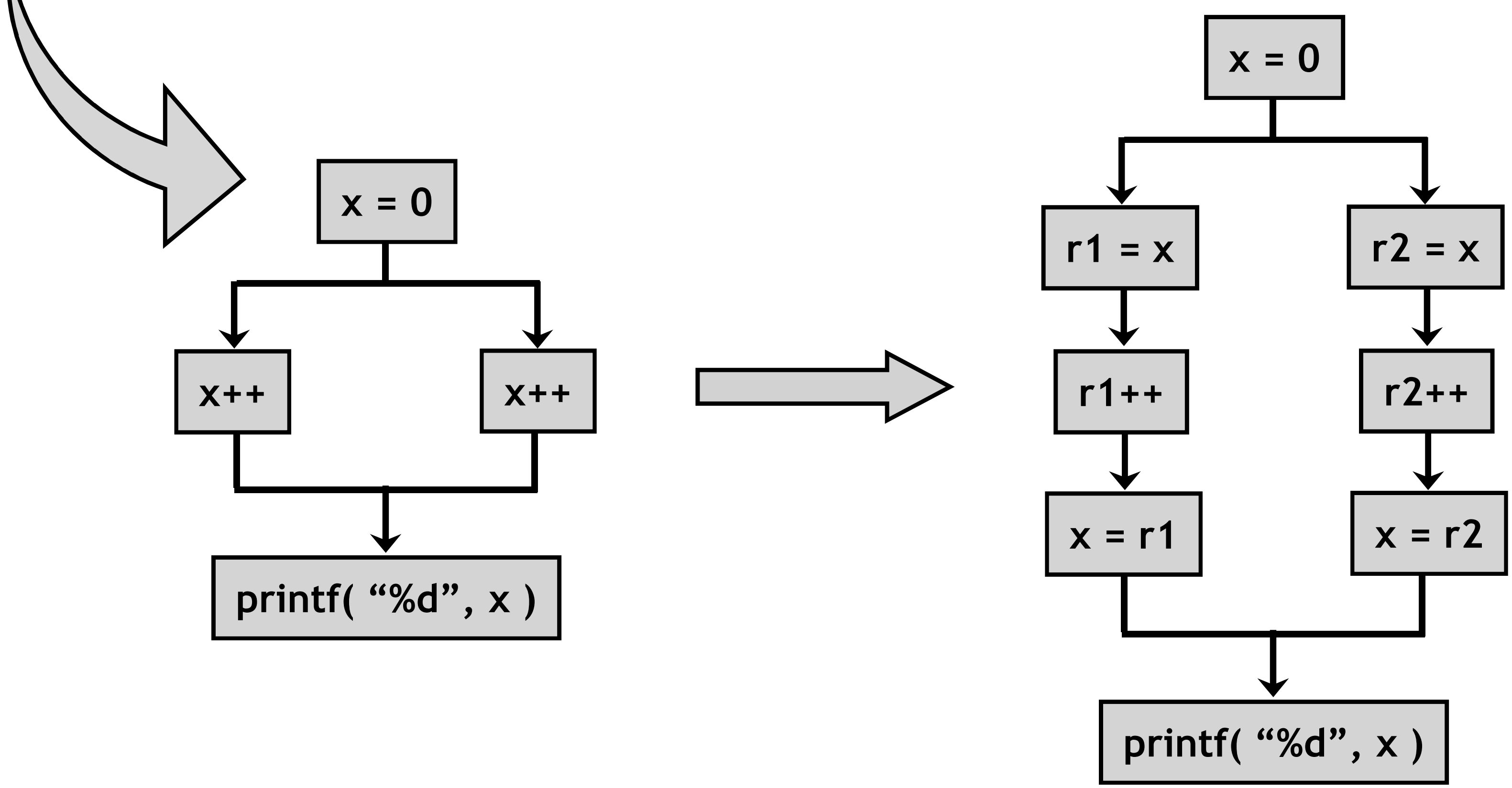
Scalability reflects a parallel algorithm's ability to utilize increasing processing elements effectively.

Races

Race Bugs

A *determinacy race* occurs if two logically parallel instructions access the same memory location and at least one of them performs a write.

```
int x = 0;
parallel for ( int i = 0; i < 2; i++ )
   x++;
printf( "%d", x );
```



Critical Sections and Mutexes

race

```
int r = 0;

parallel for ( int i = 0; i < n; i++ )
    r += eval( x[ i ] );
```

critical section

two or more strands

must not access

at the same time

```
mutex mtx;

parallel for ( int i = 0; i < n; i++ )
    mtx.lock( );
    r += eval( x[ i ] );
    mtx.unlock( );
```

mutex (mutual exclusion)

an attempt by a strand

to lock an already locked mutex

causes that strand to block (i.e., wait)

until the mutex is unlocked

Problems

– lock overhead

– lock contention

Critical Sections and Mutexes

race

```
int r = 0;

parallel for ( int i = 0; i < n; i++ )
  r += eval( x[ i ] );
```

```
mutex mtx;

parallel for ( int i = 0; i < n; i++ )
    mtx.lock( );
    r += eval( x[ i ] );
    mtx.unlock( );
```

```
mutex mtx;

parallel for ( int i = 0; i < n; i++ )
    int y = eval( x[ i ] );
    mtx.lock( );
    r += y;
    mtx.unlock( );
```

- slightly better solution
- but lock contention can still destroy parallelism

Recursive
D&C Implementation of Loops

Recursive D&C Implementation of Parallel Loops

```
parallel for ( int i = s; i < t; ++i )
        BODY( i );
```

divide-and-conquer implementation

```
void recur( int lo, int hi )
{
   if ( hi – lo > GRAINSIZE )
      {
         int mid = lo + ( hi – lo ) / 2;
         spawn recur( lo, mid );
         recur( mid, hi );
         sync;
      }
   else
      {
         for ( int i = lo; i < hi; ++i )
            BODY( i );
      }
}

recur( s, t );
```

Let $n = t - s$

$m =$ running time of a single call to BODY

Span: $T_\infty(n) = \Theta(\log n + \text{GRAINSIZE} \times m)$

Parallel Iterative MM

```
Iter-MM ( Z, X, Y )        { X, Y, Z are n × n matrices,
                             where n is a positive integer }
for i ← 1 to n do
    for j ← 1 to n do
        Z[ i ][ j ] ← 0
        for k ← 1 to n do
            Z[ i ][ j ] ← Z[ i ][ j ] + X[ i ][ k ] · Y[ k ][ j ]
```

```
Par-Iter-MM ( Z, X, Y )    { X, Y, Z are n × n matrices,
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```

Work: $T_1(n) = \Theta(n^3)$

Span: $T_\infty(n) = \Theta(n)$

Parallel Running Time: $T_p(n) = O\left(\frac{T_1(n)}{p} + T_\infty(n)\right) = O\left(\frac{n^3}{p} + n\right)$

Parallelism: $\frac{T_1(n)}{T_\infty(n)} = \Theta(n^2)$

Parallel Recursive MM

$$\overset{Z}{\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}} = \overset{X}{\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}} \times \overset{Y}{\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}}$$

$$= \begin{bmatrix} X_{11}Y_{11} + X_{12}Y_{21} & X_{11}Y_{12} + X_{12}Y_{22} \\ X_{21}Y_{11} + X_{22}Y_{21} & X_{21}Y_{12} + X_{22}Y_{22} \end{bmatrix}$$

Parallel Recursive MM

Par-Rec-MM (Z, X, Y) { X, Y, Z are $n \times n$ matrices, where $n = 2^k$ for integer $k \geq 0$ }

if $n = 1$ *then*

  $Z \leftarrow Z + X \cdot Y$

else

  spawn *Par-Rec-MM* (Z_{11}, X_{11}, Y_{11})

  spawn *Par-Rec-MM* (Z_{12}, X_{11}, Y_{12})

  spawn *Par-Rec-MM* (Z_{21}, X_{21}, Y_{11})

  Par-Rec-MM (Z_{21}, X_{21}, Y_{12})

  sync

  spawn *Par-Rec-MM* (Z_{11}, X_{12}, Y_{21})

  spawn *Par-Rec-MM* (Z_{12}, X_{12}, Y_{22})

  spawn *Par-Rec-MM* (Z_{21}, X_{22}, Y_{21})

  Par-Rec-MM (Z_{22}, X_{22}, Y_{22})

  sync

endif

Parallel Recursive MM

