

CSE 548: Analysis of Algorithms

Lecture 2

(Divide-and-Conquer Algorithms: Integer Multiplication)

Rezaul A. Chowdhury

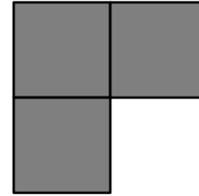
Department of Computer Science

SUNY Stony Brook

Fall 2016

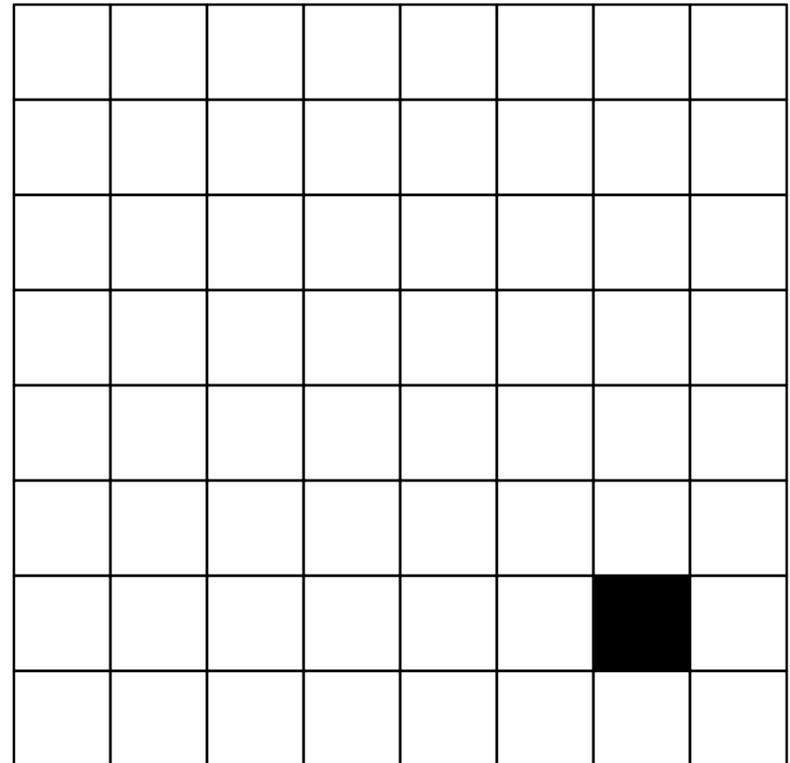
Tromino Cover

A right tromino is an L-shaped tile formed by three adjacent squares.



Puzzle: You are given a $2^n \times 2^n$ board with one missing square.

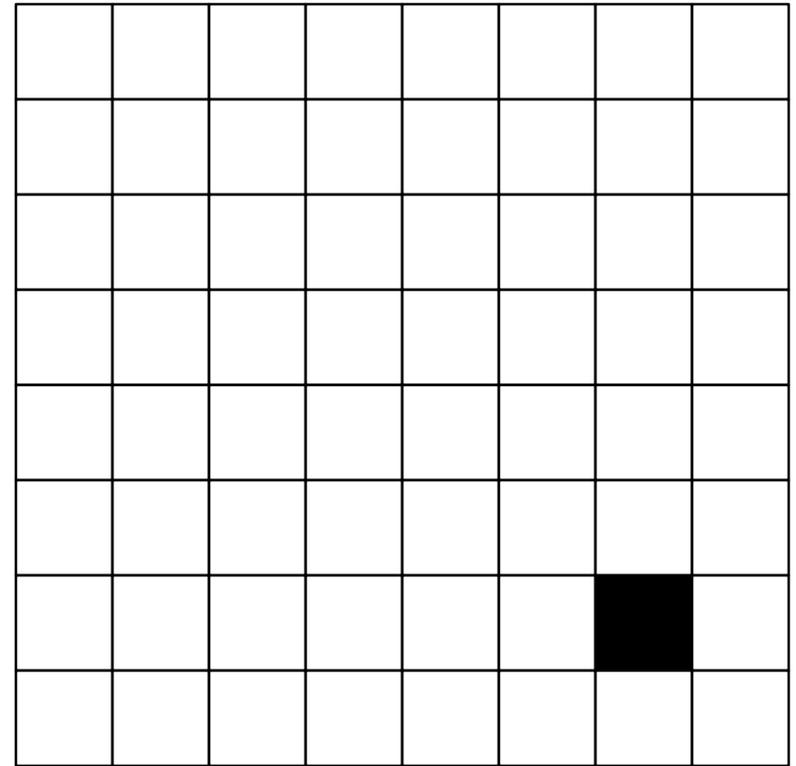
- you must cover all squares except the missing one exactly using right trominoes
- the trominoes must not overlap



$2^3 \times 2^3$ board

Tromino Cover

Steps

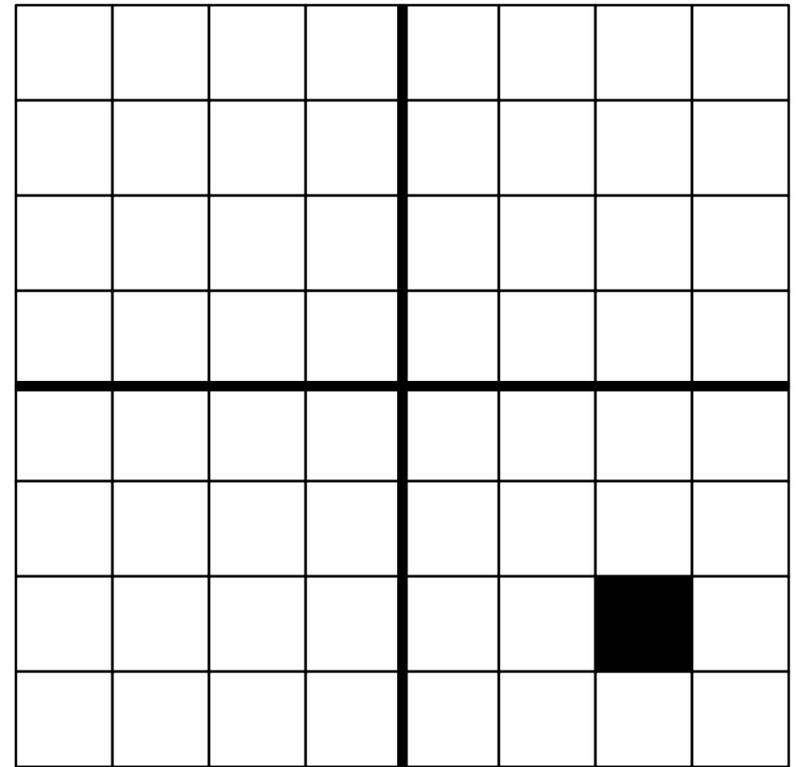


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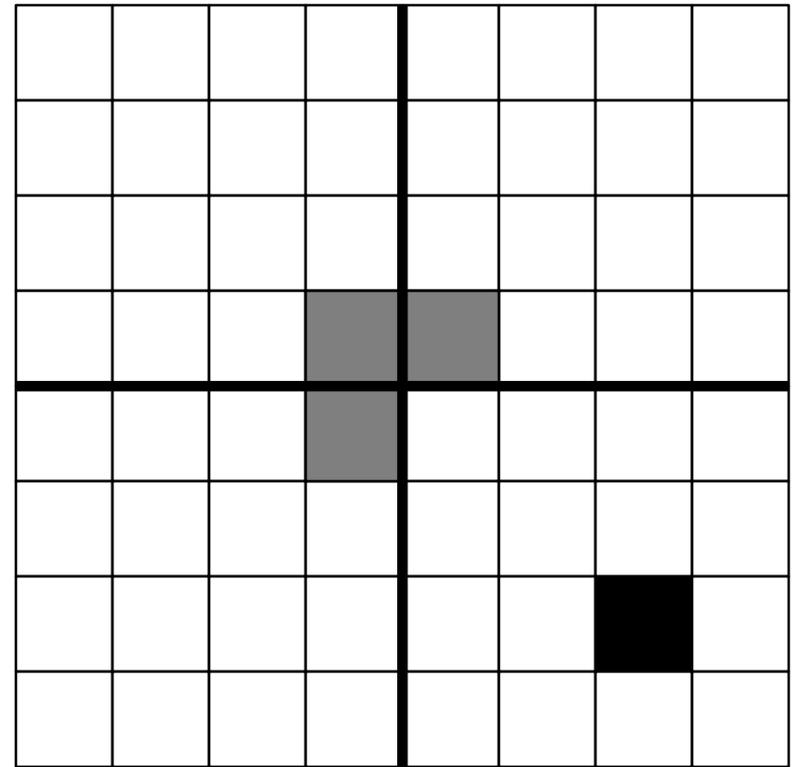