```
Par-Rec-MM ( Z, X, Y )    { X, Y, Z are n × n matrices,
                            where n = 2^k for integer k ≥ 0 }
if n = 1 then
    Z ← Z + X · Y
else
    spawn Par-Rec-MM ( Z11, X11, Y11 )
    spawn Par-Rec-MM ( Z12, X11, Y12 )
    spawn Par-Rec-MM ( Z21, X21, Y11 )
          Par-Rec-MM ( Z21, X21, Y12 )
    sync
    spawn Par-Rec-MM ( Z11, X12, Y21 )
    spawn Par-Rec-MM ( Z12, X12, Y22 )
    spawn Par-Rec-MM ( Z21, X22, Y21 )
          Par-Rec-MM ( Z22, X22, Y22 )
    sync
endif
```

Work:

$$T_1(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ 8T_1\left(\frac{n}{2}\right) + \Theta(1), & otherwise. \end{cases}$$

$$= \Theta\left(n^3\right) \qquad \text{[MT Case 1]}$$

Span:

$$T_\infty(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ 2T_\infty\left(\frac{n}{2}\right) + \Theta(1), & otherwise. \end{cases}$$

$$= \Theta(n) \qquad \text{[MT Case 1]}$$

Parallelism: $\frac{T_1(n)}{T_\infty(n)} = \Theta\left(n^2\right)$

Additional Space:

$$s_\infty(n) = \Theta(1)$$

Recursive MM with More Parallelism

$$\overset{Z}{\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}} = \begin{bmatrix} X_{11}\,Y_{11} + X_{12}\,Y_{21} & X_{11}\,Y_{12} + X_{12}\,Y_{22} \\ X_{21}\,Y_{11} + X_{22}\,Y_{21} & X_{21}\,Y_{12} + X_{22}\,Y_{22} \end{bmatrix}$$

$$= \begin{bmatrix} X_{11}Y_{11} & X_{11}Y_{12} \\ X_{21}Y_{11} & X_{21}Y_{12} \end{bmatrix} + \begin{bmatrix} X_{12}Y_{21} & X_{12}Y_{22} \\ X_{22}Y_{21} & X_{22}Y_{22} \end{bmatrix}$$

Recursive MM with More Parallelism

Par-Rec-MM2 (Z, X, Y) { X, Y, Z are $n \times n$ matrices, where $n = 2^k$ for integer $k \geq 0$ }

if $n = 1$ *then*
    $Z \leftarrow Z + X \cdot Y$
else { T is a temporary $n \times n$ matrix }
    spawn *Par-Rec-MM2* (Z_{11}, X_{11}, Y_{11})
    spawn *Par-Rec-MM2* (Z_{12}, X_{11}, Y_{12})
    spawn *Par-Rec-MM2* (Z_{21}, X_{21}, Y_{11})
    spawn *Par-Rec-MM2* (Z_{21}, X_{21}, Y_{12})
    spawn *Par-Rec-MM2* (T_{11}, X_{12}, Y_{21})
    spawn *Par-Rec-MM2* (T_{12}, X_{12}, Y_{22})
    spawn *Par-Rec-MM2* (T_{21}, X_{22}, Y_{21})
           Par-Rec-MM2 (T_{22}, X_{22}, Y_{22})
    sync
    parallel for $i \leftarrow 1$ *to* n *do*
        parallel for $j \leftarrow 1$ *to* n *do*
            $Z[\,i\,][\,j\,] \leftarrow Z[\,i\,][\,j\,] + T[\,i\,][\,j\,]$
endif

Recursive MM with More Parallelism

```
Par-Rec-MM2 ( Z, X, Y )      { X, Y, Z are n × n matrices,
                               where n = 2^k for integer k ≥ 0 }
if n = 1 then
    Z ← Z + X · Y
else          { T is a temporary n × n matrix }
    spawn Par-Rec-MM2 ( Z11, X11, Y11 )
    spawn Par-Rec-MM2 ( Z12, X11, Y12 )
    spawn Par-Rec-MM2 ( Z21, X21, Y11 )
    spawn Par-Rec-MM2 ( Z21, X21, Y12 )
    spawn Par-Rec-MM2 ( T11, X12, Y21 )
    spawn Par-Rec-MM2 ( T12, X12, Y22 )
    spawn Par-Rec-MM2 ( T21, X22, Y21 )
          Par-Rec-MM2 ( T22, X22, Y22 )
    sync
    parallel for i ← 1 to n do
        parallel for j ← 1 to n do
            Z[ i ][ j ] ← Z[ i ][ j ] + T[ i ][ j ]
endif
```

Work:

$$T_1(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ 8T_1\left(\frac{n}{2}\right) + \Theta\left(n^2\right), & otherwise. \end{cases}$$

$$= \Theta\left(n^3\right) \qquad \text{[MT Case 1]}$$

Span:

$$T_\infty(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ T_\infty\left(\frac{n}{2}\right) + \Theta(\log n), & otherwise. \end{cases}$$

$$= \Theta(\log^2 n) \qquad \text{[MT Case 2]}$$

Parallelism: $\frac{T_1(n)}{T_\infty(n)} = \Theta\left(\frac{n^3}{\log^2 n}\right)$

Additional Space:

$$s_\infty(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ 8s_\infty\left(\frac{n}{2}\right) + \Theta\left(n^2\right), & otherwise. \end{cases}$$

$$= \Theta\left(n^3\right) \qquad \text{[MT Case 1]}$$

Parallel Merge Sort

Parallel Merge Sort

```
Merge-Sort ( A, p, r )        { sort the elements in A[ p … r ] }

if p < r then
    q ← ⌊ ( p + r ) / 2 ⌋
    Merge-Sort ( A, p, q )
    Merge-Sort ( A, q + 1, r )
    Merge ( A, p, q, r )
```