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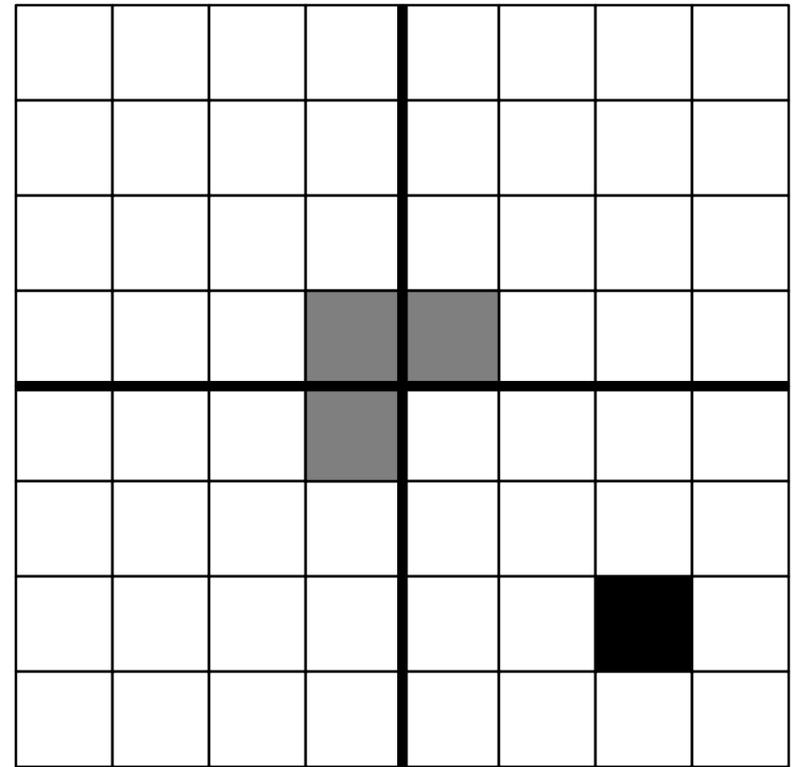
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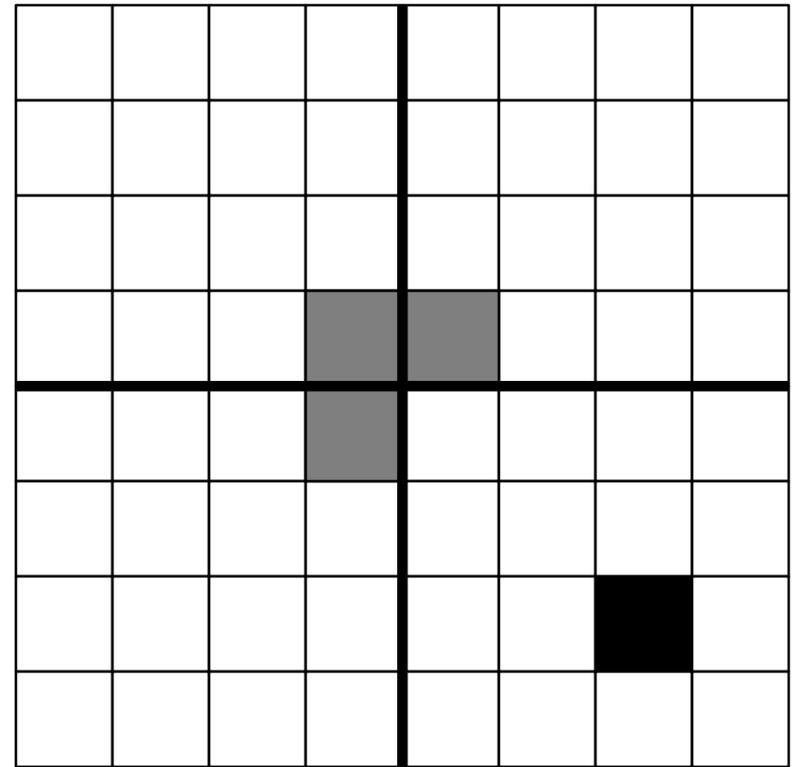
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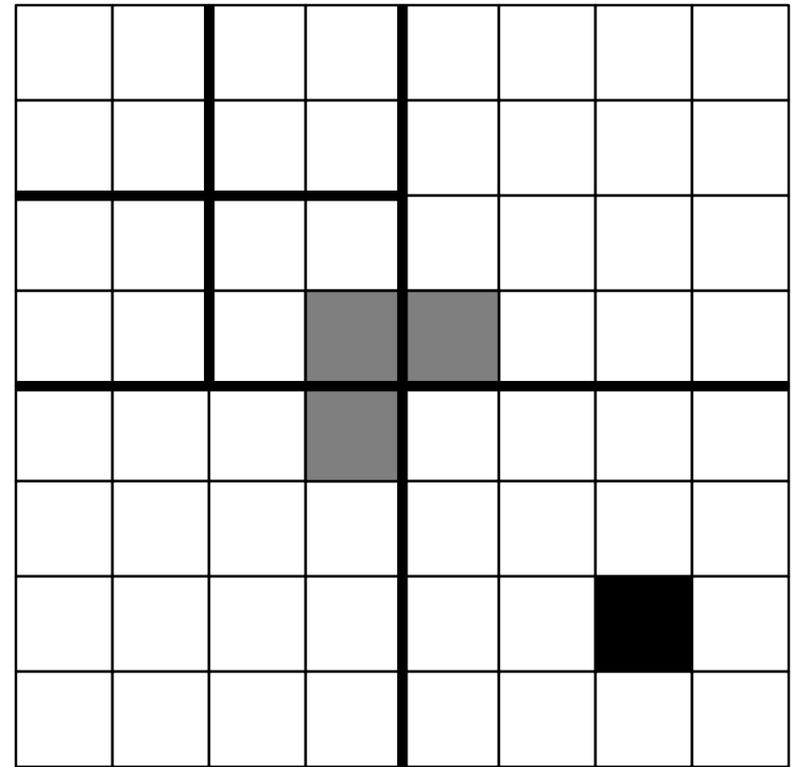
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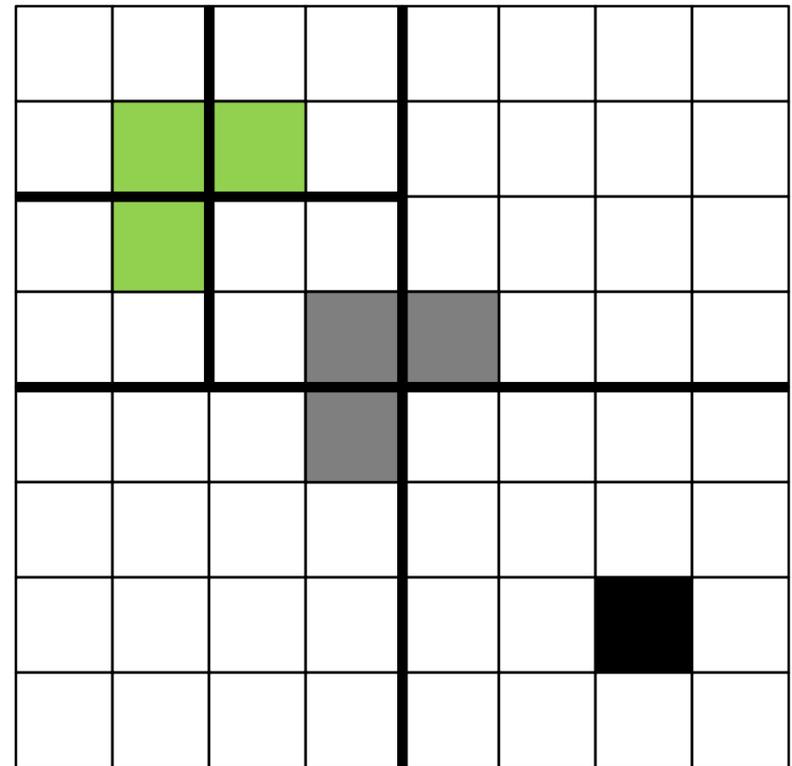
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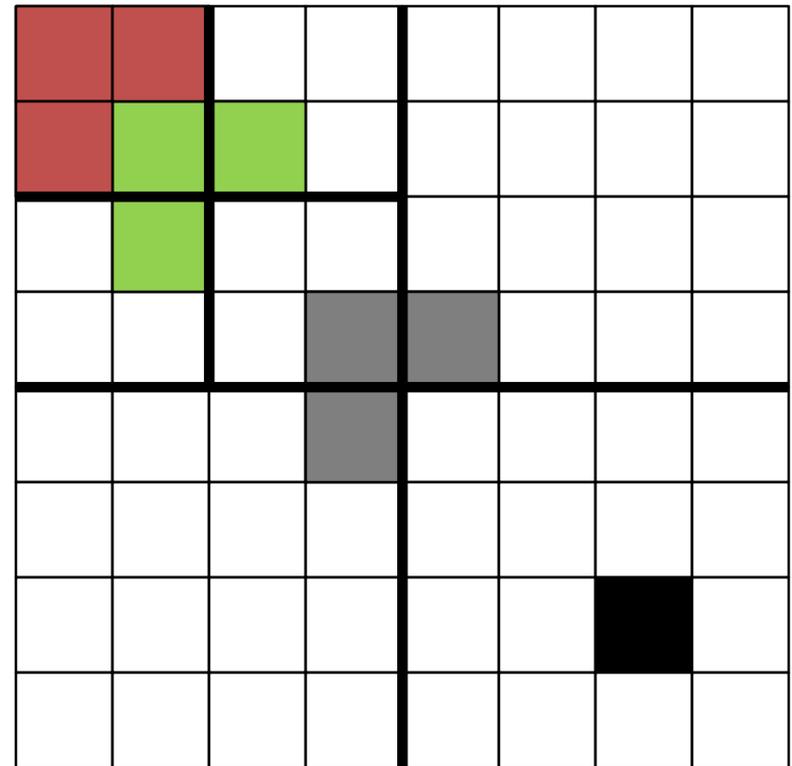