```
Par-Merge-Sort ( A, p, r )    { sort the elements in A[ p … r ] }

if p < r then
    q ← ⌊ ( p + r ) / 2 ⌋
    spawn Merge-Sort ( A, p, q )
          Merge-Sort ( A, q + 1, r )
    sync
    Merge ( A, p, q, r )
```

Parallel Merge Sort

Par-Merge-Sort (*A*, *p*, *r*) { *sort the elements in A*[*p* … *r*] }

1. *if* $p < r$ *then*
2. $q \leftarrow \lfloor (p + r) / 2 \rfloor$
3. *spawn* *Merge-Sort* (*A*, *p*, *q*)
4. *Merge-Sort* (*A*, *q* + 1, *r*)
5. *sync*
6. *Merge* (*A*, *p*, *q*, *r*)

Work: $$T_1(n) = \begin{cases} \Theta(1), & \text{if } n = 1, \\ 2T_1\left(\frac{n}{2}\right) + \Theta(n), & \text{otherwise.} \end{cases}$$

$$= \Theta(n \log n) \qquad \text{[MT Case 2]}$$

Span: $$T_\infty(n) = \begin{cases} \Theta(1), & \text{if } n = 1, \\ T_\infty\left(\frac{n}{2}\right) + \Theta(n), & \text{otherwise.} \end{cases}$$

$$= \Theta(n) \qquad \text{[MT Case 3]}$$

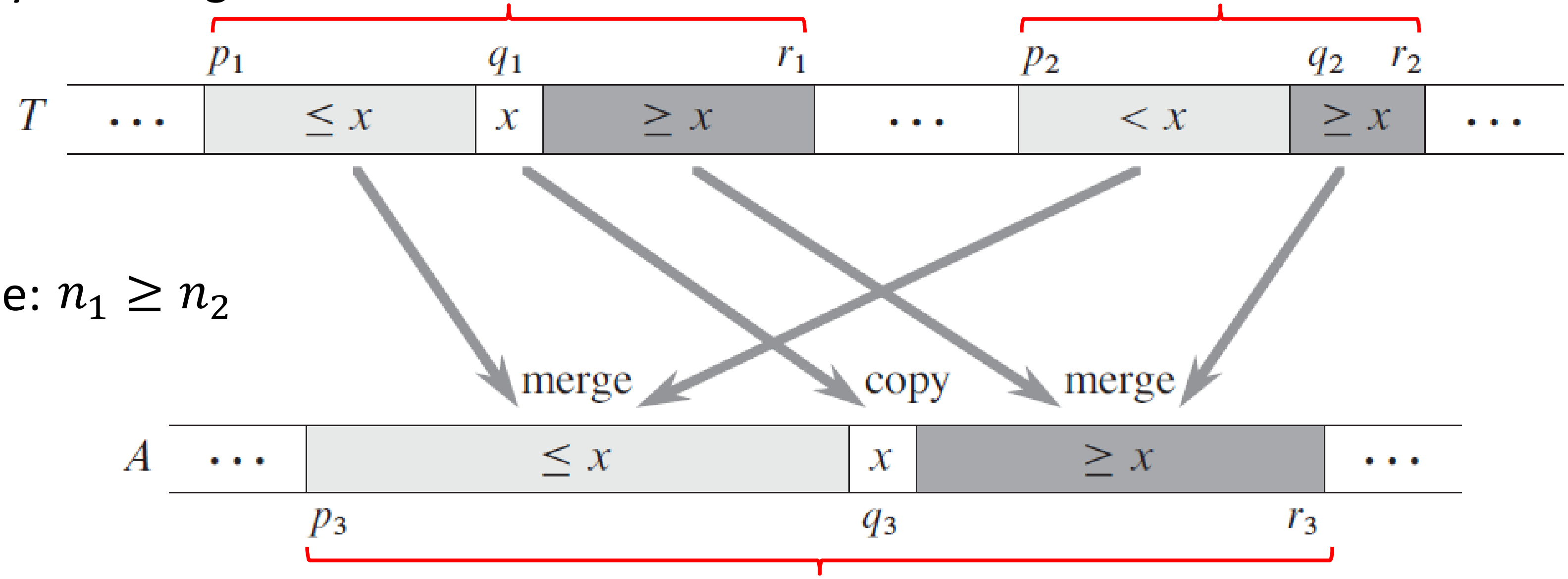
Parallelism: $$\frac{T_1(n)}{T_\infty(n)} = \Theta(\log n)$$

Parallel Merge

$n_1 = r_1 - p_1 + 1$ $\qquad$ $n_2 = r_2 - p_2 + 1$

subarrays to merge: $T[p_1..r_1]$ $\qquad$ $T[p_2..r_2]$

suppose: $n_1 \geq n_2$



merged output: $A[p_3..r_3]$

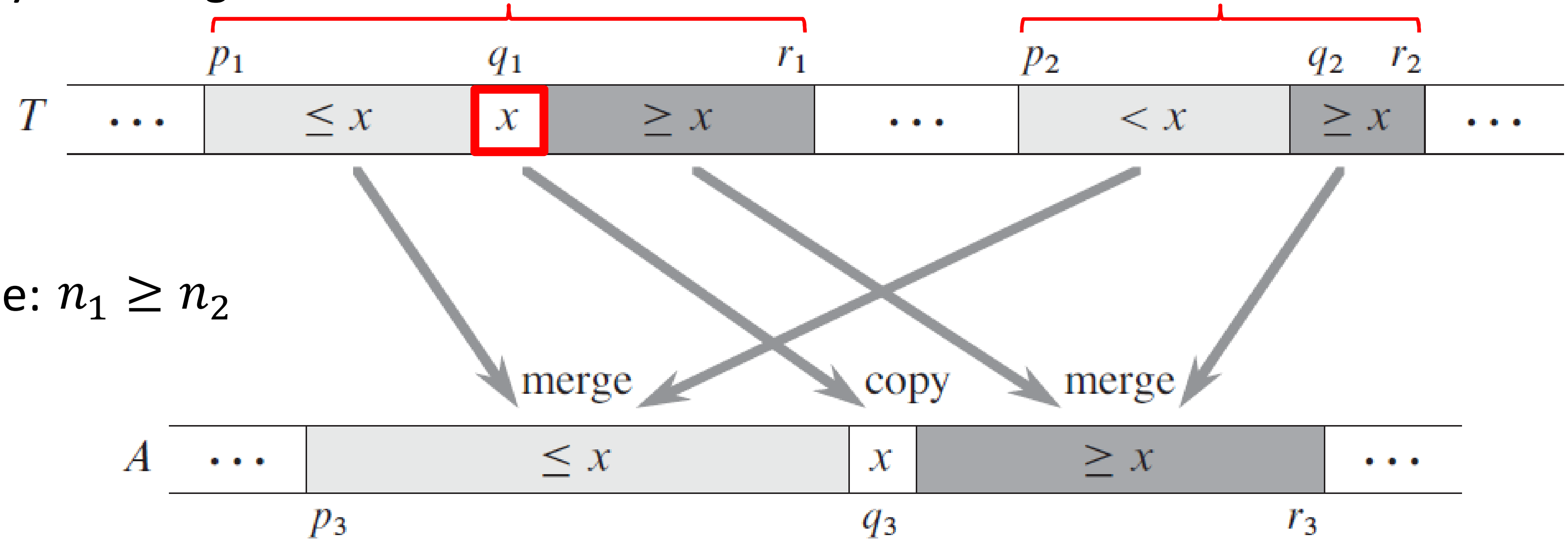
$n_3 = r_3 - p_3 + 1 = n_1 + n_2$

Source: Cormen et al., “Introduction to Algorithms”, 3rd Edition

Parallel Merge

$n_1 = r_1 - p_1 + 1$ $\qquad$ $n_2 = r_2 - p_2 + 1$

subarrays to merge: $T[p_1 \mathinner{.\,.} r_1]$ $\qquad$ $T[p_2 \mathinner{.\,.} r_2]$



suppose: $n_1 \geq n_2$

merged output: $A[p_3 \mathinner{.\,.} r_3]$

$n_3 = r_3 - p_3 + 1 = n_1 + n_2$

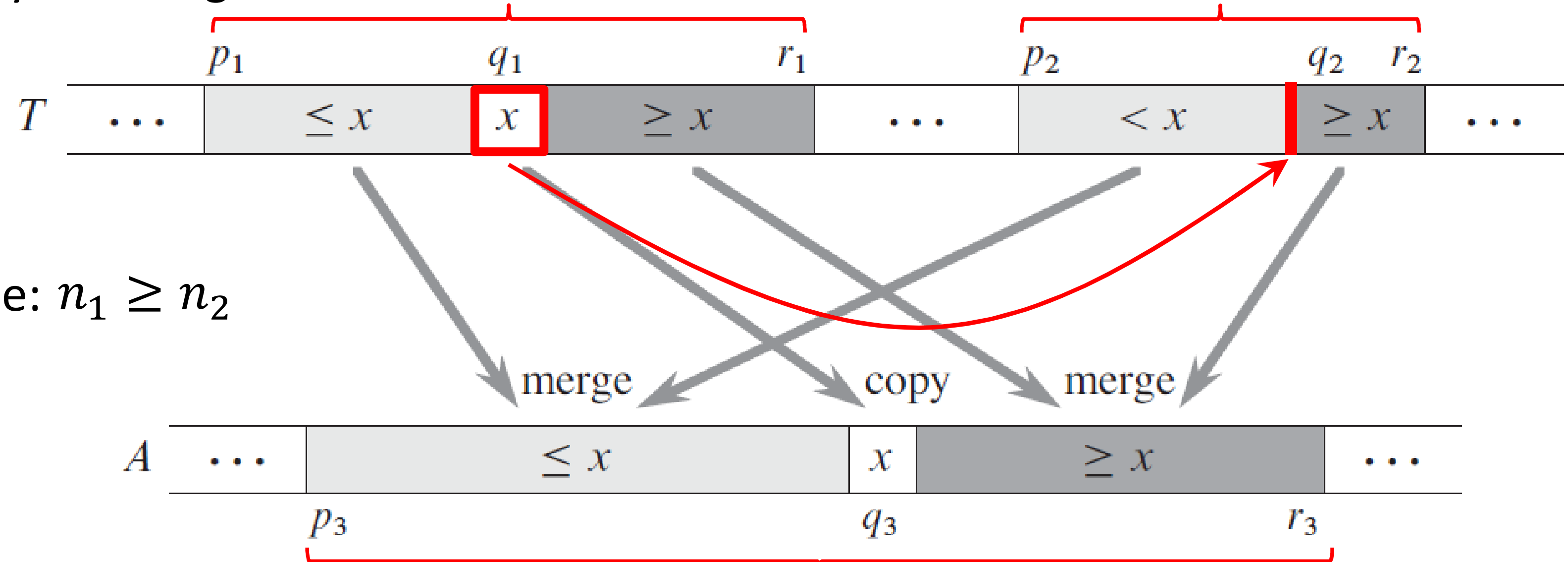
Source: Cormen et al., "Introduction to Algorithms", 3rd Edition

Step 1: Find $x = T[q_1]$, where q_1 is the midpoint of $T[p_1 \mathinner{.\,.} r_1]$

Parallel Merge

$n_1 = r_1 - p_1 + 1$ $\qquad$ $n_2 = r_2 - p_2 + 1$

subarrays to merge: $T[p_1..r_1]$ $\qquad$ $T[p_2..r_2]$



suppose: $n_1 \geq n_2$

merged output: $A[p_3..r_3]$

$n_3 = r_3 - p_3 + 1 = n_1 + n_2$

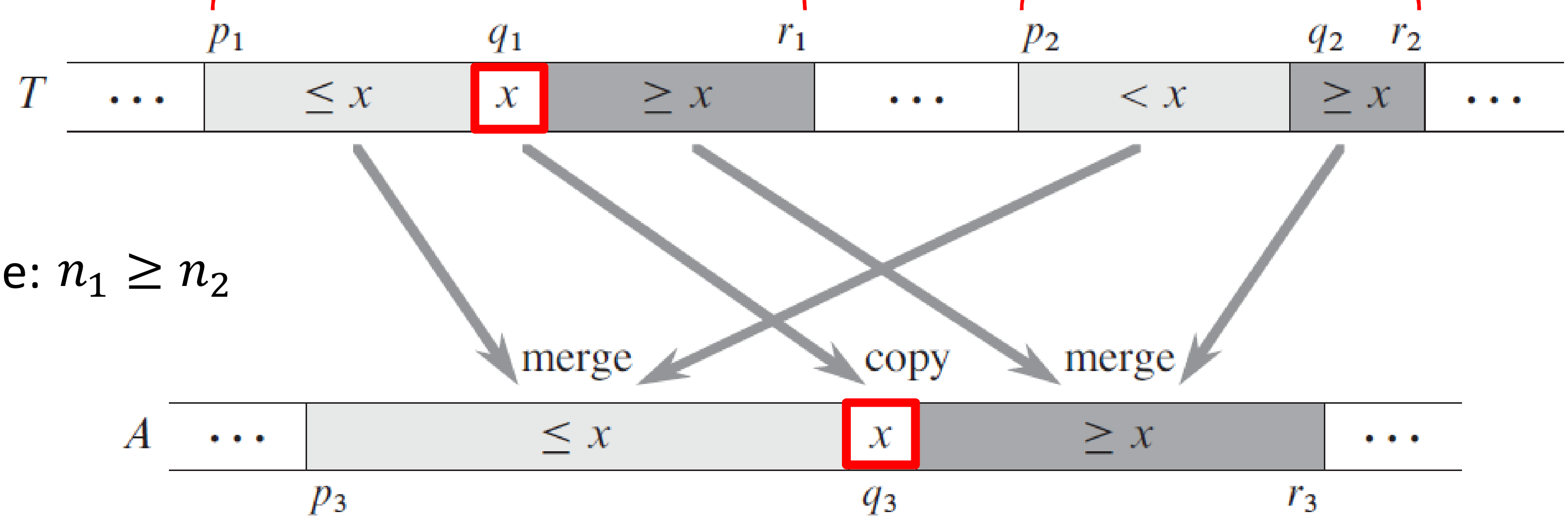
Source: Cormen et al., "Introduction to Algorithms", 3rd Edition

Step 2: Use binary search to find the index q_2 in subarray $T[p_2..r_2]$ so that the subarray would still be sorted if we insert x between $T[q_2 - 1]$ and $T[q_2]$

Parallel Merge

$n_1 = r_1 - p_1 + 1$ $\quad\quad$ $n_2 = r_2 - p_2 + 1$

subarrays to merge: $T[p_1 \mathinner{..} r_1]$ $\quad\quad$ $T[p_2 \mathinner{..} r_2]$



suppose: $n_1 \geq n_2$

merged output: $A[p_3 \mathinner{..} r_3]$

$n_3 = r_3 - p_3 + 1 = n_1 + n_2$

Source: Cormen et al., "Introduction to Algorithms", 3rd Edition

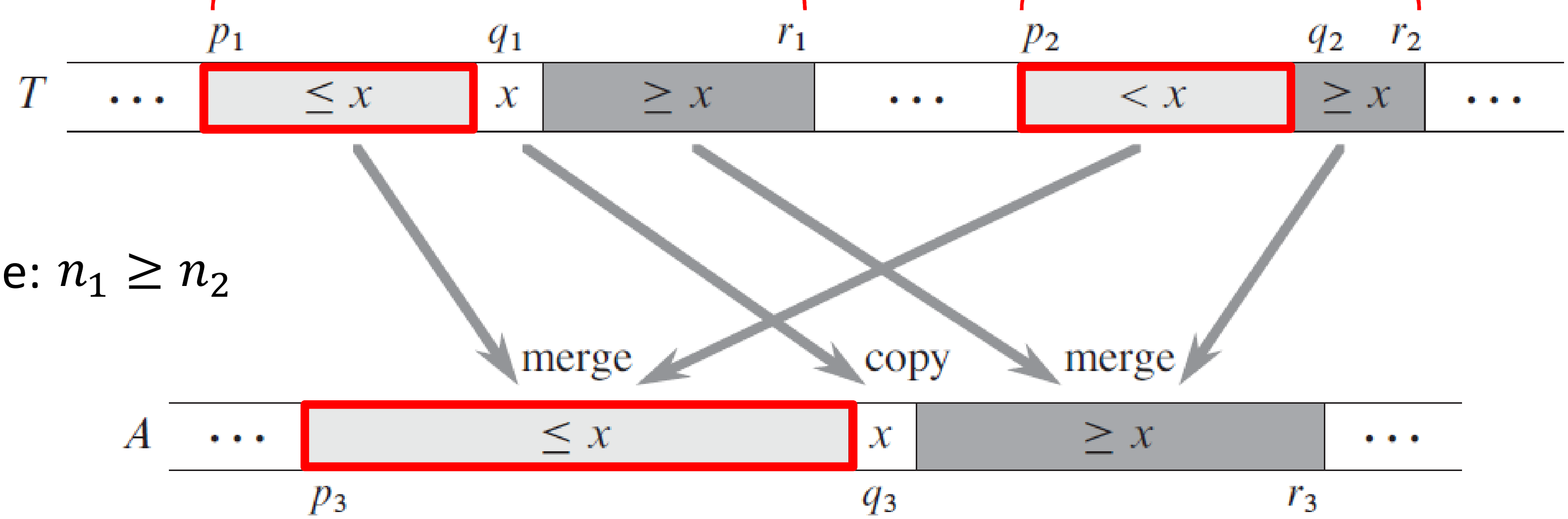
Step 3: Copy x to $A[q_3]$, where $q_3 = p_3 + (q_1 - p_1) + (q_2 - p_2)$

Parallel Merge

$n_1 = r_1 - p_1 + 1$ $\quad$ $n_2 = r_2 - p_2 + 1$

subarrays to merge: $T[p_1 \,..\, r_1]$ $\quad$ $T[p_2 \,..\, r_2]$

suppose: $n_1 \geq n_2$



merged output: $A[p_3 \,..\, r_3]$

$n_3 = r_3 - p_3 + 1 = n_1 + n_2$

Source: Cormen et al., "Introduction to Algorithms", 3rd Edition

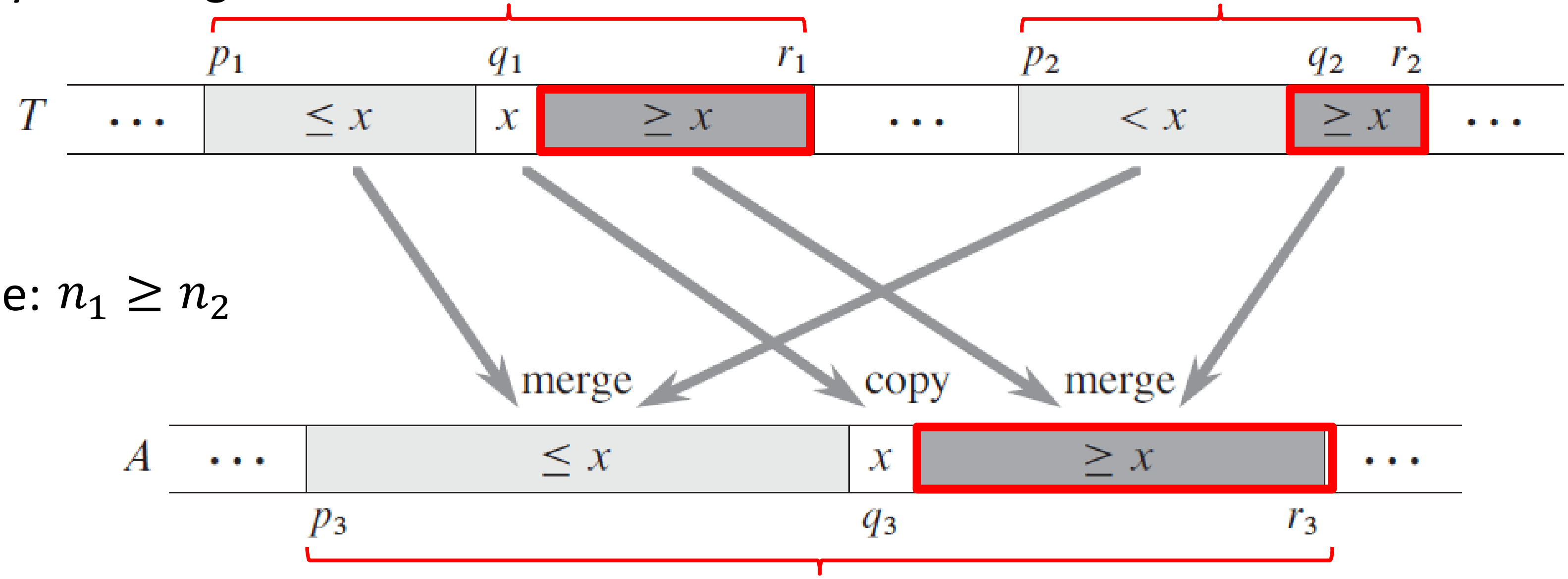
Perform the following two steps in parallel.

Step 4(a): Recursively merge $T[p_1 \,..\, q_1 - 1]$ with $T[p_2 \,..\, q_2 - 1]$, and place the result into $A[p_3 \,..\, q_3 - 1]$

Parallel Merge

$n_1 = r_1 - p_1 + 1$ $\quad$ $n_2 = r_2 - p_2 + 1$

subarrays to merge: $T[p_1..r_1]$ $\quad$ $T[p_2..r_2]$



suppose: $n_1 \geq n_2$

merged output: $A[p_3..r_3]$

$n_3 = r_3 - p_3 + 1 = n_1 + n_2$

Source: Cormen et al., "Introduction to Algorithms", 3rd Edition

Perform the following two steps in parallel.

Step 4(a): Recursively merge $T[p_1..q_1 - 1]$ with $T[p_2..q_2 - 1]$, and place the result into $A[p_3..q_3 - 1]$

Step 4(b): Recursively merge $T[q_1 + 1..r_1]$ with $T[q_2 + 1..r_2]$, and place the result into $A[q_3 + 1..r_3]$

Parallel Merge

Par-Merge ($T, p_1, r_1, p_2, r_2, A, p_3$)

$n_1 \leftarrow r_1 - p_1 + 1, \quad n_2 \leftarrow r_2 - p_2 + 1$
if $n_1 < n_2$ *then*
    $p_1 \leftrightarrow p_2, \quad r_1 \leftrightarrow r_2, \quad n_1 \leftrightarrow n_2$
if $n_1 = 0$ *then return*
else
    $q_1 \leftarrow \lfloor (p_1 + r_1) / 2 \rfloor$
    $q_2 \leftarrow$ *Binary-Search* ($T[q_1], T, p_2, r_2$)
    $q_3 \leftarrow p_3 + (q_1 - p_1) + (q_2 - p_2)$
    $A[q_3] \leftarrow T[q_1]$
    spawn *Par-Merge* ($T, p_1, q_1{-}1, p_2, q_2{-}1, A, p_3$)
    Par-Merge ($T, q_1{+}1, r_1, q_2{+}1, r_2, A, q_3{+}1$)
    sync

Parallel Merge

Par-Merge ($T, p_1, r_1, p_2, r_2, A, p_3$)

1. $n_1 \leftarrow r_1 - p_1 + 1, \quad n_2 \leftarrow r_2 - p_2 + 1$
2. *if* $n_1 < n_2$ *then*
3. $\quad p_1 \leftrightarrow p_2, \quad r_1 \leftrightarrow r_2, \quad n_1 \leftrightarrow n_2$
4. *if* $n_1 = 0$ *then return*
5. *else*
6. $\quad q_1 \leftarrow \lfloor (p_1 + r_1) / 2 \rfloor$
7. $\quad q_2 \leftarrow$ *Binary-Search* ($T[q_1], T, p_2, r_2$)
8. $\quad q_3 \leftarrow p_3 + (q_1 - p_1) + (q_2 - p_2)$
9. $\quad A[q_3] \leftarrow T[q_1]$
10. $\quad$ *spawn Par-Merge* ($T, p_1, q_1 - 1, p_2, q_2 - 1, A, p_3$)
11. $\qquad$ *Par-Merge* ($T, q_1 + 1, r_1, q_2 + 1, r_2, A, q_3 + 1$)
12. $\quad$ *sync*

We have,

$$n_2 \le n_1 \Rightarrow 2n_2 \le n_1 + n_2 = n$$

In the worst case, a recursive call in lines 9-10 merges half the elements of $T[p_1..r_1]$ with all elements of $T[p_2..r_2]$.

Hence, #elements involved in such a call:

$$\left\lfloor \frac{n_1}{2} \right\rfloor + n_2 \le \frac{n_1}{2} + \frac{n_2}{2} + \frac{n_2}{2} = \frac{n_1 + n_2}{2} + \frac{2n_2}{4} \le \frac{n}{2} + \frac{n}{4} = \frac{3n}{4}$$

Parallel Merge