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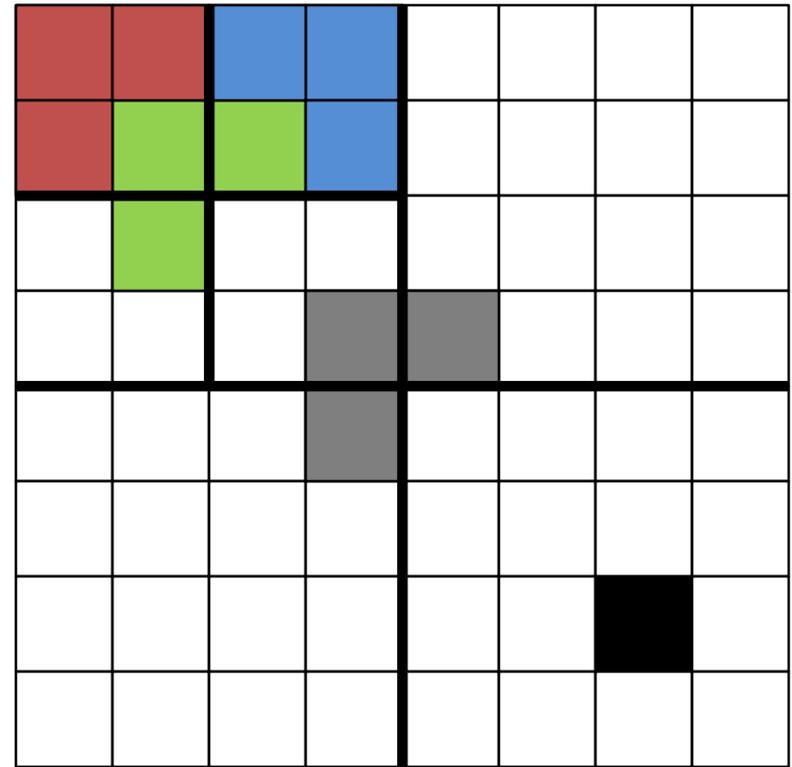
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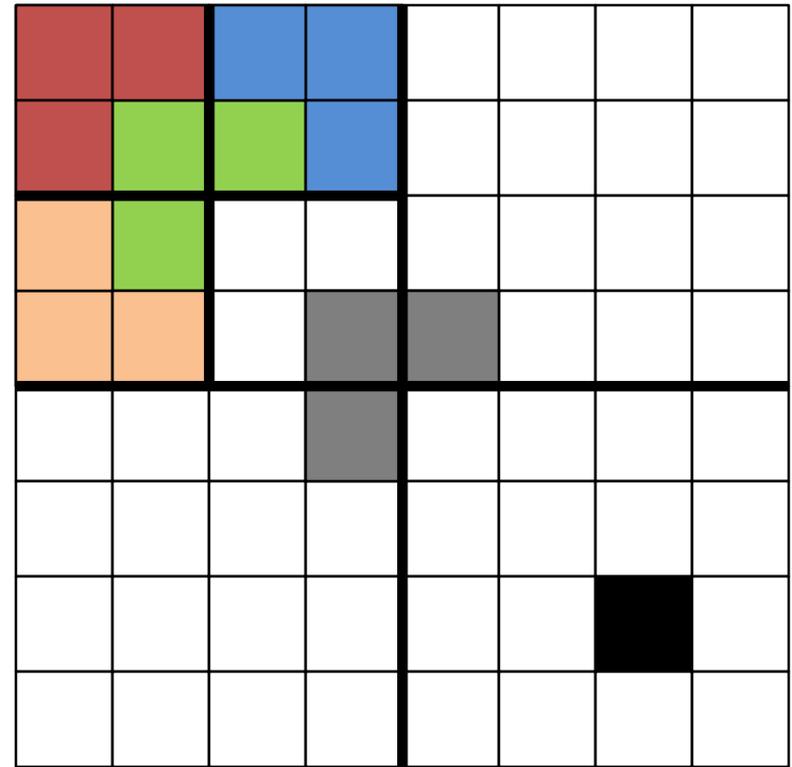
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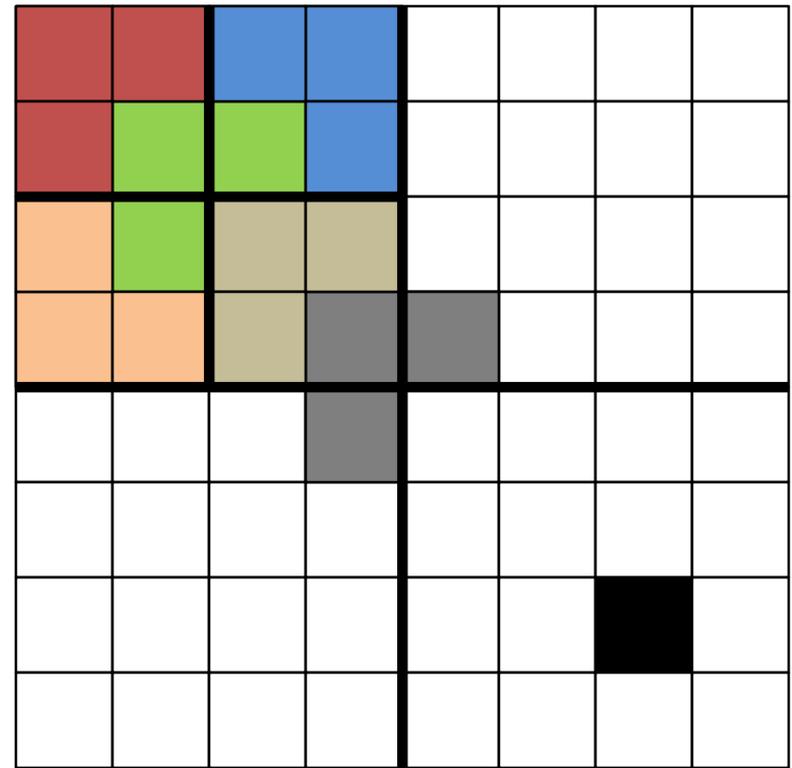
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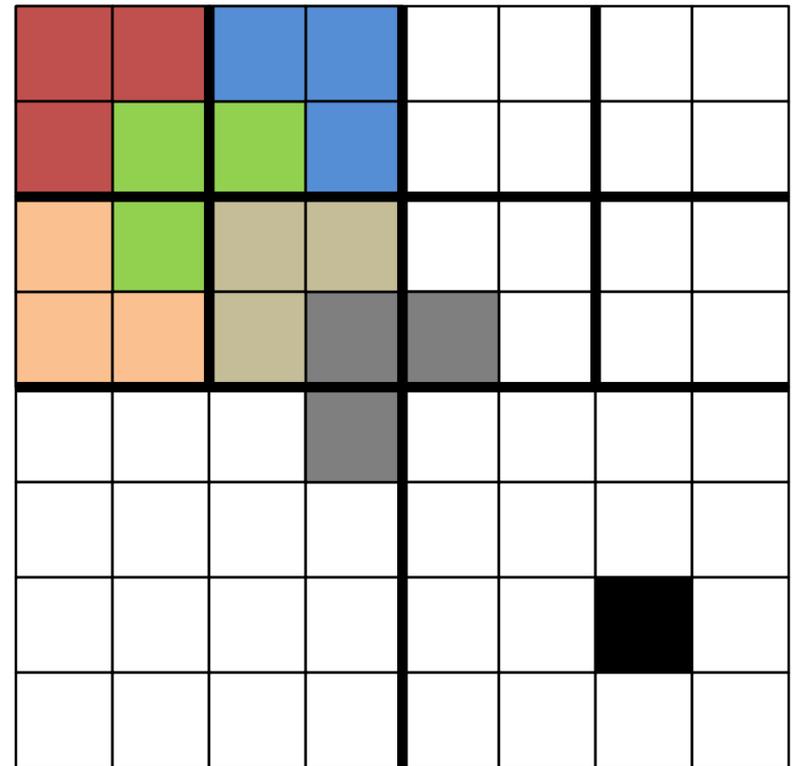