```
Par-Merge ( T, p1, r1, p2, r2, A, p3 )
n1 ← r1 - p1 + 1,   n2 ← r2 - p2 + 1
if n1 < n2 then
    p1 ↔ p2,  r1 ↔ r2,  n1 ↔ n2
if n1 = 0 then return
else
    q1 ← ⌊ ( p1 + r1 ) / 2 ⌋
    q2 ← Binary-Search (  T[ q1 ],  T,  p2, r2 )
    q3 ← p3 + ( q1 - p1 ) + ( q2 - p2 )
    A[ q3 ] ← T[ q1 ]
    spawn Par-Merge ( T, p1, q1-1, p2, q2-1, A, p3 )
          Par-Merge ( T, q1+1, r1, q2+1, r2, A, q3+1 )
    sync
```

Span:

$$T_\infty(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ T_\infty\left(\frac{3n}{4}\right) + \Theta(\log n), & otherwise. \end{cases}$$

$$= \Theta(\log^2 n) \qquad \text{[MT Case 2]}$$

Work:

Clearly, $T_1(n) = \Omega(n)$

We show below that, $T_1(n) = O(n)$

For some $\alpha \in \left[\frac{1}{4}, \frac{3}{4}\right]$, we have the following recurrence,

$$T_1(n) = T_1(\alpha n) + T_1\big((1-\alpha)n\big) + O(\log n)$$

Assuming $T_1(n) \le c_1 n - c_2 \log n$ for positive constants c_1 and c_2, and substituting on the right hand side of the above recurrence gives us: $T_1(n) \le c_1 n - c_2 \log n \ = O(n)$.

Hence, $T_1(n) = \Theta(n)$.

Parallel Merge Sort with Parallel Merge