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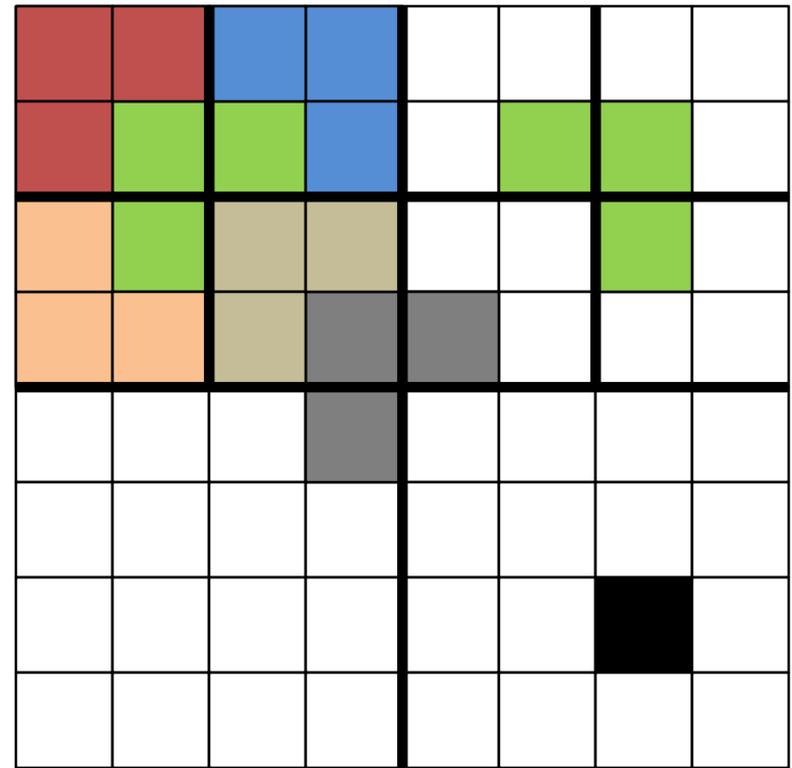
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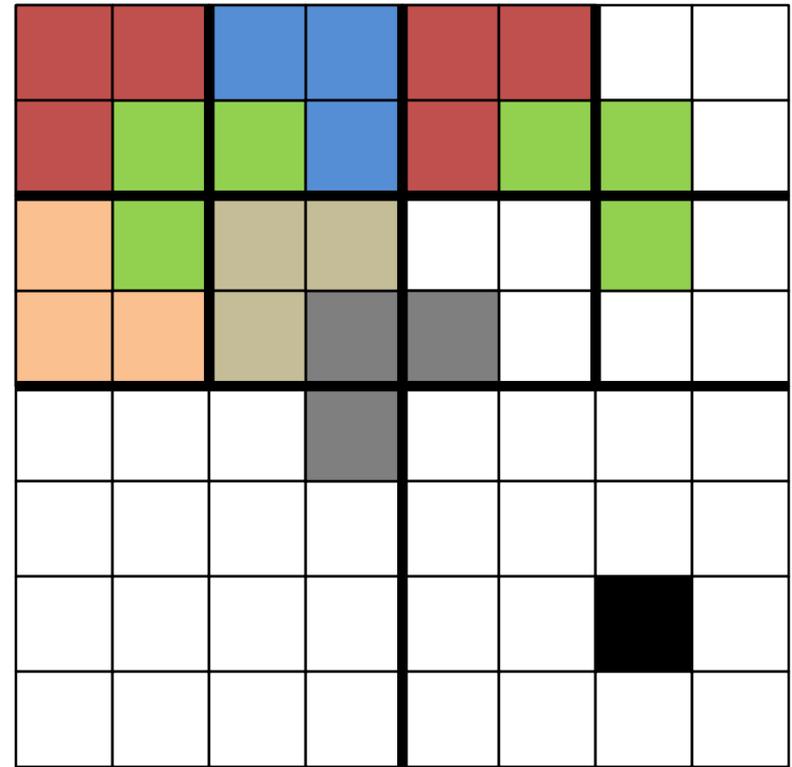
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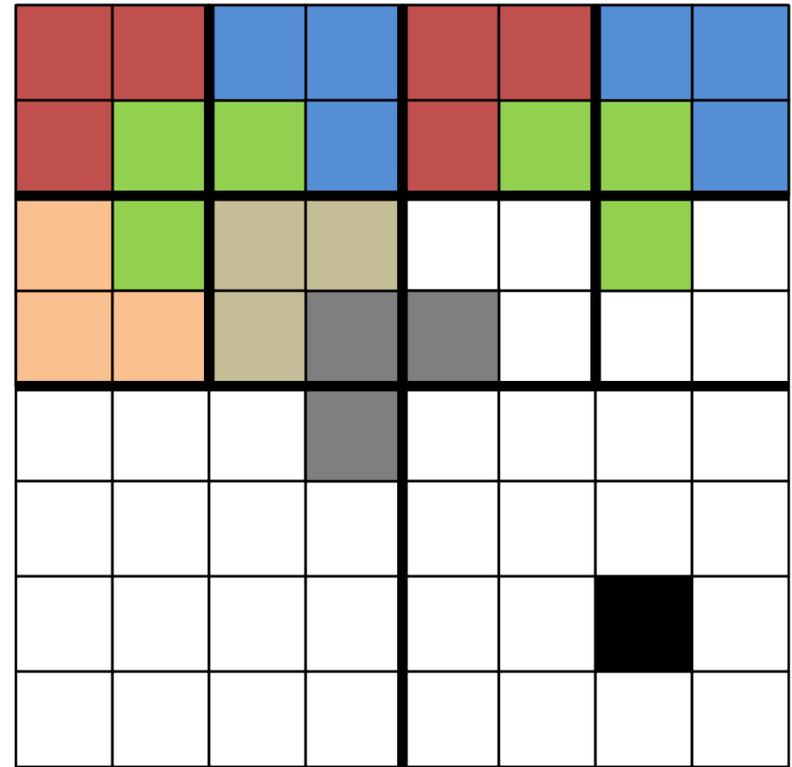
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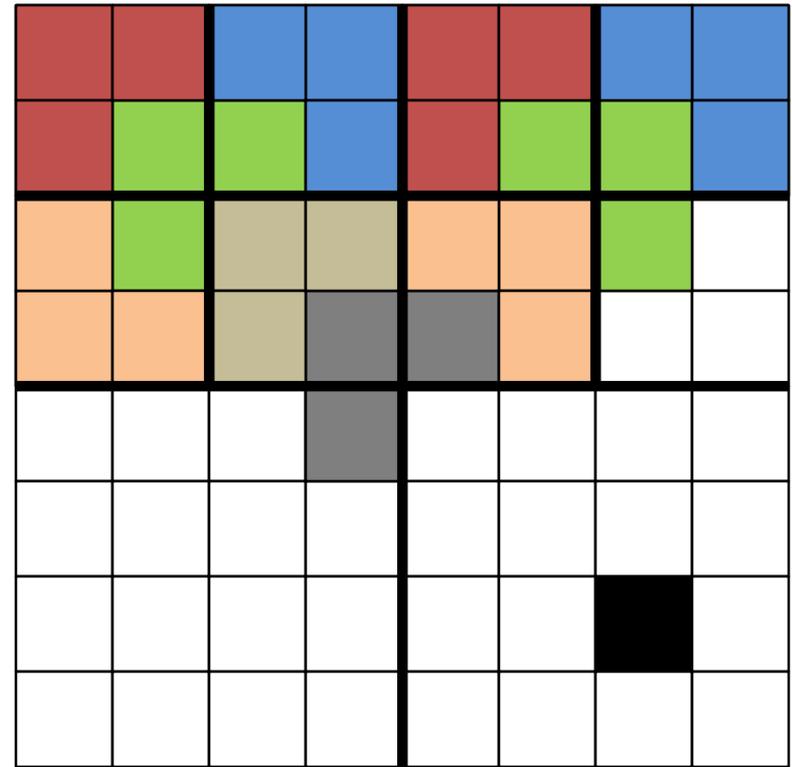
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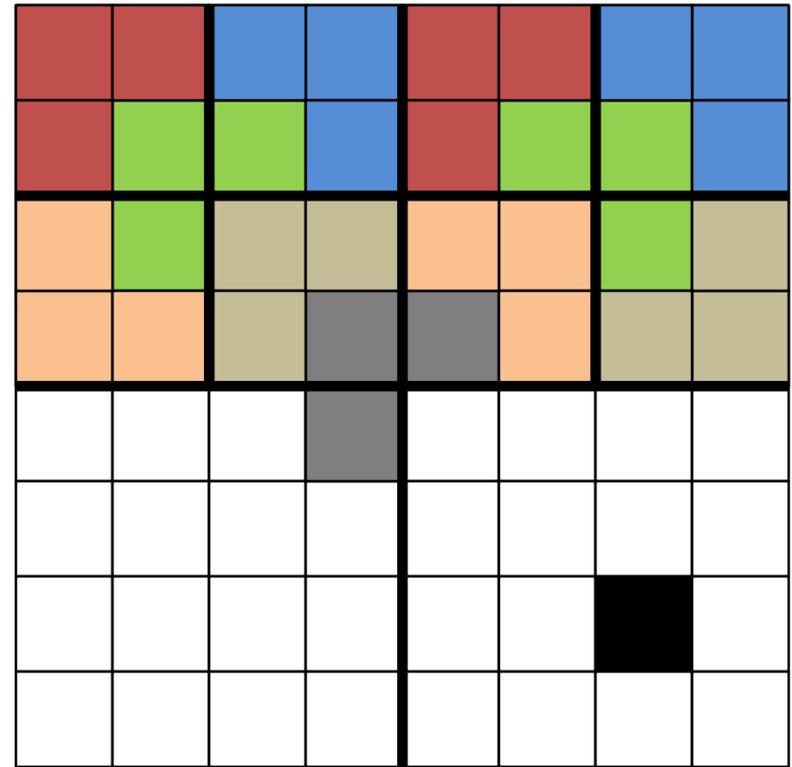
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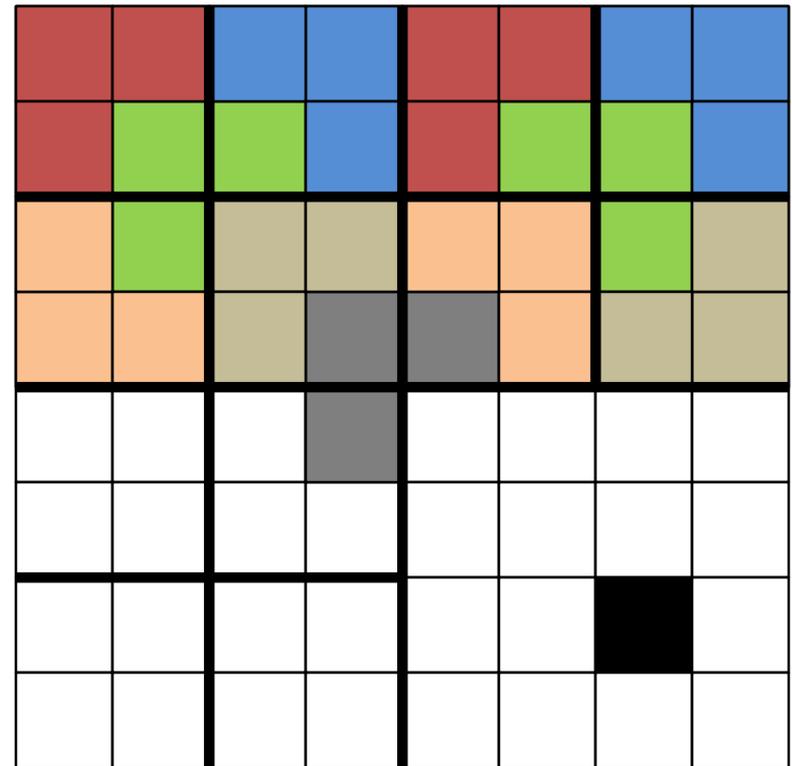
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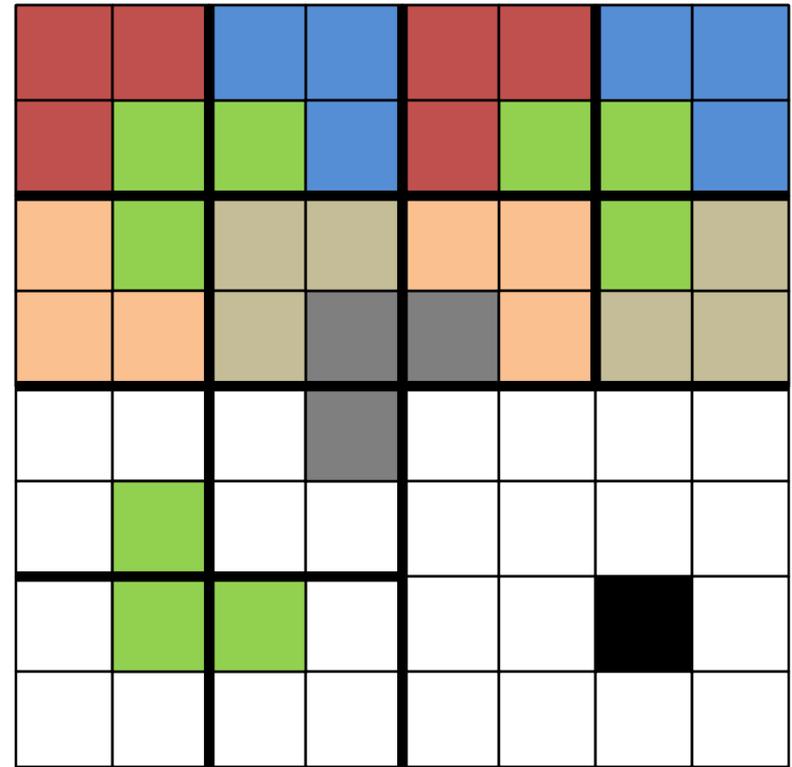
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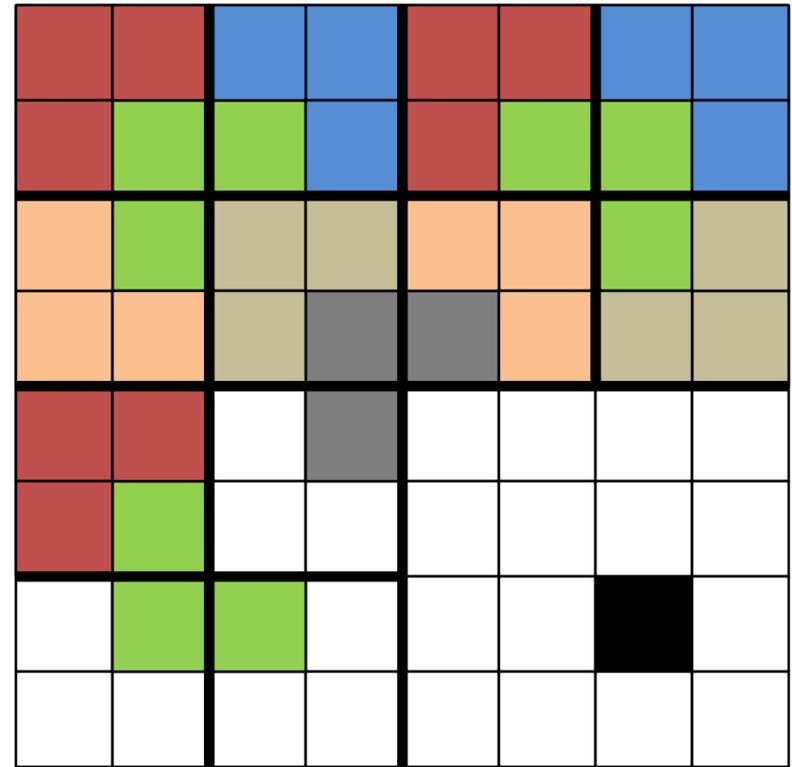
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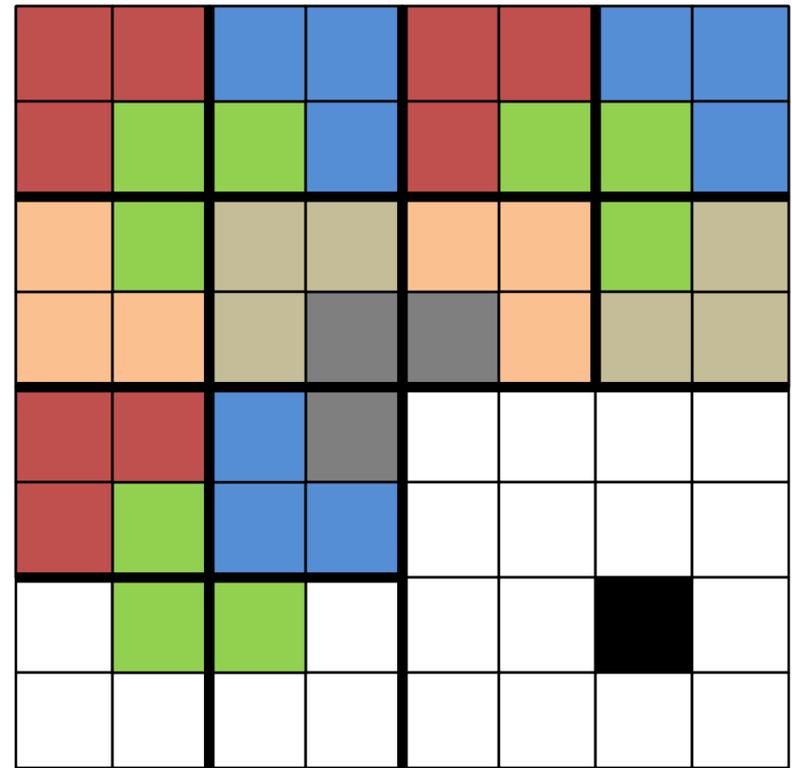
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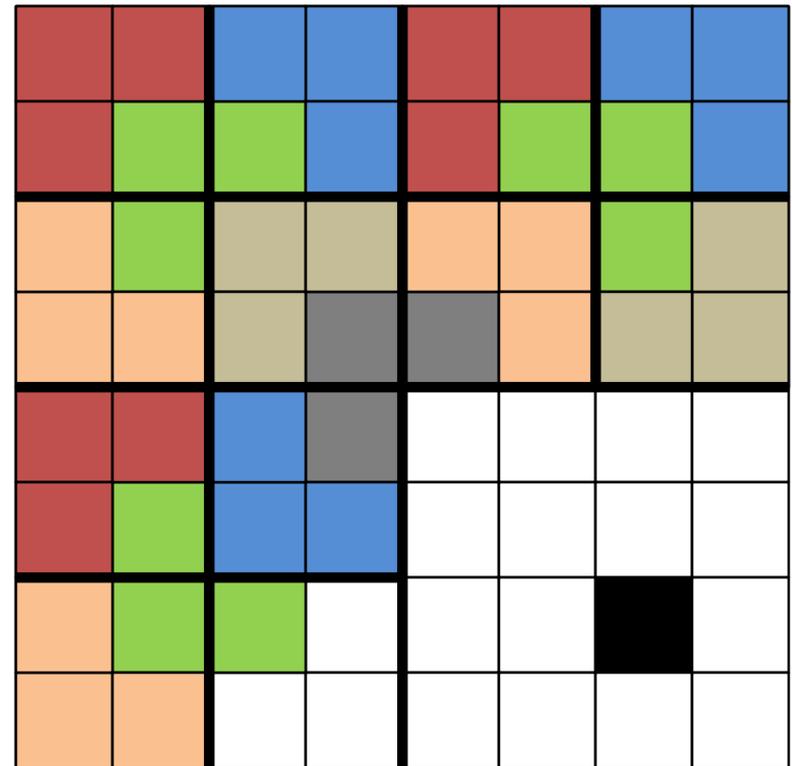