```
Par-Merge-Sort ( A, p, r )    { sort the elements in A[ p … r ] }

if p < r then
    q ← ⌊ ( p + r ) / 2 ⌋
    spawn Merge-Sort ( A, p, q )
          Merge-Sort ( A, q + 1, r )
    sync
    Par-Merge ( A, p, q, r )
```

Work: $T_1(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ 2T_1\left(\frac{n}{2}\right) + \Theta(n), & otherwise. \end{cases}$

$= \Theta(n \log n)$ [MT Case 2]

Span: $T_\infty(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ T_\infty\left(\frac{n}{2}\right) + \Theta(\log^2 n), & otherwise. \end{cases}$

$= \Theta(\log^3 n)$ [MT Case 2]

Parallelism: $\frac{T_1(n)}{T_\infty(n)} = \Theta\left(\frac{n}{\log^2 n}\right)$

Parallel Prefix Sums

Parallel Prefix Sums

Input: A sequence of n elements $\{x_1, x_2, \dots, x_n\}$ drawn from a set S with a binary associative operation, denoted by $\oplus$.

Output: A sequence of n partial sums $\{s_1, s_2, \dots, s_n\}$, where $s_i = x_1 \oplus x_2 \oplus \dots \oplus x_i$ for $1 \le i \le n$.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
5	3	7	1	3	6	2	4

$\oplus$ = binary addition

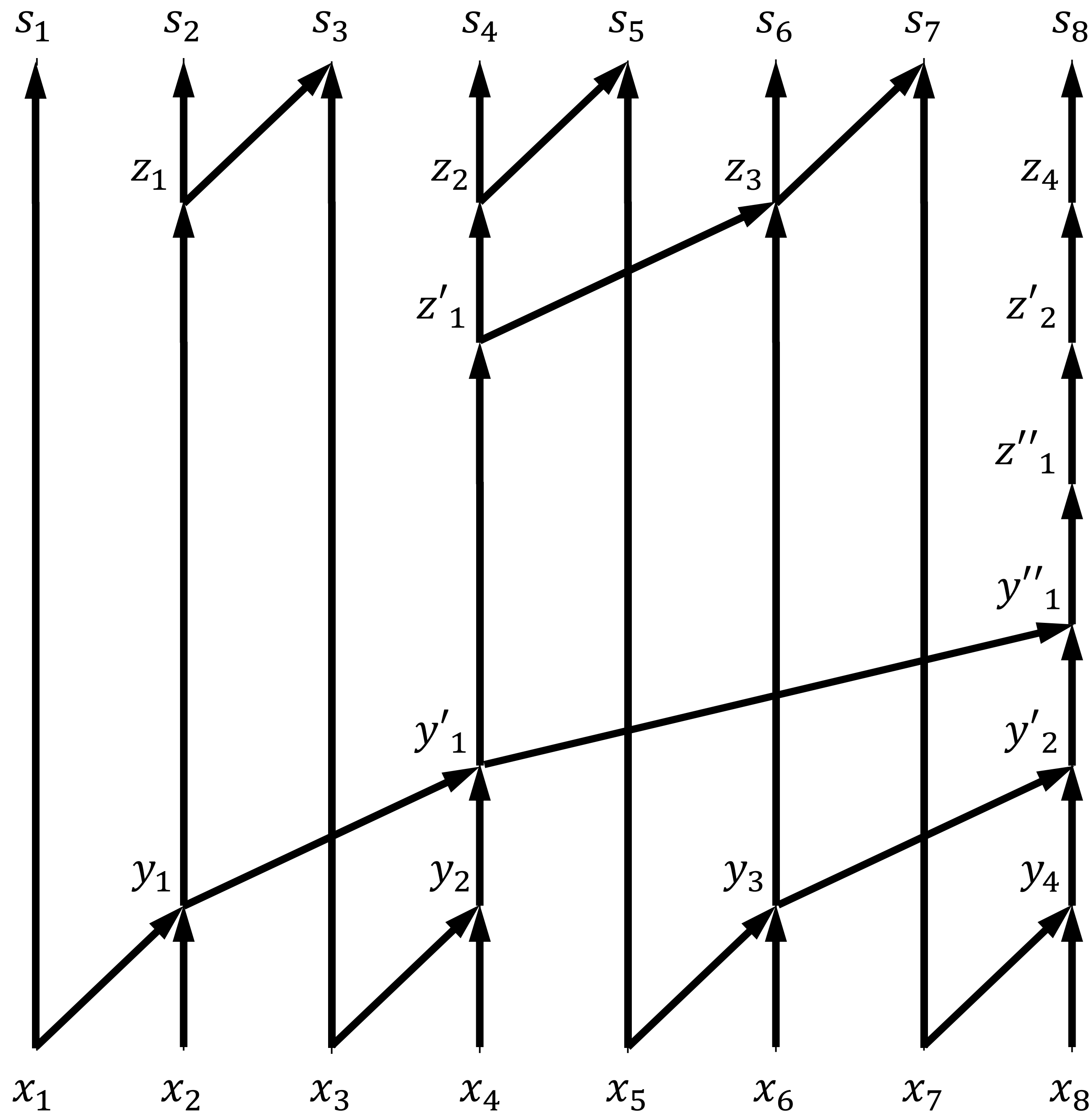
5	8	15	16	19	25	27	31
s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8

Parallel Prefix Sums

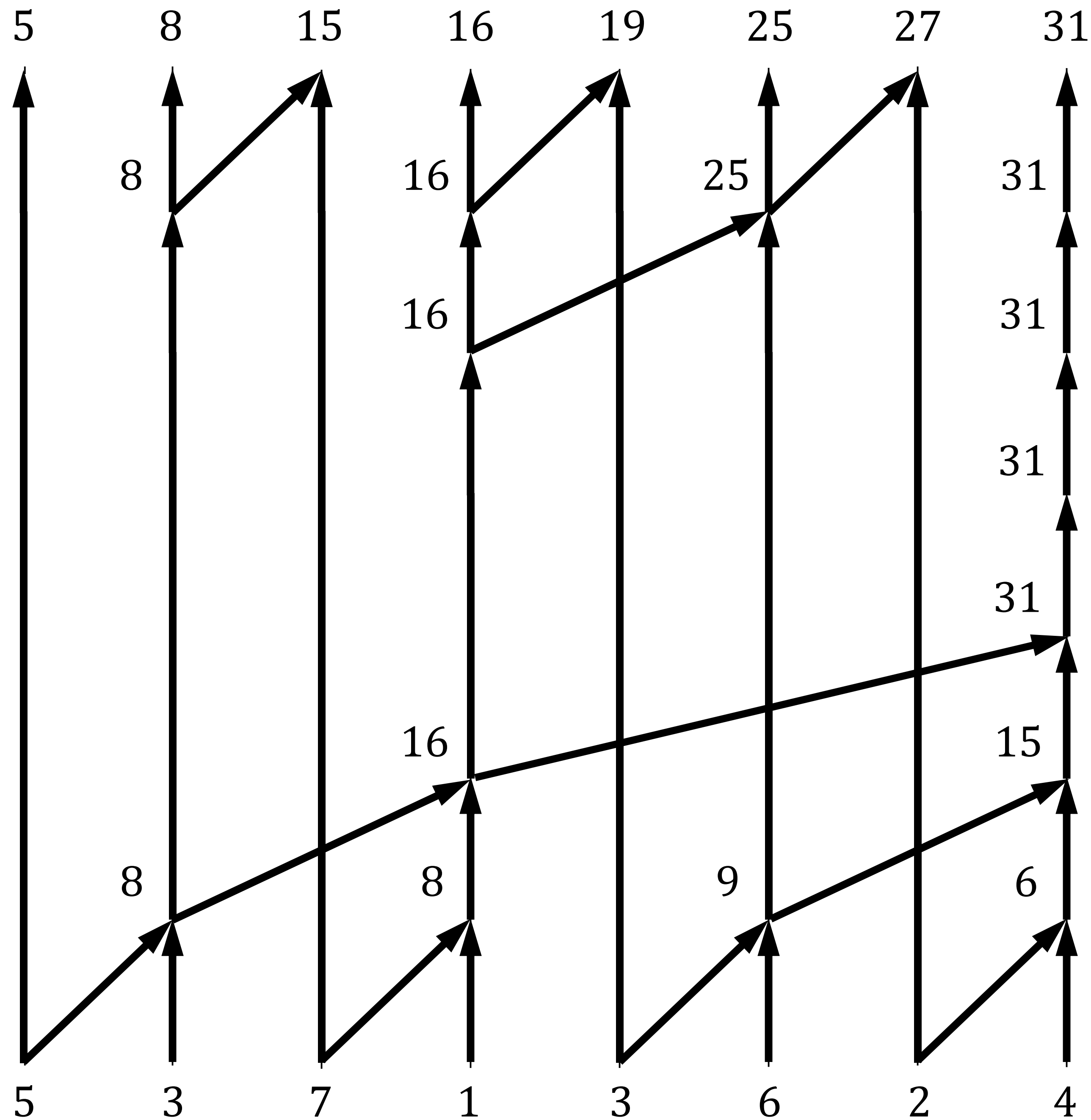
Prefix-Sum ($\langle x_1, x_2, \ldots, x_n \rangle$, $\oplus$) { $n = 2^k$ *for some* $k \geq 0$. *Return prefix sums* $\langle s_1, s_2, \ldots, s_n \rangle$ }

if $n = 1$ *then*
    $s_1 \leftarrow x_1$
else
    parallel for $i \leftarrow 1$ *to* $n/2$ *do*
        $y_i \leftarrow x_{2i-1} \oplus x_{2i}$
    $\langle z_1, z_2, \ldots, z_{n/2} \rangle \leftarrow$ *Prefix-Sum*($\langle y_1, y_2, \ldots, y_{n/2} \rangle$, $\oplus$)
    parallel for $i \leftarrow 1$ *to* n *do*
        if $i = 1$ *then* $s_1 \leftarrow x_1$
        else if $i = even$ *then* $s_i \leftarrow z_{i/2}$
            else $s_i \leftarrow z_{(i-1)/2} \oplus x_i$
return $\langle s_1, s_2, \ldots, s_n \rangle$

Parallel Prefix Sums



Parallel Prefix Sums



Parallel Prefix Sums

Prefix-Sum ($\langle x_1, x_2, \ldots, x_n \rangle, \oplus$) { $n = 2^k$ *for some* $k \geq 0$. *Return prefix sums* $\langle s_1, s_2, \ldots, s_n \rangle$ }

if $n = 1$ *then*
    $s_1 \leftarrow x_1$
else
    parallel for $i \leftarrow 1$ *to* $n/2$ *do*
        $y_i \leftarrow x_{2i-1} \oplus x_{2i}$
    $\langle z_1, z_2, \ldots, z_{n/2} \rangle \leftarrow$ *Prefix-Sum*($\langle y_1, y_2, \ldots, y_{n/2} \rangle, \oplus$)
    parallel for $i \leftarrow 1$ *to* n *do*
        if $i = 1$ *then* $s_1 \leftarrow x_1$
        else if $i = even$ *then* $s_i \leftarrow z_{i/2}$
            else $s_i \leftarrow z_{(i-1)/2} \oplus x_i$
return $\langle s_1, s_2, \ldots, s_n \rangle$

Work:

$$T_1(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ T_1\left(\frac{n}{2}\right) + \Theta(n), & otherwise. \end{cases}$$

$$= \Theta(n)$$

Span:

$$T_\infty(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ T_\infty\left(\frac{n}{2}\right) + \Theta(1), & otherwise. \end{cases}$$

$$= \Theta(\log n)$$

Parallelism: $\frac{T_1(n)}{T_\infty(n)} = \Theta\left(\frac{n}{\log n}\right)$

Observe that we have assumed here that a *parallel for loop* can be executed in $\Theta(1)$ time. But recall that *cilk_for* is implemented using divide-and-conquer, and so in practice, it will take $\Theta(\log n)$ time. In that case, we will have $T_\infty(n) = \Theta(\log^2 n)$, and parallelism $= \Theta(n/\log^2 n)$.

Parallel Partition

Parallel Partition

Input: An array $A[\ q : r\ ]$ of distinct elements, and an element x from $A[\ q : r\ ]$.

Output: Rearrange the elements of $A[\ q : r\ ]$, and return an index $k \in [\ q, r\ ]$, such that all elements in $A[\ q : k - 1\ ]$ are smaller than x, all elements in $A[\ k + 1 : r\ ]$ are larger than x, and $A[\ k\ ] = x$.