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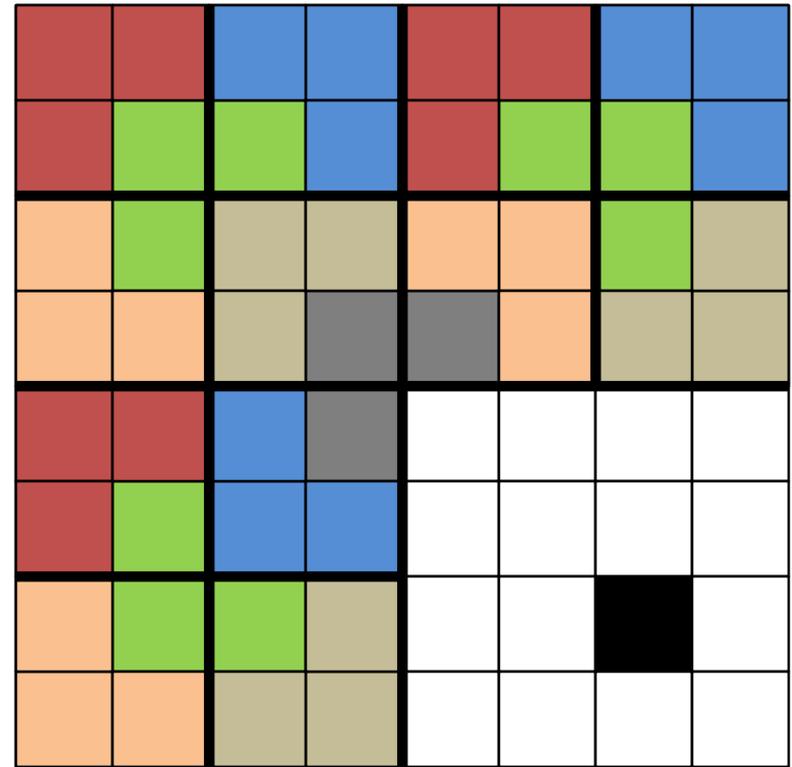
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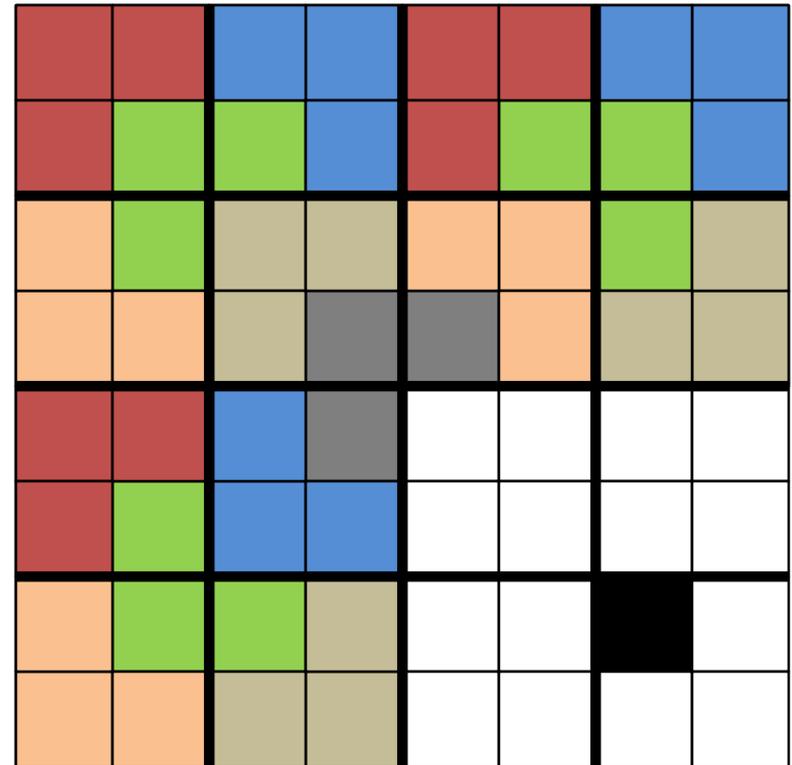
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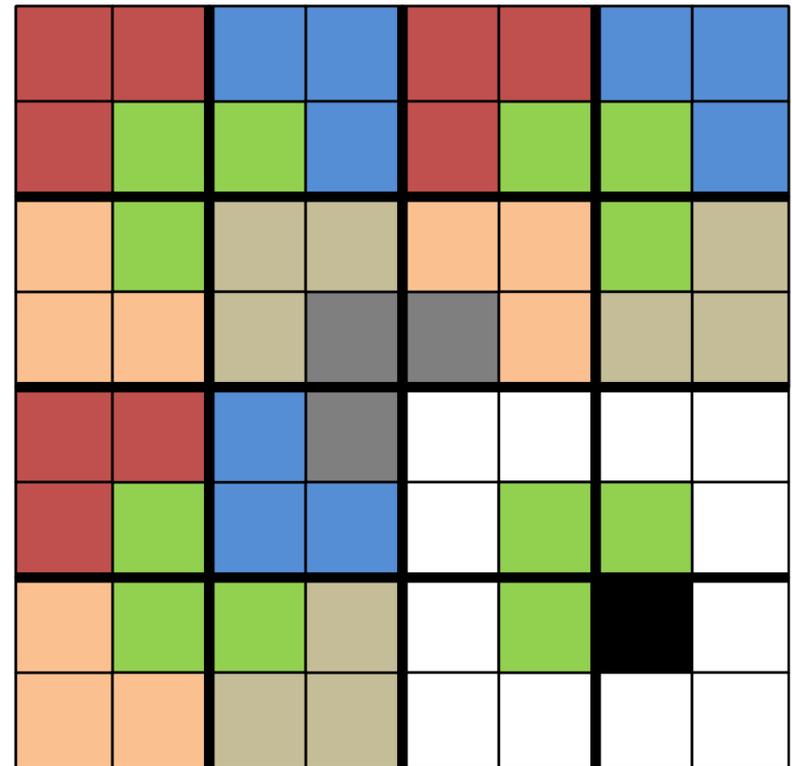
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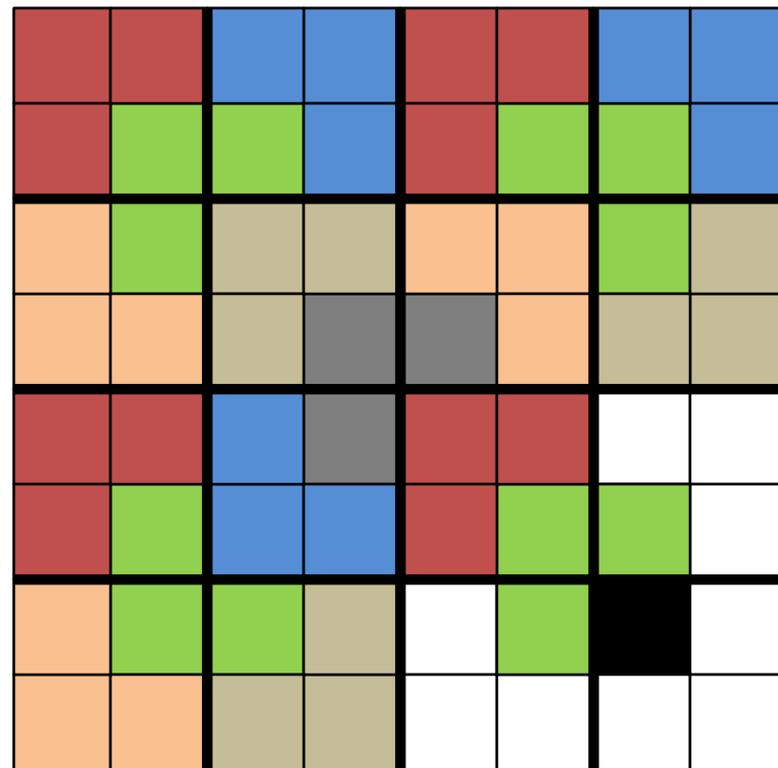
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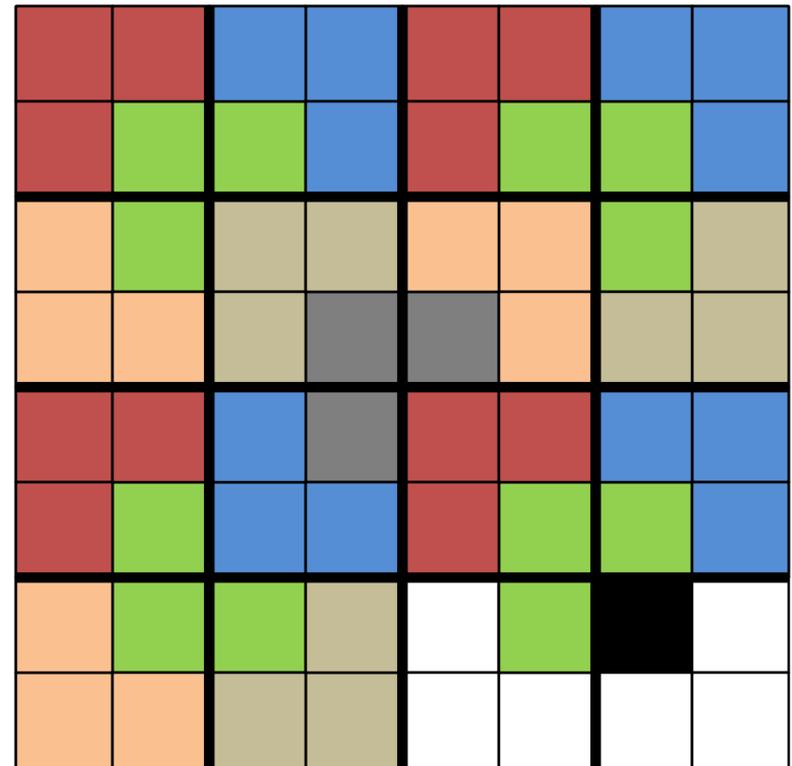
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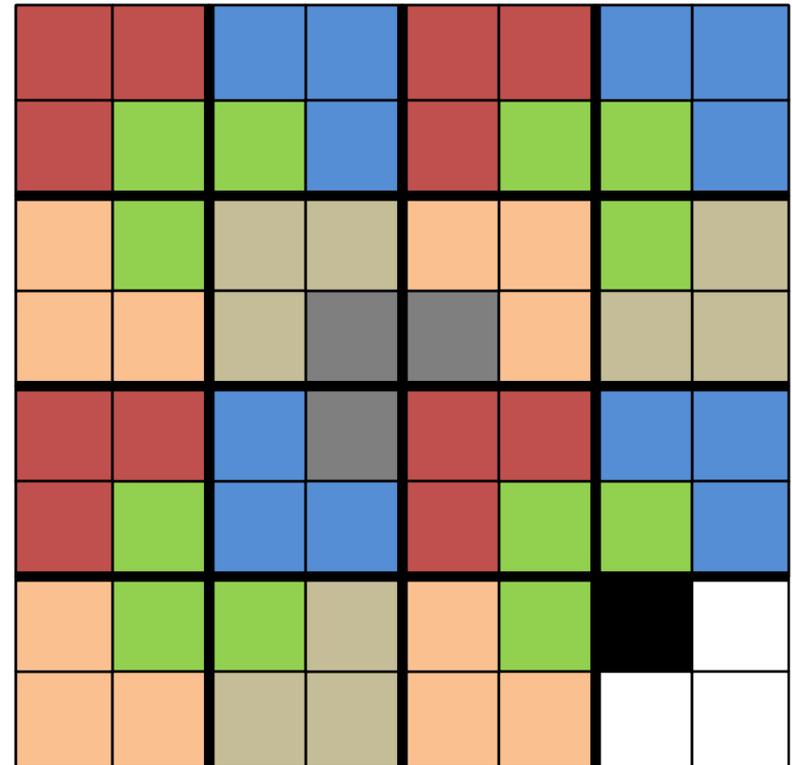
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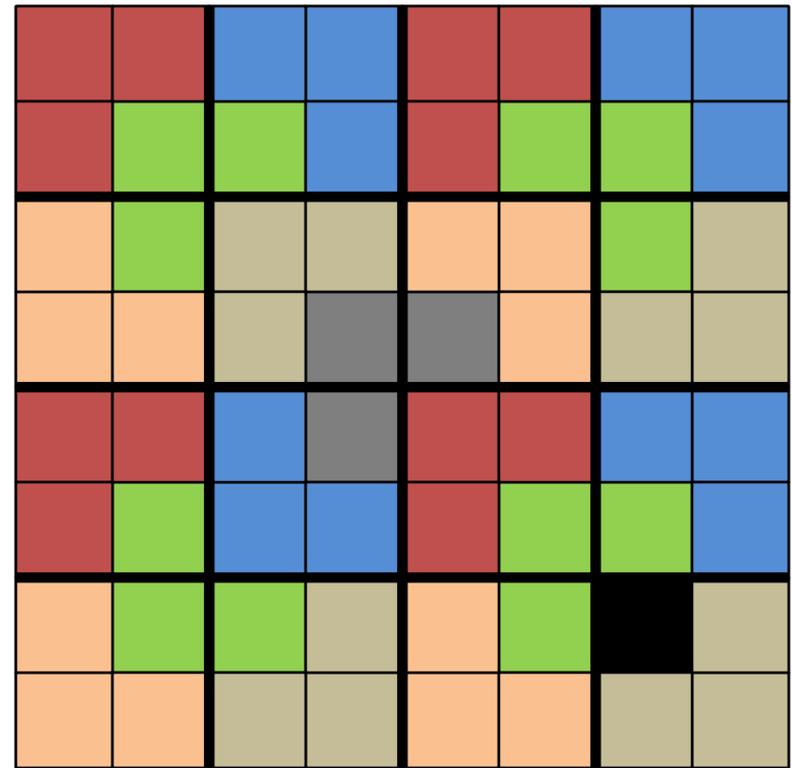
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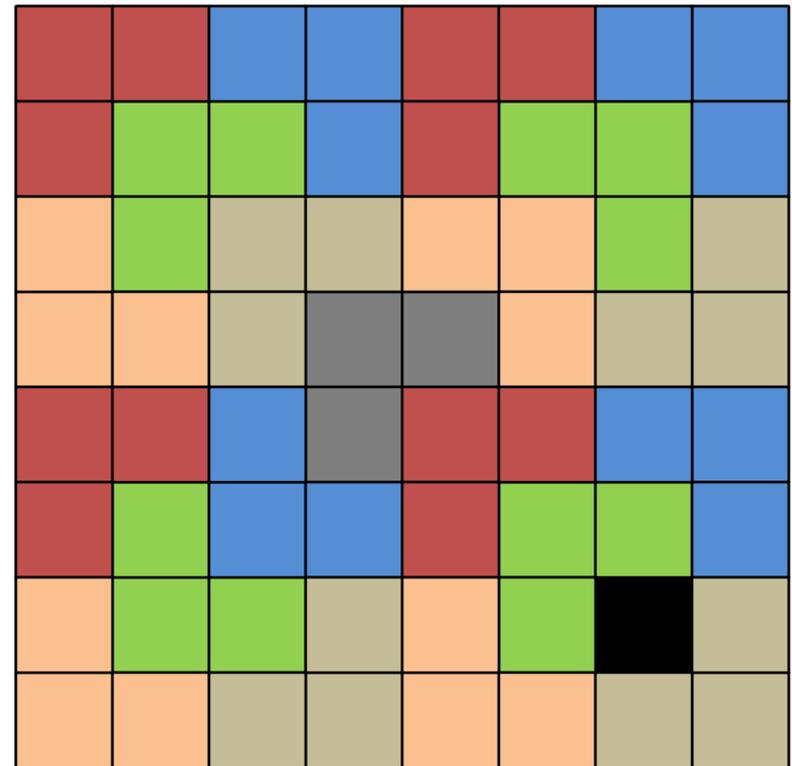
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- This algorithm design technique is called *recursive divide & conquer*.