```
Par-Partition ( A[ q : r ], x )
n ← r − q + 1
if n = 1 then return q
array  B[ 0: n − 1 ], lt[ 0: n − 1 ], gt[ 0: n − 1 ]
parallel for i ← 0 to n − 1 do
      B[ i ] ← A[ q + i ]
      if B[ i ] < x then lt[ i ] ← 1 else lt[ i ] ← 0
      if B[ i ] > x then gt[ i ] ← 1 else gt[ i ] ← 0
lt [ 0: n − 1 ] ← Par-Prefix-Sum (  lt[ 0: n − 1 ],  + )
gt[ 0: n − 1 ] ← Par-Prefix-Sum (  gt[ 0: n − 1 ],  + )
k ← q + lt [ n − 1 ],  A[ k ] ← x
parallel for i ← 0 to n − 1 do
      if B[ i ] < x then A[ q + lt [ i ] − 1 ] ← B[ i ]
      else if B[ i ] > x then A[ k + gt [ i ] ] ← B[ i ]
return k
```

Parallel Partition

***A*:**	9	5	7	11	1	3	8	14	4	21	$x = 8$

Parallel Partition

A:

9	5	7	11	1	3	8	14	4	21

***x* = 8**

B:

0	1	2	3	4	5	6	7	8	9
9	5	7	11	1	3	8	14	4	21

***lt*:**

0	1	2	3	4	5	6	7	8	9
0	1	1	0	1	1	0	0	1	0

***gt*:**

0	1	2	3	4	5	6	7	8	9
1	0	0	1	0	0	0	1	0	1

Parallel Partition

A:

9	5	7	11	1	3	8	14	4	21

***x* = 8**

B:

0	1	2	3	4	5	6	7	8	9
9	5	7	11	1	3	8	14	4	21

lt:

0	1	2	3	4	5	6	7	8	9
0	1	1	0	1	1	0	0	1	0

lt:

0	1	2	2	3	4	4	4	5	5

prefix sum

gt:

0	1	2	3	4	5	6	7	8	9
1	0	0	1	0	0	0	1	0	1

gt:

1	1	1	2	2	2	2	3	3	4

prefix sum

Parallel Partition

$x = 8$

A:	9	5	7	11	1	3	8	14	4	21

	0	1	2	3	4	5	6	7	8	9
B:	9	5	7	11	1	3	8	14	4	21

	0	1	2	3	4	5	6	7	8	9
lt:	0	1	1	0	1	1	0	0	1	0
lt:	0	1	2	2	3	4	4	4	5	5

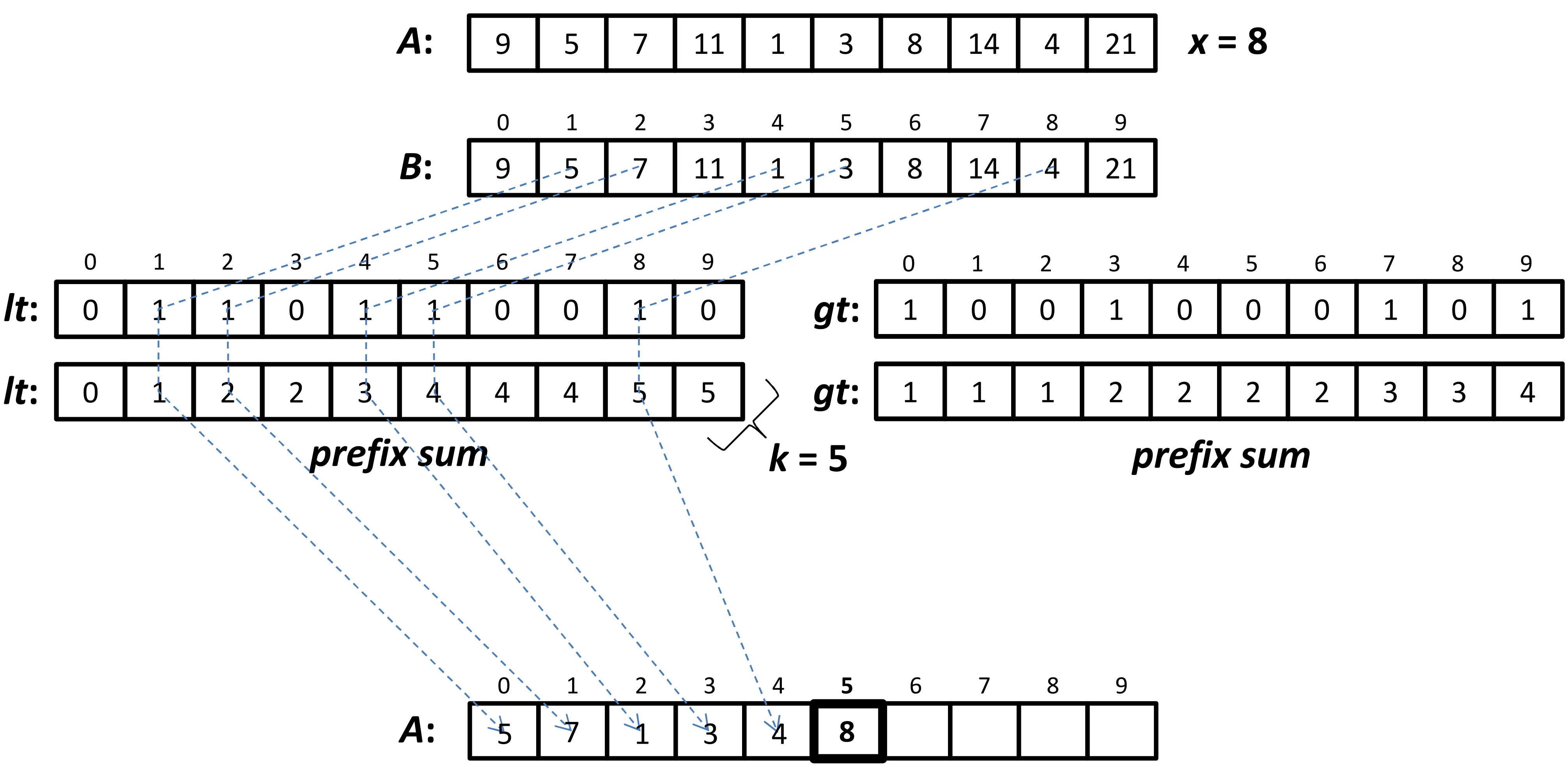
prefix sum

	0	1	2	3	4	5	6	7	8	9
gt:	1	0	0	1	0	0	0	1	0	1
gt:	1	1	1	2	2	2	2	3	3	4

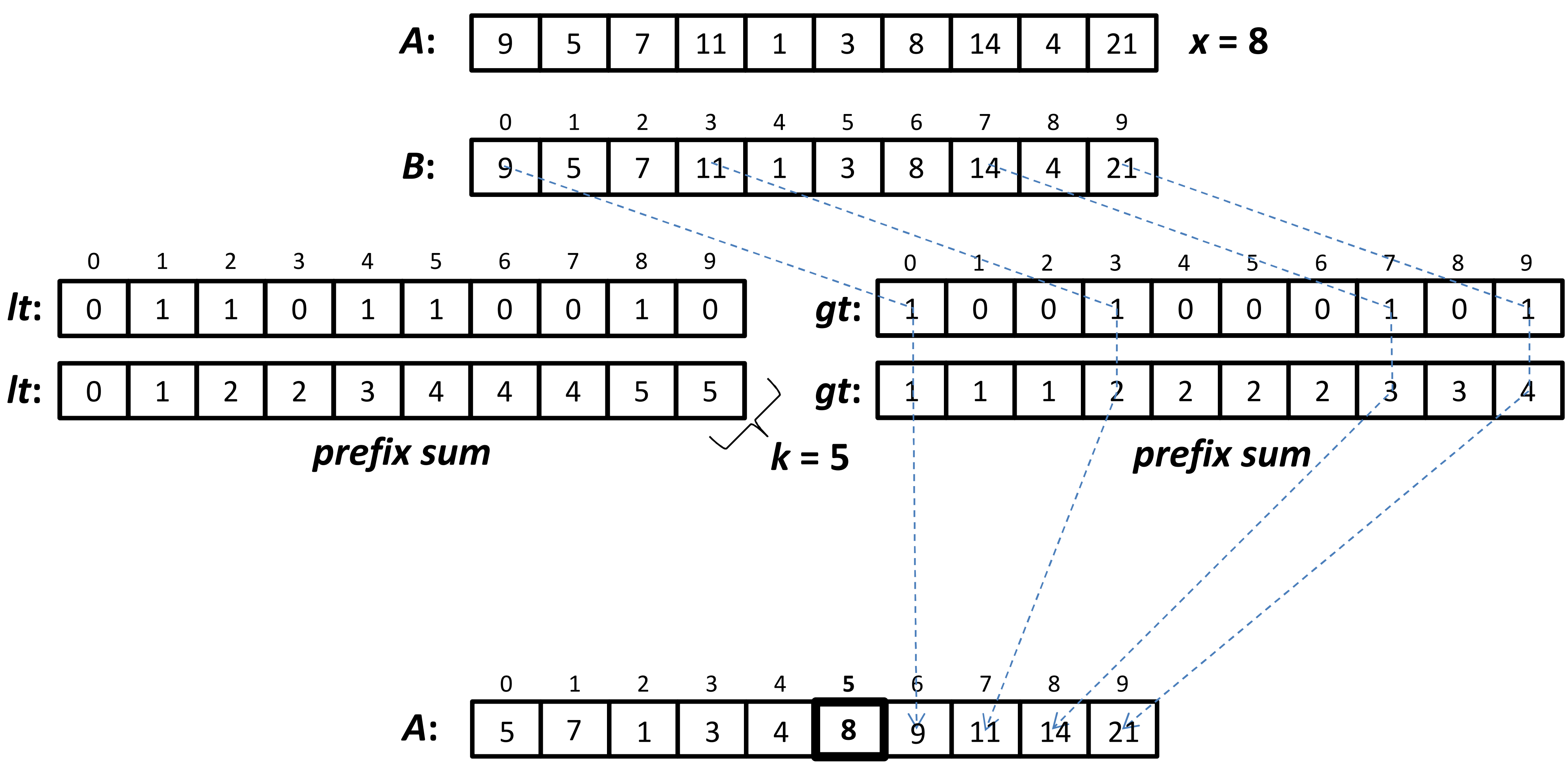
prefix sum

	0	1	2	3	4	5	6	7	8	9
A:										

Parallel Partition



Parallel Partition



Parallel Partition

***A*:**	9	5	7	11	1	3	8	14	4	21

***x* = 8**

	0	1	2	3	4	5	6	7	8	9
***B*:**	9	5	7	11	1	3	8	14	4	21

	0	1	2	3	4	5	6	7	8	9
***lt*:**	0	1	1	0	1	1	0	0	1	0
***lt*:**	0	1	2	2	3	4	4	4	5	5

prefix sum

***k* = 5**

	0	1	2	3	4	5	6	7	8	9
***gt*:**	1	0	0	1	0	0	0	1	0	1
***gt*:**	1	1	1	2	2	2	2	3	3	4

prefix sum

	0	1	2	3	4	**5**	6	7	8	9
***A*:**	5	7	1	3	4	**8**	9	11	14	21

Parallel Partition: Analysis