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A Latin Phrase

“Divide et impera”

(meaning: “divide and rule” or “divide and conquer”)

— *Philip II, king of Macedon (382-336 BC),
describing his policy toward the Greek city-states
(some say the Roman emperor Julius Caesar,
100-44 BC, is the source of this phrase)*

The strategy is to break large power alliances into smaller ones that are easier to manage (or subdue).

This is a combination of political, military and economic strategy of gaining and maintaining power.

Unsurprisingly, this is also a very powerful problem solving strategy in computer science.

Divide-and-Conquer

1. **Divide:** divide the original problem into smaller subproblems that are easier to solve
2. **Conquer:** solve the smaller subproblems
(perhaps recursively)
3. **Merge:** combine the solutions to the smaller subproblems to obtain a solution for the original problem

Integer Multiplication

Multiplying Two n -bit Numbers

$$\begin{array}{l} x = \overbrace{\left[\begin{array}{|c|c|} \hline x_L & x_R \\ \hline \end{array} \right]}^{\frac{n}{2} \text{ bits} \quad \frac{n}{2} \text{ bits}} = 2^{n/2}x_L + x_R \\ y = \overbrace{\left[\begin{array}{|c|c|} \hline y_L & y_R \\ \hline \end{array} \right]}^{n \text{ bits}} = 2^{n/2}y_L + y_R \end{array}$$

$$xy = (2^{n/2}x_L + x_R)(2^{n/2}y_L + y_R) = 2^n x_L y_L + 2^{n/2}(x_L y_R + x_R y_L) + x_R y_R$$

So # $\frac{n}{2}$ -bit products: 4

bit shifts (by n or $\frac{n}{2}$ bits): 2

additions (at most $2n$ bits long) : 3

We can compute the $\frac{n}{2}$ -bit products recursively.

Let $T(n)$ be the overall running time for n -bit inputs. Then

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 4T\left(\frac{n}{2}\right) + \Theta(n) & \text{otherwise.} \end{cases} = \Theta(n^2) \quad (\text{how? derive})$$

Multiplying Two n -bit Numbers Faster (Karatsuba's Algorithm)

$$\begin{array}{l}
 x = \overbrace{\begin{array}{|c|c|} \hline x_L & x_R \\ \hline \end{array}}^{\frac{n}{2} \text{ bits}} = 2^{n/2}x_L + x_R \\
 y = \overbrace{\begin{array}{|c|c|} \hline y_L & y_R \\ \hline \end{array}}^{\frac{n}{2} \text{ bits}} = 2^{n/2}y_L + y_R \\
 \underbrace{\hspace{10em}}_{n \text{ bits}}
 \end{array}$$

$$\begin{aligned}
 xy &= (2^{n/2}x_L + x_R)(2^{n/2}y_L + y_R) \\
 &= 2^n x_L y_L + 2^{n/2}(x_L y_R + x_R y_L) + x_R y_R \\
 &= 2^n x_L y_L + 2^{n/2}((x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R) + x_R y_R
 \end{aligned}$$

So # $\frac{n}{2}$ - or $(\frac{n}{2} + 1)$ -bit products: 3

Then the overall running time for n -bit inputs:

$$\begin{aligned}
 T(n) &= \begin{cases} \Theta(1) & \text{if } n = 1, \\ 3T\left(\frac{n}{2}\right) + \Theta(n) & \text{otherwise.} \end{cases} \\
 &= \Theta(n^{\log_2 3}) = O(n^{1.59}) \text{ (how? derive)}
 \end{aligned}$$

Algorithms for Multiplying Two n -bit Numbers

Inventor	Year	Complexity
Classical	—	$\Theta(n^2)$
Anatolii Karatsuba	1960	$\Theta(n^{\log_2 3})$
Andrei Toom & Stephen Cook (generalization of Karatsuba's algorithm)	1963 – 66	$\Theta\left(n 2^{\sqrt{2 \log_2 n}} \log n\right)$
Arnold Schönhage & Volker Strassen (Fast Fourier Transform)	1971	$\Theta(n \log n \log \log n)$
Martin Fürer (Fast Fourier Transform)	2005	$n \log n 2^{O(\log^* n)}$

Lower bound: $\Omega(n)$ (why?)