```
Par-Partition ( A[ q : r ], x )
1.  n ← r − q + 1
2.  if n = 1 then return q
3.  array B[ 0: n − 1 ], lt[ 0: n − 1 ], gt[ 0: n − 1 ]
4.  parallel for i ← 0 to n − 1 do
5.      B[ i ] ← A[ q + i ]
6.      if B[ i ] < x then lt[ i ] ← 1 else lt[ i ] ← 0
7.      if B[ i ] > x then gt[ i ] ← 1 else gt[ i ] ← 0
8.  lt [ 0: n − 1 ] ← Par-Prefix-Sum ( lt[ 0: n − 1 ], + )
9.  gt[ 0: n − 1 ] ← Par-Prefix-Sum ( gt[ 0: n − 1 ], + )
10. k ← q + lt [ n − 1 ],  A[ k ] ← x
11. parallel for i ← 0 to n − 1 do
12.     if B[ i ] < x then A[ q + lt [ i ] − 1 ] ← B[ i ]
13.     else if B[ i ] > x then A[ k + gt [ i ] ] ← B[ i ]
14. return k
```

Work:

$$\begin{aligned} T_1(n) &= \Theta(n) && \text{[lines 1 – 7]} \\ &\quad + \Theta(n) && \text{[lines 8 – 9]} \\ &\quad + \Theta(n) && \text{[lines 10 – 14]} \\ &= \Theta(n) \end{aligned}$$

Span:

Assuming $\log n$ depth for *parallel for* loops:

$$\begin{aligned} T_\infty(n) &= \Theta(\log n) && \text{[lines 1 – 7]} \\ &\quad + \Theta(\log^2 n) && \text{[lines 8 – 9]} \\ &\quad + \Theta(\log n) && \text{[lines 10 – 14]} \\ &= \Theta(\log^2 n) \end{aligned}$$

Parallelism: $\frac{T_1(n)}{T_\infty(n)} = \Theta\left(\frac{n}{\log^2 n}\right)$

Parallel Quicksort

Randomized Parallel QuickSort

Input: An array $A[\ q : r\ ]$ of distinct elements.

Output: Elements of $A[\ q : r\ ]$ sorted in increasing order of value.

```
Par-Randomized-QuickSort ( A[ q : r ] )
    n ← r − q + 1
    if n ≤ 30 then
        sort A[ q : r ] using any sorting algorithm
    else
        select a random element x from A[ q : r ]
        k ← Par-Partition ( A[ q : r ], x )
        spawn Par-Randomized-QuickSort ( A[ q : k − 1 ] )
        Par-Randomized-QuickSort ( A[ k + 1 : r ] )
        sync
```

Randomized Parallel QuickSort: Analysis

```
Par-Randomized-QuickSort ( A[ q : r ] )
1. n ← r − q + 1
2. if n ≤ 30 then
3.       sort A[ q : r ] using any sorting algorithm
4. else
5.      select a random element x from A[ q : r ]
6.      k ← Par-Partition (  A[ q : r ],  x )
7.      spawn Par-Randomized-QuickSort ( A[ q : k − 1 ] )
8.      Par-Randomized-QuickSort ( A[ k + 1 : r ] )
9.      sync
```

Lines 1—6 take $\Theta(\log^2 n)$ parallel time and perform $\Theta(n)$ work.

Also the recursive spawns in lines 7—8 work on disjoint parts of $A[\ q : r\ ]$. So the upper bounds on the parallel time and the total work in each level of recursion are $\Theta(\log^2 n)$ and $\Theta(n)$, respectively.

Hence, if D is the *recursion depth* of the algorithm, then

$$T_1(n) = \mathrm{O}(nD) \text{ and } T_\infty(n) = \mathrm{O}(D \log^2 n)$$

Randomized Parallel QuickSort: Analysis

```
Par-Randomized-QuickSort ( A[ q : r ] )
    n ← r − q + 1
    if n ≤ 30 then
        sort A[ q : r ] using any sorting algorithm
    else
        select a random element x from A[ q : r ]
        k ← Par-Partition ( A[ q : r ], x )
        spawn Par-Randomized-QuickSort ( A[ q : k − 1 ] )
        Par-Randomized-QuickSort ( A[ k + 1 : r ] )
        sync
```

We already proved that w.h.p. recursion depth, $D = \mathrm{O}(\log n)$.

Hence, with high probability,

$$T_1(n) = \mathrm{O}(n \log n) \text{ and } T_\infty(n) = \mathrm{O}(\log^3 n)$$